An introduction to the lattice approach to stabilization problems

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Many thanks!

I would like to thank Gema and Julio. . . for this very nice MAP meeting in Castro Urdiales.

Symbolic analysis: transfer matrix

- **Electric transmission line:**

$$\begin{cases} \dfrac{\partial V}{\partial x}(x,t) + L\,\dfrac{\partial I}{\partial t}(x,t) + R\,I(x,t) = 0, \\ \dfrac{\partial I}{\partial x}(x,t) + C\,\dfrac{\partial V}{\partial t}(x,t) + G\,V(x,t) = 0, \\ V(x,0) = 0, \quad I(x,0) = 0, \\ V(0,t) = u(t), \quad \lim_{x\to+\infty} V(x,t) = 0, \\ V(\overline{x},t) = y_1(t), \quad I(\overline{x},t) = y_2(t), \end{cases}$$

$$\Rightarrow \begin{cases} \widehat{y_1}(s) = e^{-\sqrt{(L\,s+R)\,(C\,s+G)}\,\overline{x}}\,\widehat{u}(s), \\ \widehat{y_2}(s) = \sqrt{\frac{C\,s+G}{L\,s+R}}\,e^{-\sqrt{(L\,s+R)\,(C\,s+G)}\,\overline{x}}\,\widehat{u}(s), \end{cases}$$

$\Rightarrow$ we obtain a **transfer matrix** $(\widehat{y_1}(s),\ \widehat{y_2}(s))^T = P\,\widehat{u}(s)$.

Fractional representations of a transfer matrix

- Let A be an **integral domain of stable transfer functions**

$$(\text{e.g.,} \quad A = RH_\infty, \quad H_\infty(\mathbb{C}_+), \quad \widehat{\mathcal{A}}).$$

- **A plant is defined by a transfer matrix** $P \in K^{q \times r}$, $K = Q(A)$.

- We can write P by means of the **fractional representations**:

$$P = D^{-1} N = \widetilde{N} \, \widetilde{D}^{-1}, \quad \begin{cases} (D \quad -N) \in A^{q \times (q+r)}, \\ (\widetilde{N}^T \quad \widetilde{D}^T)^T \in A^{(q+r) \times r}. \end{cases}$$

$$(\text{e.g.,} \quad D = d\, I_q, \quad N = d\, P, \quad \widetilde{D} = d\, I_r, \quad \widetilde{N} = d\, P).$$

3. $y = P\, u \Leftrightarrow \begin{cases} (D \quad -N) \begin{pmatrix} y \\ u \end{pmatrix} = 0, \\ \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} \widetilde{N} \\ \widetilde{D} \end{pmatrix} z, \end{cases} \Rightarrow$ **module theory**.

Example

- Let us consider the **transfer matrix**:

$$P = \begin{pmatrix} \frac{e^{-s}}{s-1} \\ \frac{e^{-s}}{(s-1)^2} \end{pmatrix}.$$

- Let us consider $A = H_\infty(\mathbb{C}_+)$ and $K = Q(A)$.

- We then have:

$$\begin{cases} y_1 = \frac{e^{-s}}{(s-1)}\, u, \\ y_2 = \frac{e^{-s}}{(s-1)^2}\, u \end{cases} \quad \Rightarrow \quad \begin{cases} \frac{(s-1)}{(s+1)}\, y_1 - \frac{e^{-s}}{(s+1)}\, u = 0, \\ \left(\frac{s-1}{s+1}\right)^2\, y_2 - \frac{e^{-s}}{(s+1)^2}\, u = 0, \end{cases}$$

$$\Rightarrow D \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = N\, u \quad \Rightarrow \quad P = D^{-1}\, N,$$

where:

$$D = \begin{pmatrix} \frac{s-1}{s+1} & 0 \\ 0 & \left(\frac{s-1}{s+1}\right)^2 \end{pmatrix} \in A^{2\times 2}, \quad N = \begin{pmatrix} \frac{e^{-s}}{s+1} \\ \frac{e^{-s}}{(s+1)^2} \end{pmatrix} \in A^2.$$

Lattices of a vector space

- Let V be a **finite-dimensional** K**-vector space.**

- **Definition**: An A-submodule M of V is a **lattice of** V if there exist L_1, L_2 two **free** A**-submodules of** V such that:

$$L_1 \subseteq M \subseteq L_2, \quad \mathrm{rk}_A(L_1) = \dim_K(V).$$

- **Proposition**: An A-submodule M of V is **a lattice of** V iff

$$K\,M \triangleq \{k\,m \,|\, k \in K,\ m \in M\} = V, \quad M \subseteq P,$$

where P is a **finitely generated** A-submodule of V.

- **Example**: Let $P \in K^{q \times r}$, then the A-module $\mathcal{L} = (I_q \quad -P)\,A^{q+r}$ is a **lattice of** the K-vector space K^q.

- **Example**: Let $P \in K^{q \times r}$, then the A-module $\mathcal{M} = A^{1 \times (q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix}$ is a **lattice of** the K-vector space $K^{1 \times r}$.

Dual of a lattice

- Let V and W be 2 finite-dimensional K-vector spaces.
- Let M (resp., N) be a **lattice** of V (resp., W).
- **Definition**: $N : M$ is the A-submodule of

$$\hom_K(V, W) = \{f : V \to W \,|\, f \text{ is a } K-\text{linear map}\}$$

formed by the K-linear maps $f : V \to W$ which satisfy:

$$f(M) \subseteq N.$$

- **Proposition**: 1. $N : M$ is **a lattice of** $\hom_K(V, W)$.

2. We have the following **bijective map**:

$$\begin{array}{rcl} N : M & \to & \hom_A(M, N) \triangleq \{f : M \to N \,|\, f \text{ is a } A-\text{linear map}\}, \\ f & \mapsto & f_{|M}. \end{array}$$

Examples

- **Example**: Let $P \in K^{q\times r}$ and $\mathcal{L} = (I_q \quad -P)\, A^{q+r}$. Then:

$$A : \mathcal{L} = \{f : K^q \to K \mid f(\mathcal{L}) \subseteq A\} = \{\lambda \in K^{1\times q} \mid \lambda\,(I_q \; -P)\, A^{q+r} \subseteq A\}$$

$$= \{\lambda \in K^{1\times q} \mid \lambda \in A^{1\times q},\ \lambda\, P \in A^{1\times r}\} = \{\lambda \in A^{1\times q} \mid \lambda\, P \in A^{1\times r}\}.$$

- **Example**: Let $P \in K^{q\times r}$ and $\mathcal{M} = A^{1\times(q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix}$. Then:

$$A : \mathcal{M} = \left\{f : K^r \to K \mid f(\mathcal{M}) \subseteq A\right\} = \left\{\lambda \in K^r \mid A^{1\times(q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix} \lambda \subseteq A\right\}$$

$$= \{\lambda \in K^r \mid \lambda \in A^r,\ P\,\lambda \in A^q\} = \{\lambda \in A^r \mid P\,\lambda \in A^q\}.$$

Weakly coprime factorizations

• **Definition**: $P \in K^{q \times r}$ admits a **weakly left-coprime factorization** if there exists $R = (D \quad -N) \in A^{q \times (q+r)}$ such that

$$\det D \neq 0, \quad P = D^{-1} N,$$

and:

$$\forall\, \lambda \in K^{1 \times q},\ \lambda\, R \in A^{1 \times (q+r)} \ \Rightarrow\ \lambda \in A^{1 \times q}.$$

• **Proposition**: $P \in K^{q \times r}$ **admits a weakly left-coprime factorization** iff $\exists\, D \in A^{q \times q}$ such that $\det D \neq 0$ and

$$A : \mathcal{L} = A : ((I_q \quad -P)\, A^{q+r}) \triangleq \{\lambda \in A^{1 \times q} \mid \lambda\, P \in A^{1 \times r}\} = A^{1 \times q}\, D,$$

i.e., $A : \mathcal{L}$ is a **free lattice of** $K^{1 \times q}$, namely, $A : \mathcal{L} \cong A^{1 \times q}$.

Example

- We consider $A = H_\infty(\mathbb{C}_+)$, $K = Q(A)$ and the transfer matrix:

$$P = \begin{pmatrix} \frac{e^{-s}}{s-1} \\ \frac{e^{-s}}{(s-1)^2} \end{pmatrix} \in K^2.$$

- We have the **fractional representation** $P = D^{-1}\,N$ of P, where:

$$D = \begin{pmatrix} \frac{s-1}{s+1} & 0 \\ 0 & \left(\frac{s-1}{s+1}\right)^2 \end{pmatrix} \in A^{2\times 2}, \quad N = \begin{pmatrix} \frac{e^{-s}}{s+1} \\ \frac{e^{-s}}{(s+1)^2} \end{pmatrix} \in A^2.$$

- $P = D^{-1}\,N$ is not a **weakly left-coprime factorization** of P as

$$\begin{pmatrix} \frac{1}{(s-1)} & -\frac{(s+1)}{(s-1)} \end{pmatrix} \begin{pmatrix} \frac{s-1}{s+1} & 0 & -\frac{e^{-s}}{s+1} \\ 0 & \left(\frac{s-1}{s+1}\right)^2 & -\frac{e^{-s}}{(s+1)^2} \end{pmatrix} = \begin{pmatrix} \frac{1}{(s+1)} & -\frac{(s-1)}{(s+1)} & 0 \end{pmatrix}$$

and:

$$\begin{pmatrix} \frac{1}{(s-1)} & -\frac{(s+1)}{(s-1)} \end{pmatrix} \in K^{1\times 2}, \; \begin{pmatrix} \frac{1}{(s+1)} & -\frac{(s-1)}{(s+1)} & 0 \end{pmatrix} \in A^{1\times 3}.$$

Coherent rings

- **Definition**: A ring A is **coherent** if, for any finitely generated ideal $I = (a_1, \ldots, a_n)$ of A, the A-module

$$S(I) = \left\{ (r_1 \ldots r_n) \in A^{1\times n} \mid \sum_{i=1}^{n} r_i \, a_i = 0 \right\}$$

is finitely generated, i.e.:

$$\exists\, m \in \mathbb{Z}_+, \quad \exists\, R \in A^{m\times n} : \; S(I) = A^{1\times m} R.$$

- **Theorem**: (McVoy-Rubel 76, Rosay 77) $H_\infty(\mathbb{C}_+)$ is **coherent**.

- **Theorem**: If A is a **coherent ring**, then we can compute a **weakly left-coprime factorization** of $P \in K^{q\times p}$ by computing

$$\mathrm{ext}^1_A(N, A),$$

where $N = D^{1\times q} / (D^{1\times (p+q)} \, R^T)$ and:

$$R = (D \quad -N) \in A^{q\times (p+q)}, \quad P = D^{-1} \, N.$$

Coherent Sylvester domains

- **Definition**: An integral domain A is a **coherent Sylvester domain** if, for every $q \in \mathbb{Z}_+$ and every $v \in A^{1\times q}$, the A-module

$$\ker_A(v.) = \{w \in A^q \mid v\, w = \sum_{i=1}^{q} v_i\, w_i = 0\} \quad \text{is } \textbf{free}.$$

- **Theorem**: $H_\infty(\mathbb{C}_+)$, $A[x]$, where A is a Bézout domain (e.g., $\mathbb{Z}$, $k[y]$, k a field) and RH_∞ are **coherent Sylvester domains**.

- **Example**: Let us consider $A = H_\infty(\mathbb{C}_+)$ and $K = Q(A)$. Then, the transfer matrix

$$P = \begin{pmatrix} \frac{e^{-s}}{s-1} \\ \frac{e^{-s}}{(s-1)^2} \end{pmatrix} \in K^2$$

admits the **weakly left-coprime factorization** $P = D'^{-1}\, N'$:

$$D' = \begin{pmatrix} \frac{1}{(s+1)} & -\frac{(s-1)}{(s+1)} \\ \frac{(s-1)}{(s+1)} & 0 \end{pmatrix} \in A^{2\times 2}, \quad N' = \begin{pmatrix} 0 \\ \frac{e^{-s}}{(s+1)} \end{pmatrix} \in A^2.$$

Weakly coprime factorizations

- **Definition**: $P \in K^{q\times r}$ admits a **weakly right-coprime factorization** if there exists $\widetilde{R} = (\widetilde{N}^T \quad \widetilde{D}^T)^T \in A^{(q+r)\times r}$ such that

$$\det \widetilde{D} \neq 0, \quad P = \widetilde{N}\,\widetilde{D}^{-1},$$

and:

$$\forall\, \lambda \in K^r,\ \widetilde{R}\,\lambda \in A^p \ \Rightarrow\ \lambda \in A^r.$$

- **Proposition**: $P \in K^{q\times r}$ **admits a weakly right-coprime factorization** iff $\exists\ \widetilde{D} \in A^{r\times r}$ such that $\det \widetilde{D} \neq 0$ and

$$A : \mathcal{M} = A : \left(A^{1\times(q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix}\right) \triangleq \{\lambda \in A^r \mid P\,\lambda \in A^q\} = \widetilde{D}\,A^r,$$

i.e., $A : \mathcal{M}$ is **free lattice of** K^r, namely, $A : \mathcal{M} \cong A^r$.

Coprime factorizations

• **Definition**: A transfer matrix $P \in K^{q \times r}$ admits a **left-coprime factorization** if there exist

$$D \in A^{q \times q}, \ N \in A^{q \times r}, \ X \in A^{q \times q}, \ Y \in A^{r \times q},$$

such that $\det D \neq 0$ and:

$$P = D^{-1} N, \quad D X - N Y = I_q.$$

• **Proposition**: $P \in K^{q \times r}$ admits a **left-coprime factorization** iff there exists $D \in A^{q \times q}$ such that $\det D \neq 0$ and

$$\mathcal{L} \triangleq (I_q \quad -P) \, A^{q+r} = D^{-1} \, A^q,$$

i.e., iff $\mathcal{L}$ is a **free lattice of** K^q, namely, $\mathcal{L} \cong A^q$.

Example

- The transfer matrix defined by $P = \begin{pmatrix} \frac{e^{-s}}{s-1} \\ \frac{e^{-s}}{(s-1)^2} \end{pmatrix} \in K^2$ admits the **left-coprime factorization** $P = D'^{-1}\, N'$ defined by

$$D' = \begin{pmatrix} \frac{1}{(s+1)} & -\frac{(s-1)}{(s+1)} \\ \frac{(s-1)}{(s+1)} & 0 \end{pmatrix} \in A^{2\times 2}, \quad N' = \begin{pmatrix} 0 \\ \frac{e^{-s}}{(s+1)} \end{pmatrix} \in A^2,$$

as we have:

$$\begin{pmatrix} \frac{1}{(s+1)} & -\frac{(s-1)}{(s+1)} \\ \frac{(s-1)}{(s+1)} & 0 \end{pmatrix} \begin{pmatrix} -2\,b\,\frac{(s-1)^2}{(s+1)^2} + 2 & b\,\frac{(s-1)}{(s+1)} \\ -2\,b\,\frac{(s-1)}{(s+1)^2} - 1 & b\,\frac{1}{(s+1)} \end{pmatrix}$$

$$- \begin{pmatrix} 0 \\ \frac{e^{-s}}{(s+1)} \end{pmatrix} \begin{pmatrix} 2\,a\,\frac{(s-1)}{(s+1)^2} & -a\,\frac{1}{(s+1)} \end{pmatrix} = I_2,$$

where a and b are defined by:

$$a = \frac{4\,e\,(5\,s-3)}{(s+1)} \in A, \quad b = \frac{(s+1)^3 - 4\,(5\,s-3)\,e^{-(s-1)}}{(s+1)\,(s-1)^2} \in A.$$

Coprime factorization

• **Definition**: A transfer matrix $P \in K^{q \times r}$ admits a **right-coprime factorization** if there exist

$$\widetilde{D} \in A^{r \times r}, \; \widetilde{N} \in A^{q \times r}, \; \widetilde{X} \in A^{r \times r}, \; \widetilde{Y} \in A^{r \times q},$$

such that $\det \widetilde{D} \neq 0$ and:

$$P = \widetilde{N}\, \widetilde{D}^{-1}, \quad -\widetilde{Y}\, \widetilde{N} + \widetilde{X}\, \widetilde{D} = I_r.$$

• **Proposition**: $P \in K^{q \times r}$ admits a **right-coprime factorization** if there exists $\widetilde{D} \in A^{r \times r}$ such that $\det \widetilde{D} \neq 0$ and

$$\mathcal{M} = A^{1 \times (q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix} = A^{1 \times r}\, \widetilde{D}^{-1},$$

i.e., iff $\mathcal{M}$ is a **free lattice of** $K^{1 \times r}$, namely, $\mathcal{M} \cong A^{1 \times r}$.

Doubly coprime factorizations

• **Definition**: $P \in K^{q \times r}$ admits a **doubly coprime factorization** over A if there exist

$$D \in A^{q \times q},\; N \in A^{q \times r},\; X \in A^{q \times q},\; Y \in A^{r \times q},$$
$$\widetilde{D} \in A^{r \times r},\; \widetilde{N} \in A^{q \times r},\; \widetilde{X} \in A^{r \times r},\; \widetilde{Y} \in A^{r \times q},$$

such that $\det D \neq 0$, $\det \widetilde{D} \neq 0$ and:

$$P = D^{-1}\, N = \widetilde{N}\, \widetilde{D}^{-1},$$

$$\left(\begin{array}{cc} D & -N \\ -\widetilde{Y} & \widetilde{X} \end{array} \right) \left(\begin{array}{cc} X & \widetilde{N} \\ Y & \widetilde{D} \end{array} \right) = I_{q+r}.$$

• **Proposition**: $P \in K^{q \times r}$ admits a **doubly coprime factorization** iff P admits a left- and a right-coprime factorization.

Bézout domains

• **Definition**: An integral domain A is called a **Bézout domain** if every finitely generated ideal of A is **principal**, namely, generated by an element of A.

• **Examples**: RH_∞, the ring $E(\mathbb{R})$ of entire functions with real coefficients and $\mathcal{E} = \mathbb{R}(s)[e^{-s}] \cap E(\mathbb{R})$ are **Bézout domains**.

• **Theorem**: We have the following equivalences:

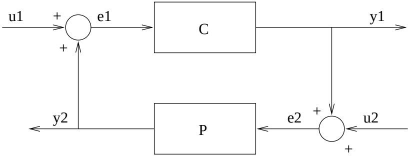
1. Every transfer matrix P with entries in K admits a **doubly coprime factorization**.

2. Every transfer function $p \in K$ admits a **coprime factorization**.

3. A is a **Bézout domain**.

Internal stabilization

- Let A be an algebra of **stable transfer function**, $K = Q(A)$.
- Let $P \in K^{q \times r}$ be a **plant** and $C \in K^{r \times q}$ a **controller**.



- The **closed-loop system** is defined by:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} I_q & -P \\ -C & I_r \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \quad \begin{cases} y_1 = e_2 - u_2, \\ y_2 = e_1 - u_1. \end{cases}$$

- **Definition**: $P \in K^{q \times r}$ is **internally stabilizable** iff there exists a **stabilizing controller** $C \in K^{r \times q}$, namely, $C \in K^{r \times q}$ satisfies:

$$\begin{pmatrix} I_q & -P \\ -C & I_r \end{pmatrix}^{-1} = \begin{pmatrix} (I_q - P\,C)^{-1} & (I_q - P\,C)^{-1}\,P \\ (I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix} \in A^{(q+r)\times(q+r)}.$$

Internal stabilizability

• **Theorem**: $P \in K^{q \times r}$ is **internally stabilizable** iff one of the following conditions is satisfied:

1. $\mathcal{L} = (I_q \ \ -P)\, A^{q+r}$ is a **projective lattice of** K^q, namely, there exists an A-module M such that:

$$\mathcal{L} \oplus M \cong A^{q+r}.$$

2. $\mathcal{M} = A^{1 \times (q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix}$ is a **projective lattice of** $K^{1 \times r}$, namely, there exists an A-module N such that:

$$\mathcal{M} \oplus N \cong A^{1 \times (q+r)}.$$

Internal stabilizability

- Let $P \in K^{q \times r}$ be an **internally stabilizable plant** and:

$$R = (I_q \quad -P) \in K^{q \times (q+r)}, \quad Q = \begin{pmatrix} P \\ I_r \end{pmatrix} \in K^{(q+r) \times r}.$$

Then, we have the following **split exact sequences**:

$$0 \longleftarrow \mathcal{L} = R\, A^{q+r} \underset{\xrightarrow{S.}}{\xleftarrow{R.}} A^{q+r} \underset{\xrightarrow{T.}}{\xleftarrow{Q.}} A : \mathcal{M} = A : \left(A^{1 \times p}\, Q\right) \longleftarrow 0,$$

$$0 \longrightarrow A : \mathcal{L} = A : (R\, A^{1 \times (q+r)}) \underset{\xleftarrow{.S}}{\xrightarrow{.R}} A^{1 \times (q+r)} \underset{\xleftarrow{.T}}{\xrightarrow{.Q}} \mathcal{M} = A^{1 \times p}\, Q \longrightarrow 0.$$

- **Corollary**: If $P \in K^{q \times r}$ is **internally stabilizable**, then we have:

1. $\mathcal{L} \oplus (A : \mathcal{M}) \cong A^{q+r}$ and $\mathcal{M} = A : (A : \mathcal{M})$.
2. $\mathcal{M} \oplus (A : \mathcal{L}) \cong A^{1 \times (q+r)}$ and $\mathcal{L} = A : (A : \mathcal{L})$.

Internal stabilizability

- **Corollary**: $P \in K^{q \times r}$ is **internally stabilizable** iff one of the following conditions is satisfied:

C1. There exists $S = (U^T \quad V^T)^T \in A^{(q+r) \times q}$ such that:

$$\begin{cases} S\,P = \begin{pmatrix} U\,P \\ V\,P \end{pmatrix} \in A^{(q+r) \times r}, \\ (I_q \quad -P)\,S = U - P\,V = I_q. \end{cases}$$

Then, $C = V\,U^{-1}$ is **a stabilizing controller of** P.

C2. There exists $T = (-\widetilde{V} \quad \widetilde{U}) \in A^{r \times (q+r)}$ such that:

$$\begin{cases} P\;T = (-P\,\widetilde{V} \quad P\,\widetilde{U}) \in A^{q \times (q+r)}, \\ T \begin{pmatrix} P \\ I_r \end{pmatrix} = -\widetilde{V}\,P + \widetilde{U} = I_r. \end{cases}$$

Then, $C' = \widetilde{U}^{-1}\,\widetilde{V}$ is **a stabilizing controller of** P.

- **Proposition**: $\exists\ S \in A^{(q+r) \times q}$, $T \in A^{r \times (q+r)}$ satisfying 1, 2 and:

$$T\,S = -\widetilde{V}\,U + \widetilde{U}\,V = 0 \Rightarrow C = V\,U^{-1} = \widetilde{U}^{-1}\,\widetilde{V}.$$

Example

- Let us consider the **transfer matrix** ($A = H_\infty(\mathbb{C}_+)$, $K = Q(A)$):

$$P = \begin{pmatrix} \frac{e^{-s}}{s-1} \\ \frac{e^{-s}}{(s-1)^2} \end{pmatrix} \in K^2.$$

- The matrix $S = (U^T \quad V^T)^T \in A^{3\times 2}$ defined by

$$S = \begin{pmatrix} \frac{2}{s+1} + b\left(\frac{s-1}{s+1}\right)^3 & 2\,b\left(\frac{s-1}{s+1}\right)^3 - 2\,\frac{(s-1)}{(s+1)} \\ b\,\frac{(s-1)^2}{(s+1)^3} - \frac{1}{s+1} & \frac{s-1}{s+1} + 2\,b\,\frac{(s-1)}{(s+1)^3} \\ -a\,\frac{(s-1)^2}{(s+1)^3} & -2\,a\,\frac{(s-1)^2}{(s+1)^3} \end{pmatrix}$$

with $a = \frac{4\,e\,(5\,s-3)}{(s+1)} \in A$ and $b = \frac{(s+1)^3 - 4\,(5\,s-3)\,e^{-(s-1)}}{(s+1)\,(s-1)^2} \in A$, satisfies

$$\begin{cases} S\,P \in A^{3\times 1}, \\ (I_2 \quad -P)\,S = U - P\,V = I_2, \end{cases}$$

$\Rightarrow C = V\,U^{-1} = -\frac{4\,(5\,s-3)\,e\,(s-1)^2}{(s+1)\,((s+1)^3 - 4\,(5\,s-3)\,e^{-(s-1)})}\,(1 \quad 2)$ **IS** P.

Projectors

• **Corollary**: $P \in K^{q \times r}$ is **internally stabilized by the controller** $C \in K^{r \times q}$ iff one of the following conditions is satisfied:

1. The matrix

$$\Pi_1 = \begin{pmatrix} (I_q - P\,C)^{-1} & -(I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & -C\,(I_q - P\,C)^{-1}\,P \end{pmatrix}$$

is a **projector of** $A^{(q+r)\times(q+r)}$, namely, $\Pi_1^2 = \Pi_1 \in A^{(q+r)\times(q+r)}$.

2. The matrix

$$\Pi_2 = \begin{pmatrix} -P\,(I_{p-q} - C\,P)^{-1}\,C & P\,(I_{p-q} - C\,P)^{-1} \\ -(I_{p-q} - C\,P)^{-1}\,C & (I_{p-q} - C\,P)^{-1} \end{pmatrix}$$

is a **projector of** $A^{(q+r)\times(q+r)}$, namely, $\Pi_2^2 = \Pi_2 \in A^{(q+r)\times(q+r)}$.

Morover, we have: $\Pi_1 + \Pi_2 = I_{q+r}$.

• **Remark**: This result was known for $A = H_\infty(\mathbb{C}_+)$. The **robustness radius** is then defined by (**loop-shaping procedure**):

$$b_{P,C} \triangleq \|\,\Pi_1\,\|_\infty^{-1} = \|\,\Pi_2\,\|_\infty^{-1}\,.$$

Prüfer domains

• **Definition**: An integral domain A is called a **Prüfer domain** if every finitely generated ideal of A is a **projective** A-module.

• **Example**: **Dedekind domains** (e.g., $\mathbb{R}[x,y]/(x^2+y^2-1)$, $\mathbb{Z}[i\sqrt{5}]$), the $\mathbb{Z}$-valued polynomials in $\mathbb{Q}[x]$, namely:

$$A = \{p \in \mathbb{Q}[x] \mid p(\mathbb{Z}) \subset \mathbb{Z}\}.$$

• **Theorem**: We have the following equivalences:

1. Every transfer matrix P with entries in K is **internally stabilizable**.

2. Every transfer function $p \in K$ is **internally stabilizable**.

3. A is a **Prüfer domain**.

SC for internal stabilizability

- **Fact 1**: P admits a **doubly coprime factorization** iff $\mathcal{L}$ and $\mathcal{M}$ are **free A-modules.**

- **Fact 2**: P is **internally stabilizable** ff $\mathcal{L}$ and $\mathcal{M}$ are **projective A-modules.**

- **Fact 3**: **A free A-module is projective.**

- **Corollary**: 1. If $P \in K^{q \times r}$ admits a **left-coprime factorization**

$$P = D^{-1} N, \quad D\, X - N\, Y = I_q,$$

then $S = ((X\, D)^T \quad (Y\, D)^T)^T$ satisfies C1

$$\Rightarrow C = (Y\, D)\, (X\, D)^{-1} = Y\, X^{-1} \in \mathrm{Stab}(P).$$

2. If $P \in K^{q \times r}$ admits a **right-coprime factorization**

$$P = \widetilde{N}\, \widetilde{D}^{-1}, \quad -\widetilde{Y}\, X + \widetilde{X}\, \widetilde{D} = I_r,$$

then $T = (-\widetilde{D}\, \widetilde{Y} \quad \widetilde{D}\, \widetilde{X})$ satisfies C2

$$\Rightarrow C = (\widetilde{D}\, \widetilde{X})^{-1}\, (\widetilde{D}\, \widetilde{Y}) = \widetilde{X}^{-1}\, \widetilde{Y} \in \mathrm{Stab}(P).$$

Stabilizable m-D linear systems

- $\overline{\mathbb{D}}^m = \{z \in \mathbb{C}^m \mid |z_i| \leq 1, \ i = 1, \ldots, m\}$ unit polydisc of $\mathbb{C}^m$.
- Let $\mathbb{R}(z_1, \ldots, z_m)_S$ be the ring of **stabilizable *m*-D systems**:

$$\mathbb{R}(z_1, \ldots, z_m)_S = \{r/s \,|\, 0 \neq s, \ r \in \mathbb{R}[z_1, \ldots, z_m], \ s(\underline{z}) = 0 \Rightarrow \underline{z} \notin \overline{\mathbb{D}^m}\}.$$

- **Z. Lin's conjecture**:

Determine whether or not an internally stabilizable linear system defined by a transfer matrix P with entries in $\mathbb{R}(z_1, \ldots, z_m)$ admits a doubly coprime factorization over $\mathbb{R}(z_1, \ldots, z_m)_S$.

- **Theorem**: (Kamen-Khargonekar-Tannenbaum, Byrnes-Spong-Tarn, 84): $\mathbb{R}(z_1, \ldots, z_m)_S$ **is a projective free ring.**

- This result is not trivial (the **proof was given by P. Deligne**).
- **Corollary**: **Z. Lin's conjecture is solved**.
- **Open question**: **Constructive proof.**

Parametrization of all stabilizing controllers

- **Theorem**: Let $P \in K^{q \times r}$ be a **stabilizable plant**.

All stabilizing controllers of P have the form

$$C(Q) = (V + Q)\,(U + P\,Q)^{-1} = (Y + Q\,P)^{-1}\,(X + Q),$$

where C_* is **a particular stabilizing controller of** P,

$$\begin{cases} U = (I_q - P\,C_*)^{-1}, \\ V = C_*\,(I_q - P\,C_*)^{-1}, \\ X = (I_r - C_*\,P)^{-1}\,C_*, \\ Y = (I_r - C_*\,P)^{-1}, \end{cases} \qquad \begin{cases} S = \begin{pmatrix} U \\ V \end{pmatrix} \in A^{(q+r) \times q}, \\ T = (-\widetilde{V} \quad \widetilde{U}) \in A^{r \times (q+r)}, \end{cases}$$

and Q is **any matrix which belongs to**:

$$\begin{aligned} \Omega &= \{L \in A^{r \times q} \mid L\,P \in A^{r \times r},\ P\,L \in A^{q \times q},\ P\,L\,P \in A^{q \times r}\} \\ &= (A : \mathcal{L}) : \mathcal{M} = (A : \mathcal{M}) : \mathcal{L} = T\,A^{(q+r) \times (q+r)}\,S, \end{aligned}$$

such that $\det(U + P\,Q) \neq 0$ and $\det(Y + Q\,P) \neq 0$.

(Ω is a **projective** A**-module of rank** $r \times q$).

The projective A-module Ω

- **Open question**: Find a **minimal family of generators** of the projective A-module Ω of rank $r \times q$, i.e., a minimal family $\{L_i\}_{1 \leq i \leq s}$ such that:

$$\forall\, L \in \Omega, \quad \exists\, \lambda_i \in A,\ i = 1, \ldots, s: \quad L = \sum_{i=1}^{s} \lambda_i\, L_i.$$

- If A has a **Krull dimension** equals to m, then we have:

$$\mu_A(\Omega) = s \leq q \times r + m.$$

- **Proposition**: If $P \in K^{q \times r}$ admits a **weakly left-coprime factorization** $P = D^{-1}\, N$, then we have:

$$\Omega = \{L \in A^{r \times q} \mid P\, L \in A^{q \times q}\}\, D.$$

- **Proposition**: If $P \in K^{q \times r}$ admits a **weakly right-coprime factorization** $P = \widetilde{N}\, \widetilde{D}^{-1}$, then we have:

$$\Omega = \widetilde{D}\, \{L \in A^{r \times q} \mid L\, P \in A^{r \times r}\}.$$

Youla-Kučera parametrization

- **Corollary**: Let $P \in K^{q \times r}$ be a plant which admits a **doubly coprime factorization** $P = D^{-1} N = \widetilde{N} \, \widetilde{D}^{-1}$:

$$\begin{pmatrix} D & -N \\ -\widetilde{Y} & \widetilde{X} \end{pmatrix} \begin{pmatrix} X & \widetilde{N} \\ Y & \widetilde{D} \end{pmatrix} = I_{q+r}.$$

Then, $\Omega = \{L \in A^{r \times q} \mid L\,P \in A^{r \times r},\ P\,L \in A^{q \times q},\ P\,L\,P \in A^{q \times r}\}$ is the **free** A-**module** defined by:

$$\Omega = \widetilde{D}\, A^{r \times q}\, D = \{L \in A^{r \times q} \mid L = \widetilde{D}\, R\, D,\ \forall\, R \in A^{r \times q}\}.$$

All stabilizing controllers of P **are then of the form**

$$C(Q) = (Y + \widetilde{D}\, Q)\,(X + \widetilde{N}\, Q)^{-1} = (\widetilde{X} + Q\, N)^{-1}\,(\widetilde{Y} + Q\, D),$$

where $Q \in A^{r \times q}$ is **any matrix such that:**

$$\det\,(X + \widetilde{N}\, Q) \neq 0, \quad \det\,(\widetilde{X} + Q\, N) \neq 0.$$

Sensitivity minimization

- Let A be a **Banach algebra** (e.g., $A = H_\infty(\mathbb{C}_+)$, $\hat{\mathcal{A}}$, W_+, $A(\mathbb{D})$).

- Let $P \in K^{q\times r}$ be a **stabilizable plant** and W_1 and W_2 two **weighted transfer matrices**. Then, we have

$$\inf_{C\in\mathrm{Stab}(P)} \| \, W_1 \, (I_q - P\,C)^{-1} \, W_2 \, \|_A = \inf_{Q\in\Omega} \| \, W_1 \, (U + P\,Q) \, W_2 \, \|_A, \quad (\star)$$

where $C_\star = V\,U^{-1}$ is **a stabilizing controller of** P and:

$$U = (I_q - P\,C_*)^{-1} \in A^{q\times q}, \quad V = C_* \, (I_q - P\,C_*)^{-1} \in A^{r\times q}.$$

- The **optimization problem** $(\star)$ is **convex**.

- If $P = D^{-1}\,N = \widetilde{N}\,\widetilde{D}^{-1}$ is a **doubly coprime factorization**

$$\begin{pmatrix} D & -N \\ -\widetilde{Y} & \widetilde{X} \end{pmatrix} \begin{pmatrix} X & \widetilde{N} \\ Y & \widetilde{D} \end{pmatrix} = I_{q+r} \Rightarrow \begin{cases} Q \in \Omega = \widetilde{D}\,A^{r\times q}\,D, \\ U + P\,Q = (X + \widetilde{N}\,R)\,D, \end{cases}$$

$$\Rightarrow \quad (\star) \quad \Leftrightarrow \quad \inf_{R\in A^{r\times q}} \| \, W_1 \, (X + \widetilde{N}\,R)\,D\,W_2 \, \|_A \, .$$

Conclusion

• We generalized the Youla-Kučera parametrization for MIMO internally stabilizable plants.

• This parametrization does not assume the existence of doubly coprime factorizations.

• **When does a stabilizable plant admit a doubly coprime factorization?** We proved that this problem is related to:

When is a certain projective A-module free?

• This has been a **difficult problem** studied for years in:

− **algebra**: algebraic K-theory (Serre's conjecture (55) $A = k[x_1, \ldots, x_n]$, k a field, solved by Quillen-Suslin (76)),

− **number theory**: number fields,

− **algebraic geometry**: function fields,

− **topology**: triviality of vector bundles,

− **operator theory**: topological K-theory (e.g., $C^\star$-algebras).

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