

## EE-559 – Deep learning

### 3.6. Back-propagation

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We want to train an MLP by minimizing a loss over the training set

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To use gradient descent, we need the expression of the gradient of the per-sample loss  $\ell_n = \ell(f(x_n; w, b), y_n)$  with respect to the parameters, e.g.

$$\frac{\partial \ell_n}{\partial w_{i,j}^{(l)}} \quad \text{and} \quad \frac{\partial \ell_n}{\partial b_i^{(l)}}.$$

For clarity, we consider a single training sample  $x$ , and introduce  $s^{(1)}, \dots, s^{(L)}$  as the summations before activation functions.

$$x^{(0)} = x \xrightarrow{w^{(1)}, b^{(1)}} s^{(1)} \xrightarrow{\sigma} x^{(1)} \xrightarrow{w^{(2)}, b^{(2)}} s^{(2)} \xrightarrow{\sigma} \dots \xrightarrow{w^{(L)}, b^{(L)}} s^{(L)} \xrightarrow{\sigma} x^{(L)} = f(x; w, b).$$

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Formally we set  $x^{(0)} = x$ ,

$$\forall l = 1, \dots, L, \quad \begin{cases} s^{(l)} = w^{(l)} x^{(l-1)} + b^{(l)} \\ x^{(l)} = \sigma(s^{(l)}) \end{cases},$$

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This is the **forward pass**.

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This generalizes to longer compositions and higher dimensions

$$J_{f_N \circ f_{N-1} \circ \dots \circ f_1}(x) = J_{f_N}(f_{N-1}(\dots(x))) \dots J_{f_3}(f_2(f_1(x))) J_{f_2}(f_1(x)) J_{f_1}(x)$$

where  $J_f(x)$  is the Jacobian of  $f$  at  $x$ , that is the matrix of the linear approximation of  $f$  in the neighborhood of  $x$ .



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To summarize: we can compute  $\frac{\partial \ell}{\partial x_i^{(L)}}$  from the definition of  $\ell$ , and recursively **propagate backward** the derivatives of the loss w.r.t the activations with

$$\frac{\partial \ell}{\partial s_i^{(l)}} = \frac{\partial \ell}{\partial x_i^{(l)}} \sigma' \left( s_i^{(l)} \right)$$

and

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And then compute the derivatives w.r.t the parameters with

$$\frac{\partial \ell}{\partial w_{i,j}^{(l)}} = \frac{\partial \ell}{\partial s_i^{(l)}} x_j^{(l-1)},$$

and

$$\frac{\partial \ell}{\partial b_i^{(l)}} = \frac{\partial \ell}{\partial s_i^{(l)}}.$$

To write all this in tensorial form, if  $\psi : \mathbb{R}^N \rightarrow \mathbb{R}^M$ , we will use the standard Jacobian notation

$$\left[ \frac{\partial \psi}{\partial \mathbf{x}} \right] = \begin{pmatrix} \frac{\partial \psi_1}{\partial x_1} & \cdots & \frac{\partial \psi_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi_M}{\partial x_1} & \cdots & \frac{\partial \psi_M}{\partial x_N} \end{pmatrix},$$

and if  $\psi : \mathbb{R}^{N \times M} \rightarrow \mathbb{R}$ , we will use the compact notation, also tensorial

$$\left[ \frac{\partial \psi}{\partial \mathbf{w}} \right] = \begin{pmatrix} \frac{\partial \psi}{\partial w_{1,1}} & \cdots & \frac{\partial \psi}{\partial w_{1,M}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi}{\partial w_{N,1}} & \cdots & \frac{\partial \psi}{\partial w_{N,M}} \end{pmatrix}.$$

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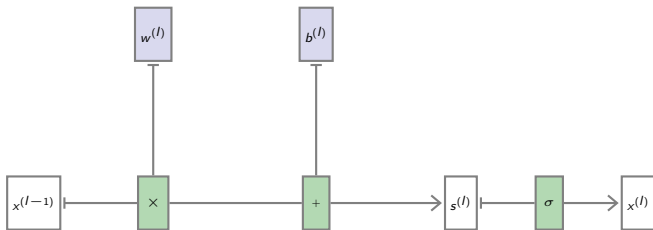
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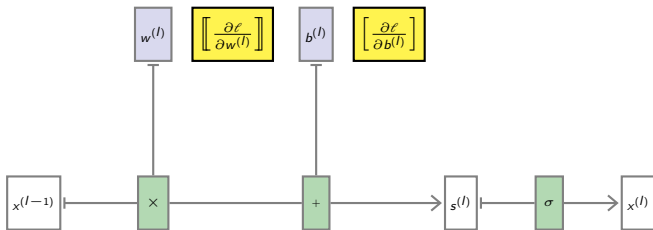
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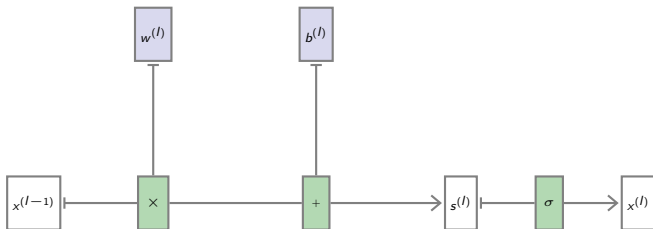
A standard notation (that we do not use here) is

$$\left[ \frac{\partial \ell}{\partial \mathbf{x}^{(l)}} \right] = \nabla_{\mathbf{x}^{(l)}} \ell \quad \left[ \frac{\partial \ell}{\partial \mathbf{s}^{(l)}} \right] = \nabla_{\mathbf{s}^{(l)}} \ell \quad \left[ \frac{\partial \ell}{\partial \mathbf{b}^{(l)}} \right] = \nabla_{\mathbf{b}^{(l)}} \ell \quad \left[ \frac{\partial \ell}{\partial \mathbf{w}^{(l)}} \right] = \nabla_{\mathbf{w}^{(l)}} \ell.$$

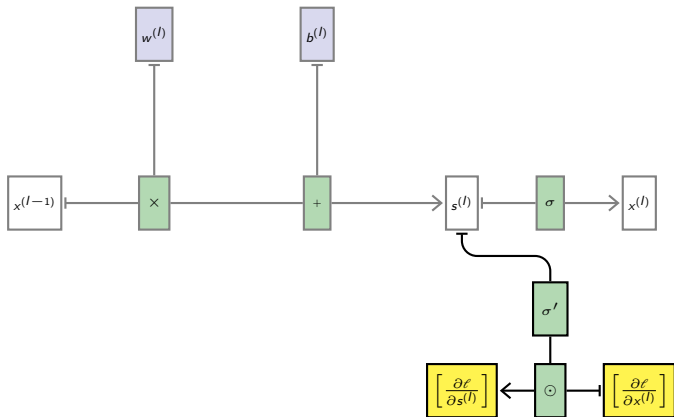




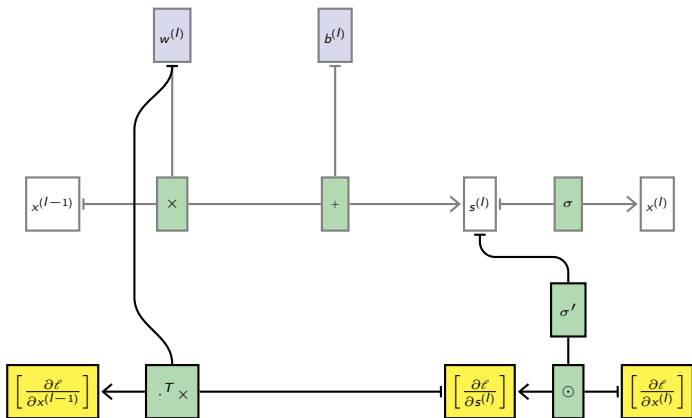




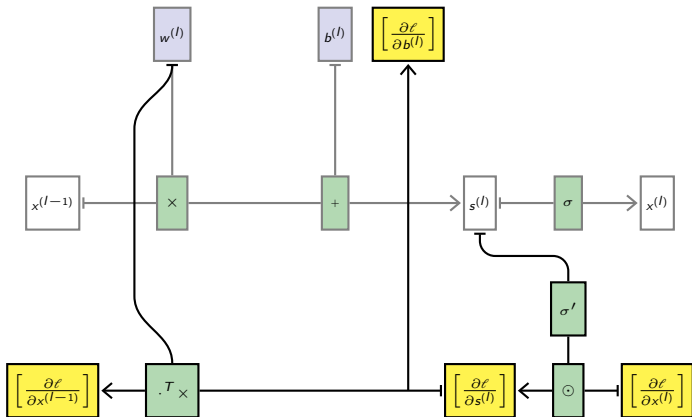
$$\left[ \frac{\partial \ell}{\partial x^{(l)}} \right]$$



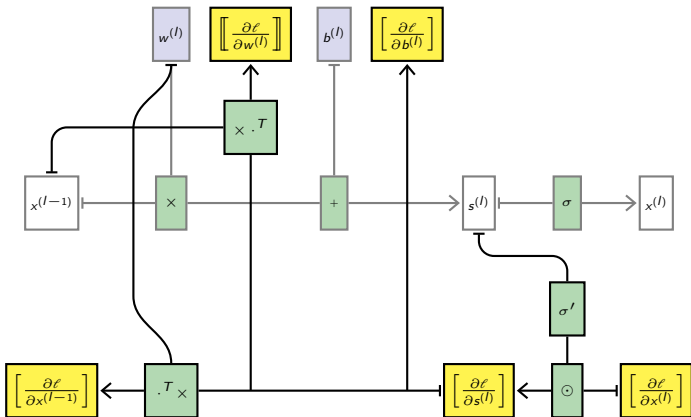
$$\frac{\partial \ell}{\partial s_i^{(l)}} = \frac{\partial \ell}{\partial x_i^{(l)}} \sigma' \left( s_i^{(l)} \right)$$



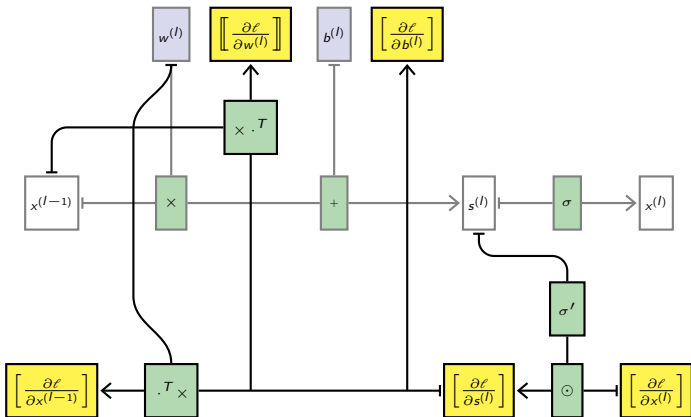
$$\frac{\partial \ell}{\partial x_j^{(l-1)}} = \sum_i w_{i,j}^{(l)} \frac{\partial \ell}{\partial s_i^{(l)}}$$



$$\frac{\partial \ell}{\partial b_i^{(l)}} = \frac{\partial \ell}{\partial s_i^{(l)}}$$



$$\frac{\partial \ell}{\partial w_{i,j}^{(l)}} = \frac{\partial \ell}{\partial s_i^{(l)}} x_j^{(l-1)}$$





## Forward pass

Compute the activations.

$$x^{(0)} = x, \quad \forall l = 1, \dots, L, \quad \begin{cases} s^{(l)} = w^{(l)} x^{(l-1)} + b^{(l)} \\ x^{(l)} = \sigma(s^{(l)}) \end{cases}$$

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## Backward pass

Compute the derivatives of the loss wrt the activations.

$$\begin{cases} \left[ \frac{\partial \ell}{\partial x^{(L)}} \right] \text{ from the definition of } \ell & \left[ \frac{\partial \ell}{\partial s^{(l)}} \right] = \left[ \frac{\partial \ell}{\partial x^{(l)}} \right] \odot \sigma'(s^{(l)}) \\ \text{if } l < L, \left[ \frac{\partial \ell}{\partial x^{(l)}} \right] = (w^{(l+1)})^T \left[ \frac{\partial \ell}{\partial s^{(l+1)}} \right] \end{cases}$$

Compute the derivatives of the loss wrt the parameters.

$$\left[ \frac{\partial \ell}{\partial w^{(l)}} \right] = \left[ \frac{\partial \ell}{\partial s^{(l)}} \right] (x^{(l-1)})^T \quad \left[ \frac{\partial \ell}{\partial b^{(l)}} \right] = \left[ \frac{\partial \ell}{\partial s^{(l)}} \right].$$

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## Gradient step

Update the parameters.

$$w^{(l)} \leftarrow w^{(l)} - \eta \left[ \frac{\partial \ell}{\partial w^{(l)}} \right] \quad b^{(l)} \leftarrow b^{(l)} - \eta \left[ \frac{\partial \ell}{\partial b^{(l)}} \right]$$

In spite of its hairy formalization, the backward pass is a simple algorithm: apply the chain rule again and again.

As for the forward pass, it can be expressed in tensorial form. Heavy computation is concentrated in linear operations, and all the non-linearities go into component-wise operations.

Regarding computation, since the costly operation for the forward pass is

$$s^{(l)} = w^{(l)}x^{(l-1)} + b^{(l)}$$

and for the backward

$$\left[ \frac{\partial \ell}{\partial x^{(l)}} \right] = \left( w^{(l+1)} \right)^T \left[ \frac{\partial \ell}{\partial s^{(l+1)}} \right]$$

and

$$\left[ \left[ \frac{\partial \ell}{\partial w^{(l)}} \right] \right] = \left[ \frac{\partial \ell}{\partial s^{(l)}} \right] \left( x^{(l-1)} \right)^T,$$

the rule of thumb is that the backward pass is twice more expensive than the forward one.

The end