Routing Games : From Altruism to Egoism

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Routing Games

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Routing Games

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Routing Background

Routing Games
System Model
Cooperation Paradigm
Numerical Investigation
What we learn!!!
Existence and Uniqueness of NEP
Non-Atomic Users
Summary

Routing

General Routing

- Input
  - network topology, link metrics, and traffic matrix
- Output
  - set of routes to carry traffic
Routing as optimization problem
  - e.g., minimum total delay in network
  - focus on global network performance (social optimal)
  - performance of individual user not important

Centralized or distributed algorithms
  - e.g., link state or distance vector
Routing as game between users
- users determine route
- decision based solely on individual performance (selfish routing)
- strongly dependent on other users decisions

Non-cooperative game (non-zero sum)
- users compete for network resources

Equilibrium point of operation
- Nash equilibrium point (NEP)
Applications of Game Theory to Network Selfish Routing

- Competitive routing in multiuser communication networks
  A. Orda, R. Rom and N. Shimkin
  IEEE/ACM Transactions on Networking, 1 (5) 1993

- How bad is selfish routing?
  T. Roughgarden and E. Tardos
  Journal of the ACM, 49 (2) 2002

- Selfish routing with atomic players
  T. Roughgarden
  ACM/SIAM Symp. on Discrete Algorithms (SODA) 2005
Parallel Links

- set of users share a set of parallel links
- each user has fixed demand (data rate)
- users decide how to split demand across links
  - minimize individual cost
- link has a load dependent cost (e.g., delay)
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Routing Games

Background

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Investigation

What we learn !!!!

Existence and

Uniqueness of

NEP

Non - Atomic

Users

Summary

System Model

- **Network** : a graph \( \mathcal{G} = (\mathcal{V}, \mathcal{L}) \)
  - \( \mathcal{V} \) is a set of nodes
  - \( \mathcal{L} \subseteq \mathcal{V} \times \mathcal{V} \) is set of directed links.
- \( \mathcal{I} = \{1, 2, ..., I\} \) is a set of users which share the network \( \mathcal{G} \).
- \( f_i^l \) is flow of user \( i \) in link \( l \).
- Each user \( i \) has a throughput demand rate \( r_i \) (which can be split among various path).
- **Strategy** : \( f^i = (f_i^l)_{l \in \mathcal{L}} \) is the routing strategy of user \( i \).

Assumptions :

- At least one link exist between each pair of nodes (in each direction).
- Flow is preserved at all nodes.
Nash Equilibrium

Cost/Utility function $J^i(f) = \sum_i f^i T_i(f)$.

Each user seeks to minimize the cost function $J^i$, which depends upon routing strategy of user $i$ as well as on the routing strategy of other users.

Nash Equilibrium

A vector $\tilde{f}^i$, $i = 1, 2, \ldots, I$ is called a Nash equilibrium if for each user $i$, $\tilde{f}^i$ minimizes the cost function given that other users’ routing decisions are $\tilde{f}^j$, $j \neq i$. In other words,

$$\tilde{J}^i(\tilde{f}^1, \tilde{f}^2, \ldots, \tilde{f}^I) = \min_{f^i \in F^i} \tilde{J}^i(\tilde{f}^1, \tilde{f}^2, \ldots, f^i, \ldots, \tilde{f}^I),$$

$$i = 1, 2, \ldots, I,$$  \hspace{1cm} (1)

where $F^i$ is the routing strategy space of user $i$. 
Consider the following network topology

Load Balancing Network

\[ \hat{J}^i = \sum_{l \in \{1, \ldots, 4\}} f_l^i T_l(f_l) \]

Parallel Link Network

\[ \hat{J}^i = \sum_{l \in \{1, 2\}} f_l^i T_l(f_l) \]
Consider the following Cost function.

**Linear Cost Function**
- Used in Transportation Networks
- \( T_l(f_l_i) = a_i f_l_i + g_i \) for link \( i = 1, 2 \), where as,
- \( T_l(f_l_j) = c f_l_j + d \) for link \( j = 3, 4 \).

**M/M/1 Delay Cost Function**
- Used in Queueing Networks
- \( T_l(f_l_i) = \frac{1}{C_l_i - f_l_i} \), where the \( C_l_i \) and \( f_l_i \) denote the total capacity and total flow of the link \( l_i \).

For parallel link topology only link \( l_i, i = 1, 2 \) exist while for load balancing topology link \( l_i, i = 3, 4 \) also exist.
Related work

For Selfish Users

Orda et al


Kameda et al


- Orda et al has shown unique Nash equilibrium for Parallel link network with MM1 cost function.
- Kameda et al also claim unique Nash equilibrium for Load balancing network with MM1 cost function.
- Braess-like paradox is observed by Kameda et al in Load balancing network with MM1 cost function.
What happens with "User Cooperation"?
**Degree of Cooperation**

**Definition**

Let $\alpha^i = (\alpha^i_1, .., \alpha^i_{|I|})$ be the *degree of Cooperation* for user $i$. The new operating cost function $\hat{J}^i$ of user $i$ with Degree of Cooperation, is a convex combination of the cost of user from set $I$,

$$\hat{J}^i(f) = \sum_{k \in I} \alpha^i_k J^k(f); \quad \sum_k \alpha^i_k = 1, i = 1, ... |I|$$

- **Non cooperative user** : $\alpha^i_k = 0$ for all $k \neq i$ $\Rightarrow$ User $i$ takes into account of only its cost
- **Cooperative (Equally cooperative)** : $\alpha^i_j = \frac{1}{|P|}$, where, $j \in P$, $P \subseteq I$ $\Rightarrow$ User $i$ takes into account the cost of each users $j$ (including itself).
- **Beyond Cooperation - Altruistic user** : $\alpha^i_i = 0$ $\Rightarrow$ User $i$ takes into account the cost of only other users
Each user still seeks to minimize the operating cost function $\hat{j}_i$.

**Non-Cooperative Framework**

We can benefit to apply the properties of non-cooperative games. e.g. (Nash Equilibrium etc.)
Consider the following network topology

**Load Balancing Network**

\[
\hat{J}^i = \sum_{l \in \{1,\ldots,4\}} \sum_{k \in \{1,2\}} \alpha_{ik}^l T_l(f_l)
\]

**Parallel Link Network**

\[
\hat{J}^i = \sum_{l \in \{1,2\}} \sum_{k \in \{1,2\}} \alpha_{ik}^l T_l(f_l)
\]
Related work

On Various degree of Cooperation
Michiardi Pietro, Molva Refik A game theoretical approach to evaluate cooperation enforcement mechanisms in mobile ad hoc networks WiOpt’03

On Altruism
Handbook of the Economics of Giving, Altruism and Reciprocity, Volume 1, 2006, Edited by Serge-Christophe Kolm and Jean Mercier Ythier

”Motivationally, altruism is the desire to enhance the welfare of others at a net welfare loss to oneself.”
Load Balancing Network with Linear link Cost

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What we learn ! ! ! ! !
Existence and Uniqueness of NEP
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Summary

Cost at Nash Equilibria
Flow at Nash Equilibria

Parameters: \( a = 1, c = 0, d = 0.5, \)
Cooperation: \{ Symmetrical: \( \alpha^1 = \alpha^2, \) Asymmetrical: \( 0 \leq \alpha^1 \leq 1, \alpha^2 = 1 \) \}
Some strange observation

- Multiple Nash equilibrium ...
Cooperation Paradox

Cost at Nash Equilibrium

Parameters: $a = 1$, $c = 0$, $d = 0.5$.

Cooperation Paradox: Cooperation improves the cost.

- Selfishness is not good always :)
Braess like Paradox

Cost at Nash Equilibrium

Parameters: $a_1 = a_2 = 4.1$, $d = 0.5$
Symmetrical: $\alpha^1 = \alpha^2 = 0.93$
Braess Paradox: Additional resources degrades the performance.
Parallel Link Network with Linear link Cost

Cost at Nash Equilibrium

Flow at Nash Equilibrium

Parameters: $a = 1, c = 0, d = 0.5$. 
Load balancing network with M/M/1 link cost

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Observation Summary

- Uniqueness of NEP is lost
- Paradox in Cooperation
- Braess like paradox
Assumptions on Cost function

Consider the following assumption on the Cost function \( J^i \)

**Type G function- Assumptions**

- **G1**: \( J^i(f) = \sum_{l \in L} \hat{J}^i_l(f_l) \). Each \( \hat{J}^i_l \) satisfies:
- **G2**: \( J^i_l : [0, \infty) \rightarrow (0, \infty) \) is continuous function.
- **G3**: \( J^i_l \) is convex in \( f^i_j \) for \( j = 1, \ldots, |I| \).
- **G4**: Wherever finite, \( J^i_l \) is continuously differentiable in \( f^i_l \), denote \( K^i_l = \frac{\delta \hat{J}^i_l}{\delta f^i_l} \).

Existence of NEP is shown to exist in Orda et al for Selfish users operating on parallel link.
Existence and Uniqueness of NEP

Cost functions

\[ \hat{J}_i(f) = \sum_{l \in \mathcal{L}} (\alpha^i f^i_l + (1 - \alpha^i)f_i^{-i})T_l(f_i) \]

\[ = \sum_{l \in \mathcal{L}} (\alpha^i f_l + (1 - 2\alpha^i)f_i^{-i})T_l(f_i) \]

Existence can be studied as in Orda et al. (Shown to exist.)

Uniqueness of NEP

- for \( \alpha^i \leq 0.5 \) - Unique - Extended from Orda et al
- for \( \alpha^i > 0.5 \) - Not Unique (Because \( K^i_l(f_i^{-i}, f_i) \) is not strictly increasing function in \( f_i^{-i} \) and \( f_i \)).
Still some unique NEP can be obtained for \(\alpha > 0.5\)

**Theorem**

Consider the cost function of type B. Let \(\hat{f}\) and \(f\) be two Nash equilibria such that there exists a set of links \(\overline{L}_1\) such that 
\[
\{f^i_l > 0 \text{ and } \hat{f}^i_l, i \in I\} \text{ for } l \in \overline{L}_1, \text{ and } \{f^i_l = \hat{f}^i_l = 0, i \in I\} \text{ for } l \notin \overline{L}_1.
\]

Then \(\hat{f} = f\).

Unique NEP can be seen for some \(\alpha\).
Mixed Equilibrium

Network is shared by two types of users:

- **group users**: have to route a large amount of jobs; Seek Wardrop equilibria.
- **individual users**: have a single job to route; Seek Nash equilibria.


- Unique equilibria with M/M/1 cost function.
Mixed Equilibrium

Cost function

\( J^i : F \rightarrow [0, \infty) \) is the cost function for each user \( i \in \mathcal{N} \).

\( \mathcal{F}_p : F \rightarrow [0, \infty) \), is the cost function of path \( p \) for each individual user.

The aim of each user is to minimize its cost, i.e., for \( i \in \mathcal{N} \),

\[
\min_{f^i} J^i(f^i)
\]

and for individual user, \( \min_{p \in \mathcal{P}} \mathcal{F}_p(f^i) \). Let \( f_p \) be the amount of individual users that choose path \( p \).

Definition

\( f \in F \) is a Mixed Equilibrium (M.E.) if

\[
\forall i \in \mathcal{N}, \forall g^i \text{s.t.} (f^{-i}, g^i) \in F, \hat{J}^i(f) \leq \hat{J}^i(f^{-i}, g^i)
\]

\[
\forall p \in \mathcal{P}, \mathcal{F}_p(f) - A \geq 0; (\mathcal{F}_p(f) - A)f_p = 0
\]

where \( A = \min_{p \in \mathcal{P}} \mathcal{F}_p(f) \)
Mixed Equilibrium with Cooperation

We obtain closed form solutions with cooperation ($\alpha$) for a parallel link network with M/M/1 cost function.

**When Both link is used at Wardrop equilibrium**:

$$
\begin{align*}
(M_1, N_1) & \quad \text{if } a_1 < M_1 < b_1; \\
0, -cc & \quad \text{otherwise,} \\
(r_1, r_1 - cc) & \quad \text{if } r_1 < \min \left( r_2 + C_2 - C_1, \frac{\alpha(C_2 - C_1) + 2\alpha r_2}{2\alpha - 1} \right), \\
(r_1, r_1 - cc) & \quad \text{if } r_1 < \min \left( \frac{\alpha(C_2 - C_1)}{1 - 2\alpha}, r_2 - (C_2 - C_1) \right),
\end{align*}
$$

where

$$
M_1 = \frac{-\alpha(C_2 - C_1) + r_1(2\alpha - 1)}{2(2\alpha - 1)}, \quad N_1 = \frac{(C_1 - C_2)(1 - \alpha) + (2\alpha - 1)r_2}{2(2\alpha - 1)},
$$

$$
a_1 = \max \left( -\frac{C_2 - C_1}{2} - \frac{r_2 - r_1}{2}, 0 \right), \quad b_1 = \min \left( -\frac{C_2 - C_1}{2} + \frac{r_1 + r_2}{2}, r_1 \right),
$$

$$
cc = -\frac{C_2 - C_1}{2} - \frac{r_2 - r_1}{2}, \quad dd = -\frac{C_2 - C_1}{2} + \frac{r_2 + r_1}{2},
$$

**When only one link (link 1) is used at Wardrop equilibrium**:

**When only one link (link 2) is used at Wardrop equilibrium**:
Parameters: \( C_1 = 4, C_2 = 3, r^1 = 1.2, r^2 = 1 \)

Multiple Equilibria
Concluding Remarks

We parameterize the "degree of Cooperation" to capture the behavior in the regime from altruistic to egocentric and identify some strange behavior:

- Loss of uniqueness
- Cooperation paradox - Typically caused due to several equilibria.
- Braess Paradox - Typically caused due inefficiency.
Many questions are raised

- How does the system behave when the users cooperate with more fairness, e.g., $\alpha$ fairness?
- How does the cooperation behaves for an hierarchical routing game (Stackelberg games)?
- How does the similar routing games behave in dynamic environment?
- Few more - Measure of inefficiency (e.g., price of anarchy vs price of stability), Selection of desired equilibria, Convergence to desired equilibria.
Questions?
Routing : different methods

Optimization problem :

- single control objective
  eg. optimization of average network delay
- Either centralized or distributed control
- Passive Users

Game theoretic : resource shared by a group of active users

- Each user optimize its own cost/performance
- A non-cooperative game
- Existence, uniqueness, paradoxes?
Assumptions on Cost function

**Type B function- Assumptions**

B1 : \[ J^i(f) = \sum_{l \in L} f^i_l T_l(f_l) \]
B2 : \[ T_l : [0, \infty) \to (0, \infty]. \]
B3 : \[ T_l(f_l) \] is positive, strictly increasing and convex.
B4 : \[ T_l(f_l) \] is continuously differentiable.

**Type C function**

C1 : \[ \hat{J}^i(f^i_l, f_l) = f^i_l T_l(f_l) \] is a type-B cost function.
C2 : \[ T_l = \begin{cases} \frac{1}{C_l-f_l} & f_l < C_l \\ \infty & f_l > C_l \end{cases} \]

Where \( C_l \) is the capacity of the link \( l \).

Note that type C is a special kind of type B function which correspond to M/M/1 delay function.