Formalization of Rings with Explicit Divisibility in Type Theory

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Introduction

- SSReflect
- Rings with explicit divisibility
  - GCD domains
  - Bézout domains
  - Euclidean rings
- Smith normal form
  - Constructive PIDs

SSReflect

- Extension to Coq
- George Gonthier: formalization of the four color theorem
- Cayley-Hamilton theorem, decidability of ACF...

- Feit-Thompson theorem (part of the classification of finite simple groups)

SSReflect

- Small Scale Reflection
- New tactics and tacticals
- Library of mathematical theories

DvdRing

- A ring R has *explicit divisibility* if it has a divisibility test that give witnesses:

$$a \mid b \leftrightarrow \exists x.\, b = xa$$

- $\mathbb{Z}$ and $k[x]$ where k is a field (e.g. $\mathbb{Q}$)

DvdRing

- Demo!

GCD domains

- GCD domain: Every pair of elements have a greatest common divisor

$$\forall a\, b. \exists g \,.\, g \mid a \wedge g \mid b \wedge \forall g' \,.\, g' \mid a \wedge g' \mid b \to g' \mid g$$

Properties of GCD domains

Definition

The gcd of the coefficients of $p \in R[x]$ is called the *content* of p, written $cont(p)$

Definition

$p \in R[x]$ is *primitive* if $cont(p) = 1$

Theorem

Gauss lemma: $cont(pq) = cont(p)cont(q)$

Theorem

Every polynomial $p \in R[x]$ *can be written as* $p = cont(p)q$ *with* q *primitive*

Properties of GCD domains

- Using this one can give an algorithm for computing the gcd of $p, q \in R[x]$
- Give a proof that if R is a GCD domain then $R[x]$ also is
- Don't use field of fractions!
- Can compute gcd in $\mathbb{Z}[x_1, \ldots, x_n]$ and $k[x_1, \ldots, x_n]$

Bézout domains

- Non-Noetherian analogue of principal ideal domains
- PID: Every ideal is principal
  - Quantification over all ideals
- Bézout domain: Every finitely generated ideal is principal
- Equivalent definition:

$$\forall a\, b. \exists x\, y.\; ax + by = gcd(a, b)$$

Euclidean rings

- Euclidean norm, $f : R \to \mathbb{N}$
- Euclidean division: $\forall a\, b.\, \exists q\, r.\, a = bq + r$ and either $f(r) < f(b)$ or $r = 0$
- Examples: $\mathbb{Z}$ with absolute value and $k[x]$ with degree

Bézout domains and Euclidean rings

Theorem

Every Euclidean ring is a Bézout domain

Theorem

Every Bézout domain is a GCD domain

Theorem

$\mathbb{Z}$ *is a Euclidean ring*

Theorem

$k[x]$ *is a Euclidean ring*

Smith normal form

- Generalization of Gauss elimination algorithm
- Elements from a PID and not just a field
- Compute homology groups of simplicial complexes
  - "Homology is a rigorous mathematical method for detecting and categorizing holes in a shape." - Wikipedia

Smith normal form

- Let A be a nonzero $m \times n$ matrix over a PID. There exists invertible $m \times m$ and $n \times n$ matrices S,T such that

$$SAT = \begin{pmatrix} \alpha_1 & 0 & 0 & & \cdots & & 0 \\ 0 & \alpha_2 & 0 & & \cdots & & 0 \\ 0 & 0 & \ddots & & & & 0 \\ & & & \alpha_r & & & \vdots \\ \vdots & & & & 0 & & \\ & & & & & \ddots & \\ 0 & & & \cdots & & & 0 \end{pmatrix}$$

and $\alpha_i \mid \alpha_{i+1}$, $1 \leq i < r$

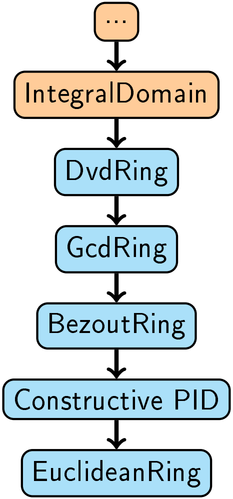
Smith normal form

- Open question for Bézout domains
- Need constructive approximation of PIDs

Constructive PIDs

- Mines, Richman, Ruitenburg: Bézout domains such that if we have a sequence $u(n)$ with $u(n+1) \mid u(n)$ then there exists k such that $u(k) \mid u(k+1)$
- In type theory this can be represented as that strict divisibility is well founded

Summary



...
IntegralDomain
DvdRing
GcdRing
BezoutRing
Constructive PID
EuclideanRing

Conclusions and further work

- Have: Divisibility theory in SSReflect
  - Formalized algorithms for $\mathbb{Z}$ and $k[x]$
- Todo: R GCD domain implies $R[x]$ GCD domain
- Have: Smith normal form algorithm in Haskell
- Todo: Formalize Smith normal form algorithm

Questions?

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