Restricted coloring problems on graphs with few P_4's

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Abstract

In this paper, we obtain polynomial time algorithms to determine the acyclic chromatic number, the star chromatic number and the harmonious chromatic number of P_4-tidy graphs and $(q, q-4)$-graphs, for every fixed q. These classes include cographs, P_4-sparse and P_4-lite graphs. We also obtain a polynomial time algorithm to determine the Grundy number of $(q, q-4)$-graphs. All these coloring problems are known to be NP-hard for general graphs.

Keywords: Acyclic, star, harmonious and greedy colorings, $(q, q-4)$-graphs, P_4-tidy graphs, polynomial time algorithms.

1 Introduction

Let $G = (V, E)$ be a finite undirected graph, without loops and multiple edges. The complete bipartite graph with partitions of size m and n is denoted by $K_{m,n}$. A $K_{1,n}$ is called a star. A P_4 is an induced path with four vertices. A cograph is any P_4-free graph. The graph terminology used here follows [4].

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A k-coloring of G is a partition $\{V_1, \dots, V_k\}$ of $V(G)$ into stable sets. The stable sets $V_1, \dots, V_k$ are called *color classes* and we say that a vertex in V_i is colored i. The chromatic number $\chi(G)$ of G is the smallest integer k such that G admits a k-coloring.

An *acyclic* coloring is a coloring such that every cycle receives at least three colors (that is, every pair of color classes induces a forest). A *star* coloring is an acyclic coloring such that every P_4 receives at least three colors (that is, every pair of color classes induces a forest of stars). An *harmonious* coloring is a star coloring such that every pair of color classes induces at most one edge.

It is easy to see that any coloring of a split or chordal graph is acyclic. In 2009, Lyons [11] proved that every acyclic coloring of a cograph is also a star coloring.

The acyclic, star and harmonious chromatic numbers of G, denoted respectively by $\chi_a(G)$, $\chi_s(G)$, $\chi_h(G)$, are the minimum number of colors k such that G admits an acyclic, star and harmonious coloring with k colors. By definitions, $\chi(G) \leq \chi_a(G) \leq \chi_s(G) \leq \chi_h(G)$.

In 2004, Albertson et al. [1] proved that computing the star chromatic number is NP-hard even for planar bipartite graphs. In 2004, Fertin, Raspaud and Reed give exact values of $\chi_s(G)$ for several graph classes [9]. In 2007, Asdre et al. [3] proved that determining the harmonious chromatic number is NP-hard for interval graphs, permutation graphs and split graphs.

Other coloring parameters we are interested in define maximization problems, as the Grundy number. A greedy coloring of G is a coloring such that every vertex colored k has a neighbor colored i for every $i = 1, 2, \dots, k-1$. The Grundy number of G, $\Gamma(G)$, is the largest integer k such that G admits a greedy coloring with k colors. Determining the Grundy number is NP-complete even for complements of bipartite graphs [12].

Many NP-hard problems were proved to be polynomial time solvable for cographs. For example, Lyons [11] obtained a polynomial time algorithm to find an optimal acyclic and an optimal star coloring of a cograph. However, it is known that computing the harmonious chromatic number of a disconnected cograph is NP-hard [8].

Some superclasses of cographs, defined in terms of the number and structure of its induced P_4's, can be completely characterized by their modular or primeval decomposition. Among these classes, we cite P_4-sparse graphs, P_4-lite graphs, P_4-tidy graphs and $(q, q-4)$-graphs.

Babel and Olariu [6] defined a graph as $(q, q-4)$-*graph* if no set of at most q vertices induces more than $q-4$ distinct P_4's. Cographs and P_4-sparse graphs are precisely $(4, 0)$-graphs and $(5, 1)$-graphs respectively. P_4-lite graphs are special $(7,3)$-graphs. We say that a graph is P_4-tidy if, for every P_4 induced by $\{u,v,x,y\}$, there exists at most one vertex z such that $\{u,v,x,y,z\}$ induces more than one P_4. In this paper, we prove the following result:

Theorem 1.1 (main theorem) *Let q be a fixed integer. There exists linear time algorithms to obtain*

- $\chi_a(G)$ *and* $\chi_s(G)$*, if* G *is a* P_4*-tidy or a* $(q,q-4)$*-graph;*
- $\chi_h(G)$*, if* G *is a* P_4*-tidy or a* $(q,q-4)$*-graph, and* G *is connected.*
- $\Gamma(G)$*, if* G *is a* $(q,q-4)$*-graph;*

2 Primeval and Modular decomposition

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two vertex disjoint graphs. The disjoint union of G_1 and G_2 is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The join is the graph $G_1 \vee G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{uv : \ u \in V_1, \ v \in V_2\})$.

A *spider* is a graph whose vertex set has a partition (R, C, S), where $C = \{c_1, \ldots, c_k\}$ and $S = \{s_1, \ldots, s_k\}$ for $k \geq 2$ are respectively a clique and a stable set; s_i is adjacent to c_j if and only if $i = j$ (a thin spider), or s_i is adjacent to c_j if and only if $i \neq j$ (a thick spider); and every vertex of R is adjacent to each vertex of C and non-adjacent to each vertex of S. A *quasi-spider* is a graph obtained from a spider (R, C, S) by replacing at most one vertex from $C \cup S$ by a K_2 (the complete graph on two vertices) or a $\overline{K_2}$ (the complement of K_2).

Jamison and Olariu [6] proved an important structural theorem for $(q,q-4)$-graphs, using their primeval decomposition. A graph is *p-connected* if, for every bipartition of the vertex set, there is a crossing P_4. A *separable p-component* is a maximal p-connected subgraph with a particular bipartition (H_1, H_2) such that every crossing P_4 $wxyz$ satisfies $x, y \in H_1$ and $w, z \in H_2$.

Theorem 2.1 (Characterization of $(q,q-4)$-graphs [6]) *A graph G is a $(q,q-4)$-graph if and only if exactly one of the following holds:*

(a) G or $\overline{G}$ is disconnected and each one of its components is a $(q,q-4)$-graph;

(b) G is a spider (R, C, S) and $G[R]$ is a $(q,q-4)$-graph;

(c) G contains a separable p-component H, with bipartition (H_1, H_2) and $|V(H)| \leq q$, such that $G - H$ is a $(q,q-4)$-graph and every vertex of $G - H$ is adjacent to every vertex of H_1 and non-adjacent to every vertex of H_2.

Using the modular decomposition of P_4-tidy graphs, Giakoumakis et al. proved a similar result for this class [10].

Theorem 2.2 (Characterization of P_4-tidy graphs [10]) *A graph G is a P_4-tidy graph if and only if exactly one of the following holds:*

(a) G or $\overline{G}$ is disconnected and each one of its components is a P_4-tidy graph;

(b) G is a quasi-spider (R, C, S) and $G[R]$ is a P_4-tidy graph;

(c) G is isomorphic to P_5, $\overline{P_5}$, C_5 or a single vertex.

As a consequence, a $(q, q-4)$-graph (resp. a P_4-tidy graph) G can be decomposed by successively applying Theorem 2.1 (resp. Theorem 2.2) as follows: If (a) holds, apply the theorem to each component of G or $\overline{G}$ (operations disjoint union and join). If (b) holds, apply the theorem to $G[R]$ (operation spider or quasi-spider). Finally, if (c) holds and G is a $(q, q-4)$-graph, then apply the theorem to $G - H$ (operation small subgraph).

The idea now is to consider the graph by the means of its decomposition tree obtained as described. According to the coloring parameter to be determined, the tree will be visited in an up way or bottom way fashion. We notice that the primeval and modular decomposition of any graph can be obtained in polynomial time [6].

3 Dealing with Disjoin Union, Join and Spiders

We start by recalling a result from [11] for acyclic and star chromatic numbers.

Lemma 3.1 (χ_a and χ_s for union and join [11]) *Given graphs G_1 and G_2, let $G' = G_1 \cup G_2$ and $G^* = G_1 \vee G_2$. We have that $\chi_a(G') = \max\{\chi_a(G_1), \chi_a(G_2)\}$, $\chi_s(G') = \max\{\chi_s(G_1), \chi_s(G_2)\}$, $\chi_a(G^*) = \min\{\chi_a(G_1) + |V(G_2)|, \chi_a(G_2) + |V(G_1)|\}$ and $\chi_s(G^*) = \min\{\chi_s(G_1) + |V(G_2)|, \chi_s(G_2) + |V(G_1)|\}$.*

The two following lemmas deal with spiders and quasi-spiders. Consider $\chi_a(G[R]) = \chi_s(G[R]) = 0$ whenever $R = \emptyset$.

Lemma 3.2 (χ_a and χ_s for spiders) *Let G be a spider (R, C, S), where $|C| = |S| = k$. Then $\chi_a(G) = \chi_a(G[R]) + k$ and $\chi_s(G) = \chi_s(G[R]) + k$, unless $R = \emptyset$ and G is thick, when in this case, $\chi_s(G) = k + 1$.*

Lemma 3.3 (χ_a and χ_s for quasi-spiders) *Let G be a quasi-spider (R, C, S) such that $\min\{|C|, |S|\} = k$ and $\max\{|C|, |S|\} = k + 1$. Let $H = K_2$ or $H = \overline{K_2}$ be the subgraph that replaced a vertex of $C \cup S$. Then*

$$\chi_a(G) = \begin{cases} \chi_a(G[R]) + k + 1, & \textit{if } V(H) \in C, \\ \chi_a(G[R]) + k + 1, & \textit{if } H = K_2 \textit{ and } G \textit{ is thick and } R = \emptyset, \\ \chi_a(G[R]) + k, & \textit{otherwise}, \end{cases}$$

$$\chi_s(G) = \begin{cases} \chi_s(G[R]) + k, & \text{if } V(H) \in S \text{ and } G \text{ is thin,} \\ \chi_s(G[R]) + k, & \text{if } V(H) \in S \text{ and } G \text{ is thick and } R \neq \emptyset, \\ \chi_s(G[R]) + k + 2, & \text{if } V(H) \in C \text{ and } G \text{ is thick and } R = \emptyset, \\ \chi_s(G[R]) + k + 1, & \text{otherwise.} \end{cases}$$

Lemma below determines the harmonious chromatic number for join and spider operations. Recall that χ_h for union operation is NP-hard [8].

Lemma 3.4 (χ_h for join and quasi-spiders) *Let G be a graph with n vertices. If G is the join of two graphs G_1 and G_2, then $\chi_h(G) = n$. If G is a quasi-spider (R, C, S) with $k = \max\{|C|, |S|\}$, then $\chi_h(G) = |R| + k + 1$, if G is thin, or $\chi_h(G) = n$, for otherwise.*

Some results are known for the Grundy number. In [13], it is proved that $\Gamma(G_1 \cup G_2) = \max\{\Gamma(G_1), \Gamma(G_2)\}$ and that $\Gamma(G_1 \vee G_2) = \Gamma(G_1) + \Gamma(G_2)$. In 2009, Araujo and Linhares Sales [2] proved that, if G is a spider (R, C, S) and $\Gamma(R)$ is given, then $\Gamma(G)$ can be determined in linear time. They also obtain a linear time algorithm to determine $\Gamma(G)$ for P_4-tidy graphs.

4 Coloring $(q, q-4)$-graphs

Theorem below proves that determining the chromatic numbers χ_a, χ_s, χ_h and Γ for item (c) of Theorem 2.1 is linear time solvable, if q is a fixed integer.

Theorem 4.1 *Let G be a $(q, q-4)$-graph which contains a separable p-component H, with bipartition (H_1, H_2) and at most q vertices, such that every vertex from $V(G) - H$ is adjacent to all vertices in H_1 and non-adjacent to all vertices in H_2. Then, given $\chi_a(G-H)$, $\chi_s(G-H)$, $\chi_h(G-H)$ and $\Gamma(G-H)$, we can obtain $\chi_a(G)$, $\chi_s(G)$, $\chi_h(G)$ and $\Gamma(G)$ in linear time.*

In the complete version of this paper, each parameter is treated separately and the proofs of each theorem are extensive and extremely technical. However, they all take advantage on the fact that H is a small subgraph with size at most q (fixed). Therefore, it is possible to enumerate all colorings of H and match each one with a coloring of $G - H$, in order to choose the optimal combination in constant time. Observe that the time complexity for this procedure depends only on q other than n.

Theorem 4.1, together with the lemmas and theorems of Sections 2 and 3, complete the proof of Theorem 1.1.

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