A priori-based mesh adaptation for third-order accurate Euler simulation

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Motivation of the study

- Propagation of waves in large spaces.
- Mesh adaptation should follow the wave propagation.
- Adjoint-based adaptation criterion will concentrate on zone of interest for a chosen functional output.
- An accurate approximation scheme needs be chosen.
1. Interpolation error analysis
2. Reconstruction error analysis
3. CENO2 Scheme
4. Approximation error analysis
5. Optimal metric
6. Resolution of optimum
7. Numerical experiments
8. Concluding remarks
1. Interpolation error analysis (1)

\[ |(u - \Pi_2 u)(\delta x)| \approx |P_u(\delta x)|. \]

\( \delta x = (\delta x, \delta y) \) defines mesh size.

The generic error model is a homogeneous polynomial of degree \( k = 3 \) in 2 variables:

\[ P_u(x) = \sum_{i=0}^{i=3} \binom{k}{i} a(u)_i x^i y^{k-i} \]

\( a(u)_i \) are third derivatives of \( u \).
1. Interpolation error analysis (2):
Modeling of the quadratic order interpolation error

Local optimization problem

**Main Idea:** $(P_u)^{\frac{3}{2}}$ being homogeneous with a quadratic form with respect to mesh size, let us approach locally the variations of $P_u$ by a quadratic definite positive form taken at power $\frac{3}{2}$, i.e., for $(x, y) \in \mathbb{R}^2$,

$$|P_u(x, y)| \leq \left( t(x, y) S_{u}^{\text{loc}}(x, y) \right)^{\frac{3}{2}}$$

⇒ **Find the optimal local metric** $S_{u}^{\text{loc}}$ (optimal local directions and sizes) to construct a majorant of the local error model $P_u$.

The optimal local metric $S_{u}^{\text{loc}}$ is the one whose unit ball is the **maximum area ellipse** included in the isoline 1 of $P_u$. 
1. Interpolation error error analysis (3): Example

\[ P_u = 6x^3 - 8.594x^2y - 6.98xy^2 - 100y^3 \]

- Optimal local metric in \( P_u \)
- in \( (t(x, y) S_u^{loc}(x, y))^{3/2} \)

Superposition of the iso-lines of \( P_u \) and \( (t(x, y) S_u^{loc}(x, y))^{3/2} \)
Construction of optimal local metrics

**Binary decomposition by Sylvester’s theorem** [Brachat-Mourrain, 1999]

\[ P_u(x, y) = \sum_{j=1}^{2(ran)} \lambda_j (\alpha_j x + \beta_j y)^3 ; \beta_i = 1 \]

<table>
<thead>
<tr>
<th>Real case</th>
<th>Complex case</th>
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<td>( S^{loc}_u = t^c Q_u \left( \begin{array}{cc}</td>
<td>\lambda_1</td>
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\[ Q_u = \left( \begin{array}{cc} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{array} \right) ; \ t^c Q_u = \left( \begin{array}{cc} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{array} \right) ; \ t^c Q_u = \left( \begin{array}{cc} \bar{\alpha}_1 & \bar{\alpha}_2 \\ \bar{\beta}_1 & \bar{\beta}_2 \end{array} \right) \]

In the complex case, \( \lambda_2 = \bar{\lambda}_1 \) and \( \alpha_2 = \bar{\alpha}_1 \).

\( S^{loc}_u \) is real symmetric in both cases.
1. Interpolation error error analysis (6)

Optimal estimate

Find the maximal ellipse inside the \(|P_u(x)| = 1\) level set.

\[
\Rightarrow \quad \tilde{S}_u = c_{opt} S_{u}^{loc} = t R_u \begin{pmatrix} \tilde{\mu}_1 & 0 \\ 0 & \tilde{\mu}_2 \end{pmatrix} R_u
\]

\(c_{opt} = 1\) real case ; \(c_{opt} = 2^{-\frac{1}{3}}\) complex case.

\[
\Rightarrow \quad |P_u(x, y)| \leq \left( t(x, y) \tilde{S}_u (x, y) \right)^{\frac{3}{2}}
\]
Mesh parametrization by a Riemannian metric

For any \( x \) in \( \Omega \), \( \mathcal{M}(x) \) is a symmetric matrix.

\[
\mathcal{M} \mapsto \text{Mesh}(\mathcal{M})
\]

Such that any edge \( ab \) of \( \text{Mesh}(\mathcal{M}) \) has a unit \( \mathcal{M} \)-length:

\[
\int_0^1 \sqrt{\mathcal{M}(a + t\mathcal{M}(ab))\mathcal{M}(ab)} \, dt = 1.
\]

\[
\mathcal{M}(x) = d_{\mathcal{M}}(x)\mathcal{R}(x)\Lambda(x)\mathcal{R}^t(x) \quad \text{with} \quad \det \Lambda = 1.
\]

- The rotation \( \mathcal{R} = (e_\xi, e_\eta) \) parametrises the two orthogonal stretching directions of the metric.

- \( \Lambda \) is a diagonal matrix with eigenvalues \( \lambda_1 = m_\eta/m_\xi \) and \( \lambda_2 = m_\xi/m_\eta \) where \( m_\xi \) and \( m_\eta \) represent the two directional local mesh sizes in the characteristic/stretching directions of \( \mathcal{M} \). \( \lambda_1/\lambda_2 \) is the stretching ratio.

- \( d_{\mathcal{M}} = m_\xi^{-1} m_\eta^{-1} \) is the node density.
Find the optimal Riemannian metric $\mathcal{M}$ defined on the computational domain for smallest weighted-$L^1$ error

$$\min_{\mathcal{M}} \int_{\Omega} g(x) \left( \text{trace}(\mathcal{M}^{-\frac{1}{2}} \tilde{\mathcal{S}}_u | \mathcal{M}^{-\frac{1}{2}}) \right)^{\frac{3}{2}} dx$$

with $\int_{\Omega} d(\mathcal{M}) dx = N$

$$\Rightarrow \mathcal{M}_{opt} = N \left( \int_{\Omega} \kappa \; dx \right)^{-1} \kappa \;^t \mathcal{R}_u \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_{1}^{-1} \end{pmatrix} \mathcal{R}_u$$

$$\nu_1 = \left( \frac{\tilde{\mu}_1}{\tilde{\mu}_2} \right)^{-\frac{1}{2}} \quad \kappa = (2g \tilde{\mu}_1 \tilde{\mu}_2)^{\frac{3}{5}}.$$
2. Reconstruction error error analysis

**Vertex, dual cell, 2-exact quadratic reconstruction**

Given $\bar{u}_i$ on each cell $i$ of centroid $G_i$, find the $c_{i,\alpha}$, $|\alpha| \leq 2$ s.t.

$$R_2^0 \bar{u}_i(x) = \bar{u}_i + \sum_{|\alpha| \leq 2} c_{i,\alpha} [(X - G_i)^\alpha - \int_{Cell_i} (X - G_i)^\alpha d\mathbf{x}]$$

$$\overline{R_2^0 \bar{u}} = \int_{Cell_i} R_2^0 \bar{u}_{i,j} d\mathbf{x} = \bar{u}_i \quad \sum_{j \in N(i)} (\overline{R_2^0 \bar{u}_{i,j}} - \bar{u}_j)^2 = \text{Min}.$$
2. Reconstruction error analysis (2)

Same treatment as interpolation

\[ R_2: \text{2D quadratic ENO reconstruction:} \]

\[ |u(x) - R_2^0 u(x)| \approx |\sum_{i=0}^{i=3} \binom{k}{i} a_i x^i y^{k-i}| \leq \left( t(x, y) \tilde{S}_u (x, y) \right)^{3/2} \]

\( a(u)_i \) are third derivatives of \( u \).
3. CENO2 Scheme (1)

Variational statement of a model problem

\[ B(u, v) = \int_{\Omega} v \nabla \cdot \mathcal{F}(u) \, d\Omega - \int_{\Gamma} v \mathcal{F}_\Gamma(u) \, d\Gamma, \]

\( B \) is linear with respect to \( v \).

Find \( u \in \mathcal{V} \) such that \( B(u, v) = 0 \ \forall \ v \in \mathcal{V} \).

CENO discrete statement (after C. Groth), vertex version

\( \mathcal{V}_0 = \{ v_0, V_0 | \text{Cell}_i = \text{const} \ \forall \ i \ \text{vertex} \} \)

Find \( u_0 \in \mathcal{V}_0 \) such that \( B(R_2^0 u_0, v_0) = 0 \ \forall \ v_0 \in \mathcal{V}_0 \)
2-exact flux integration

The integral on a cell interface \( C_{ij} = C_i \cap C_j \) is split into the integrals on the two segments of \( C_{ij} \).

On each segment \( C_{ij}^{(1)} \) and \( C_{ij}^{(2)} \) a numerical integration with two Gauss points (two Riemann solvers) is applied.
4. *A priori* error analysis (1)

Cf. “*A posteriori* analysis for a large set of *p*-order reconstruction-based Godunov methods” Barth-Larson 2002.

Scalar output \( j(u) = (g, u) \).

Projection \( \pi_0 : \mathcal{V} \rightarrow \mathcal{V}_0, \quad v \mapsto \pi_0 v \)

\[ \forall \ C_i, \text{dual cell}, \quad \pi_0 v|_{C_i} = \int_{C_i} v dx / \text{meas}(C_i). \]

Output error \( \delta j = (g, R_2^0 \pi_0 u - R_2^0 u_0) \).

The adjoint state \( u_0^* \in \mathcal{V}_0 \) is the solution of:

\[ \frac{\partial B}{\partial u}(R_2^0 u_0)(R_2^0 v_0, u_0^*) = (g, R_2^0 v_0), \quad \forall \ v_0 \in \mathcal{V}_0. \]
4. *A priori* error analysis (2)

Simpler case:

\[ B(u, \nu) = F(\nu) \] where \( B \) is bilinear, \( F \) is linear, for example:

\[ B(u, \nu) = \int_{\Omega} \nu \text{div}(\mathbf{V}u) d\Omega + \int_{\Gamma} \nu \mathbf{V} \cdot \mathbf{n} d\Gamma \text{ and } F(\nu) = \int_{\Gamma} u_B \nu \mathbf{V} \cdot \mathbf{n} d\Gamma. \]

\[
\begin{align*}
B(\nu, u_0^*) & = (g, \nu) \quad \text{(discr. adj. eq.)} \\
B(R_0^2 u_0, \nu) & = F(\nu) \quad \text{(discr. state eq.)} \\
B(u, \nu) & = F(\nu) \quad \text{(cont. state eq.)}
\end{align*}
\]

\[
(g, R_0^2 \pi_0 u - R_0^2 u_0) = B(R_0^2 \pi_0 u - R_0^2 u_0, u_0^*) \quad \text{(discr. adj. eq.)}
\]

\[
\begin{align*}
\approx & \quad B(R_0^2 \pi_0 u, u_0^*) - B(R_0^2 u_0, u_0^*) \\
\approx & \quad B(R_0^2 \pi_0 u, u_0^*) - F(u_0^*) \quad \text{(discr. state eq.)} \\
\approx & \quad B(R_0^2 \pi_0 u, u_0^*) - B(u, u_0^*) \quad \text{(cont. state eq.)} \\
\approx & \quad B(R_0^2 \pi_0 u - u, u_0^*)
\end{align*}
\]

The error is directly expressed in terms of the reconstruction error for exact solution.
4. *A priori* error analysis (3)

Unsteady Euler: \( W = (\rho, \rho u, \rho v, \rho E) \)

For the case of Euler eqs, we get after some calculations:

\[
|B(R_{2}^{0} \pi_{0} W - W, W_{0}^{*})| \approx \leq 2 \int_{\Omega} \sum_{q} K_{q}(W, W^{*}) |R_{2}^{0} \pi_{0} u_{q} - u_{q}| \, d\Omega
\]

with \((u_{q})_{q=1,8} = (W, W_{t})\),

and in which the \( K_{q}(W, W^{*}) \) are built from space derivatives of \( W^{*} \) and Euler fluxes derivatives with respect to dependant variable \( W \).
5. Optimal metric (1)

Using the previous reconstruction error estimate:

\[ |R_2^0 \pi_0 u_q - u_q| \approx \left( \text{trace}(\mathcal{M}^{-1/2} S_q \mathcal{M}^{-1/2}) \right)^{\frac{3}{2}}. \]

Optimization problem

After the *a priori* analysis, we have to minimise the following error:

\[ \mathcal{E} = \int \sum_q K_q(W, W^*) \left( \text{trace}(\mathcal{M}^{-1/2} S_q \mathcal{M}^{-1/2}) \right)^{\frac{3}{2}} dxdy \]

\[ = \int \left( \text{trace}(\mathcal{M}^{-1/2} S\mathcal{M}^{-1/2}) \right)^{\frac{3}{2}} dxdy \]

\[ = \int d\mathcal{M}^{-\frac{3}{2}} \left( \text{trace}[(\mathcal{R}_\mathcal{M} \Lambda_\mathcal{M} \mathcal{R}_\mathcal{M}^T)^{-\frac{1}{2}} |S|(\mathcal{R}_\mathcal{M} \Lambda_\mathcal{M} \mathcal{R}_\mathcal{M}^T)^{-\frac{1}{2}}] \right)^{\frac{3}{2}} dxdy \]

with constraint \( \int d\mathcal{M} dxdy = N \).
Minimize \[ \mathcal{E}(\mathcal{M}) = \int d\mathcal{M}^{-\frac{3}{2}} \left( \text{trace}\left[ (\mathcal{R}_{\mathcal{M}} \Lambda_{\mathcal{M}} \mathcal{R}_{\mathcal{M}}^T)^{-\frac{1}{2}} |S|(\mathcal{R}_{\mathcal{M}} \Lambda_{\mathcal{M}} \mathcal{R}_{\mathcal{M}}^T)^{-\frac{1}{2}} \right] \right)^{\frac{3}{2}} dx dy \]

with constraint \[ \int d\mathcal{M} dx dy = N. \]

Optimal solution:

\( \forall x : S = \mathcal{R}_S \bar{\Lambda}_S \mathcal{R}_S^T \Rightarrow \mathcal{R}_{\mathcal{M}_{\text{opt}}} = \mathcal{R}_S , \Lambda_{\mathcal{M}_{\text{opt}}} = \bar{\Lambda}_S^{-1}/\text{det}(S) \)

\[ \text{det}(S)^{\frac{3}{2}} (d_{\mathcal{M}_{\text{opt}}})^{-\frac{5}{2}} = \text{const}. \]

\[ \Rightarrow d_{\mathcal{M}_{\text{opt}}} = N \left( \int \text{det}(S)^{\frac{3}{5}} dx dy \right)^{-1} \text{det}(S)^{\frac{3}{5}} \]

\[ \mathcal{M}_{\text{opt}} = d_{\mathcal{M}_{\text{opt}}} \mathcal{R}_{\mathcal{M}_{\text{opt}}} \Lambda_{\mathcal{M}_{\text{opt}}} \mathcal{R}_{\mathcal{M}_{\text{opt}}}^T \]
5. Optimal metric (3): isotropic case

Minimize \( \mathcal{E}(\mathcal{M}) = \int d^{\frac{3}{2}}_M \left( \text{trace}\left[ (\lambda_M)^{-\frac{1}{2}} |S| (\lambda_M)^{-\frac{1}{2}} \right]\right)^{\frac{3}{2}} dxdy \)

with constraint \( \int d_M dxdy = N \).

Optimal solution:

\( \lambda_{M_{\text{opt}}} = \text{trace}[S]^{-1}/\text{det}(S) \)
\( \text{det}(S)^{\frac{3}{2}} (d_{M_{\text{opt}}})^{-\frac{5}{2}} = \text{const.} \)

\( \Rightarrow d_{M_{\text{opt}}} = N \left( \int \text{det}(S)^{\frac{3}{5}} dxdy \right)^{-1} \text{det}(S)^{\frac{3}{5}} \)

\( \mathcal{M}_{\text{opt}} = d_{M_{\text{opt}}} \lambda_{M_{\text{opt}}} \)
6. Resolution of optimum, unsteady case

The continuous model giving the adapted mesh involves a **state system**, an **adjoint system** and the **optimality relation** giving $\mathcal{M}_{opt}$.

We discretise it.

The time discretisation of the metric is made of coarser time intervals.

We solve it.

by the Global Unsteady Fixed Point algorithm of Belme et al. JCP2012.
6. Resolution of optimum, unsteady case

Fixed-point loop \( j \)

0 \( t_i \) \( \Delta t_i \) \( t_{i+1} \) \( T = t_{n_{\text{adapt}}} \)

Solution state and adjoint state sampling

Solve state once to get checkpoints

\( \tilde{\Psi}(W) = 0 \)

\( \tilde{\Psi}^*(W, W^*) = 0 \)

\( n_{\text{adapt}} = 5 \)

\( n_k = 11 \)
7. Numerical experiments

We present a first series of experiments where the propagation of an acoustical perturbation is followed by the mesh adaptator in order to minimise the mesh effort.

1. Noise wall problem
2. Diffraction problem
7. Numerical experiments: Noise wall problem (1)

Propagation of an acoustic wave from a source while observing the impact on the detector.

Acoustic source:

\[ r = -Ae^{-B\ln(2)[x^2+y^2]}\cos(2\pi f). \]

Functional:

\[ j(W) = \int_0^T \int_M \frac{1}{2} (p - p_{air})^2 dM dt. \]
7. Numerical experiments: Noise wall problem (2)

Density field evolving in time on uniform mesh (line 1), adapted one (middle line), with corresponding adapted meshes (last line):
7. Numerical experiments: Noise wall problem (3)

Scalar output comparison: MUSCL (-) vs CENO (—) schemes.

![Graph showing scalar output comparison between MUSCL and CENO schemes.](image-url)
7. Numerical experiments: Diffraction problem (1)

The diffraction of an acoustic wave travelling from a source location to a detector situated in a small region under the "step".

Acoustic source:

\[ r = -A e^{-B \ln(2) [x^2 + y^2]} \cos(2\pi f). \]

Functional:

\[ j(W) = \int_0^T \int_{\mathcal{M}} \frac{1}{2} (p - p_{air})^2 d\mathcal{M} dt. \]
Density field evolving in time on uniform mesh (line 1), adapted one (middle line), with corresponding adapted meshes (last line):
Our goal-oriented method ensures the accuracy of the functional output \( j(W) = \int_0^T \int_M \frac{1}{2} (p - p_{\text{air}})^2 dM dt \).

It is thus interesting to analyse the integrand \( k(t) = \int_M \frac{1}{2} (p - p_{\text{air}})^2 dM \) of \( j(W) \) on the micro \( M \) for different sizes of uniform vs. adapted meshes.
7. Numerical experiments: Diffraction problem (4)

Functional time integrand calculation on different sizes of non-adapted meshes (28K, 40K, 68K) vs. adapted meshes (mean sizes: 2620, 4892, 6130).

The convergence order is found to be 0.6 for uniform meshes and 1.98 for the adapted one.
7. Numerical experiments: 3rd order error vs 2nd order error

The previous computations were performed with 2nd-order error, made of 2nd-order derivatives, used with the 3rd-order CENO2 scheme.

In a preliminary use of 3rd-order estimate, relying on 3rd-order derivatives, we compare two calculations:
- one with **3rd-order scheme and 2nd-order error**, 
- the second with **3rd-order scheme and 3rd-order error**.

The test case is the translation by Euler of a density Gaussian from left to right in a rectangle, with a functional observation with a Gaussian weight at rectangle center.
The integrand of the functional is a function of time which culminate when the center of density Gaussian coincides with the center of Gaussian which is the functional weight (center of rectangle).

Left, 350 equivalent nodes, right 1500 equivalent nodes.
Here now are the results with the new 3rd-order estimate.

Left, 350 equivalent nodes, right 1500 equivalent nodes.
A third-order (spatially) accurate goal-oriented mesh adaptation method has been built on the basis of:

- The extension of Hessian analysis to higher order interpolation of Mbinky et al..
- A novel *a priori* analysis.
- The Unsteady Global Fixed-Point mesh adaptation algorithm of Belme et al.

Numerical experiments are yet only 2D and at the very beginning.
8. Concluding remarks (2)

Next studies will address:

- Steady test cases.
- Flows with singularities.
- 3D extension.
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THANK YOU!