

Factorization, reduction and decomposition problems

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Factorization, reduction and decomposition problems

- Let D be an Ore algebra.
- Let $R \in D^{q \times p}$ be a matrix of functional operators.
- Questions:
 1. $\exists R_1 \in D^{r \times p}, R_2 \in D^{q \times r} : R = R_2 R_1 ?$
 2. $\exists W \in GL_p(D), V \in GL_q(D)$ s.t. $V R W = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} ?$
 3. $\exists W \in GL_p(D), V \in GL_q(D)$ s.t. $V R W = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix} ?$

Outline

- **Type of systems:** OD and PD/difference/differential time-delay. . . linear systems: **linear functional systems**.
- **General topic:** **algebraic study of linear functional systems** coming from mathematical physics, engineering sciences, control theory. . .
- **Techniques:** **module theory** and **homological algebra**.
- **Applications:** equivalences of systems, symmetries, quadratic first integrals/conservation laws, decoupling problems. . .
- **Implementation:** package **OREMORPHISMS**:

`http://www-sop.inria.fr/members/Alban.Quadrat/OreMorphisms/index.html`

Jacobson/Smith normal form

- Let D be a principal ideal domain.
- **Theorem:** $\forall R \in D^{q \times p}$, $\exists V \in GL_q(D)$, $\exists U \in GL_p(D)$:

$$\bar{R} = VRU = \begin{pmatrix} \alpha_1 & 0 & \dots & \dots & 0 & \dots & 0 \\ 0 & \alpha_2 & & & \vdots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \vdots \\ 0 & \dots & 0 & \alpha_r & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & \dots & 0 \\ \vdots & & & & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & \dots & 0 \end{pmatrix},$$

where $\alpha_1 \parallel \alpha_2 \parallel \dots \parallel \alpha_r \neq 0$.

Example: Smith normal form

- Let us consider 2 pendulum of the same length mounted on a car:

$$\begin{cases} \ddot{x}_1(t) + \alpha x_1(t) - \alpha u(t) = 0, \\ \ddot{x}_2(t) + \alpha x_2(t) - \alpha u(t) = 0, \end{cases} \quad \alpha = \frac{g}{l}.$$

- Let us consider the principal ideal domain $D = \mathbb{Q}(\alpha) [\partial; \text{id}, \frac{d}{dt}]$.

$$P = \sum_{i=0}^n a_i \partial^i \in D, \quad a_i \in \mathbb{Q}(\alpha).$$

$$\begin{aligned} \begin{pmatrix} -\alpha & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \partial^2 + \alpha & 0 & -\alpha \\ 0 & \partial^2 + \alpha & -\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & \partial^2 + \alpha \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \partial^2 + \alpha & 0 \end{pmatrix}. \end{aligned}$$

Example: Jacobson normal form

- Let us consider the **time-varying linear system**:

$$\begin{cases} t \dot{y}_1(t) - y_1(t) - t^2 \dot{y}_2(t) + u_1(t) = 0, \\ \dot{y}_1(t) + t \dot{y}_2(t) - y_2(t) + u_2(t) = 0. \end{cases}$$

- We consider the **left principal ideal domain** $D = \mathbb{Q}(t) [\partial; \text{id}, \frac{d}{dt}]$.

$$\begin{pmatrix} t\partial - 1 & -t^2\partial & 1 & 0 \\ \partial & t\partial - 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -t\partial + 1 & t^2\partial \\ 0 & 1 & -\partial & -t\partial + 1 \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

- Implementation** in the package **JACOBSON**.

- Let us consider the **first order** OD system:

$$\partial y = E(t) y \quad (*)$$

- Does it exist an **invertible change of variables** $y = P(t) z$ s.t.

$$(*) \Leftrightarrow \partial z = F(t) z, \quad F = P^{-1} (E P - \partial P),$$

is either of the **form**:

$$F = \begin{pmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{pmatrix} \quad \text{or} \quad F = \begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix} ?$$

$$\partial I - F(t) = \begin{pmatrix} \partial I - F_{11} & -F_{12} \\ 0 & \partial I - F_{22} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \partial I - F_{11} & 0 \\ 0 & \partial I - F_{22} \end{pmatrix}.$$

- If $E(t) = E \in \mathbb{R}^{n \times n}$, then $F = P^{-1} E P$: **Jordan normal form**.

Eigenring: example

- Let us consider the **system** $\dot{y}(t) = E(t)y(t)$, where:

$$E(t) = \begin{pmatrix} t(2t+1) & -2t^3 - 2t^2 + 1 \\ 2t & -t(2t+1) \end{pmatrix}.$$

- The **eigenring** of the system $\partial y(t) = E(t)y(t)$ is:

$$\mathcal{E} = \{P \in \mathbb{Q}(t)^{2 \times 2} \mid \dot{P}(t) = E(t)P(t) - P(t)E(t)\}.$$

- Computing the **rational solutions** of $\dot{P} = [E, P]$, we then get:

$$\mathcal{E} = \left\{ P = \begin{pmatrix} a_1 - a_2(t+1) & a_2 t(t+1) \\ -a_2 & a_2 t + a_1 \end{pmatrix} \mid a_1, a_2 \in \mathbb{Q} \right\}.$$

- P is **isospectral** because (E, P) is a **Lax pair**:

$$\det(P - \lambda I_2) = (\lambda - a_1)(\lambda - a_1 + a_2).$$

Eigenring: example

- Computing a **Jordan normal form** of P , we obtain

$$J = V^{-1} P V = \begin{pmatrix} a_1 & 0 \\ 0 & a_1 - a_2 \end{pmatrix},$$

$$V = \begin{pmatrix} -t & 1+t \\ -1 & 1 \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} 1 & -(t+1) \\ 1 & -t \end{pmatrix}.$$

- Let us denote by $z = V^{-1} y = (y_1 - (t+1)y_2 \quad y_1 - t y_2)^T$:

$$\dot{y}(t) = E(t) y(t) \quad \Leftrightarrow \quad \dot{z}(t) = \begin{pmatrix} -t & 0 \\ 0 & t \end{pmatrix} z(t)$$

$$\Rightarrow \begin{cases} z_1(t) = C_1 e^{-t^2/2}, \\ z_2(t) = C_2 e^{t^2/2}, \end{cases} \Rightarrow \begin{cases} y_1(t) = -C_1 t e^{-t^2/2} + C_2 (t+1) e^{t^2/2}, \\ y_2(t) = -C_1 e^{-t^2/2} + C_2 e^{t^2/2}. \end{cases}$$

Finitely presented left D -modules

- Let D be an Ore algebra, $R \in D^{q \times p}$ and a left D -module \mathcal{F} .
- Let us consider the system $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\eta = 0\}$.
- Let us consider the left D -homomorphism:

$$\begin{aligned} D^{1 \times q} &\longrightarrow D^{1 \times p} \\ \lambda = (\lambda_1, \dots, \lambda_q) &\longmapsto \lambda R. \end{aligned}$$

- As in number theory or algebraic geometry, we associate with the system $\ker_{\mathcal{F}}(R.)$ the finitely presented left D -module:

$$M = D^{1 \times p} / (D^{1 \times q} R).$$

- Theorem: (Malgrange) We have the following isomorphism:

$$\ker_{\mathcal{F}}(R.) \cong \operatorname{hom}_D(M, \mathcal{F}) = \{f : M \rightarrow \mathcal{F} \mid f \text{ left } D\text{-linear}\}.$$

Homomorphisms of finitely presented modules

- Let D be an Ore algebra of functional operators.
- Let $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$ be two matrices.
- Let us consider the finitely presented left D -modules:

$$M = D^{1 \times p} / (D^{1 \times q} R), \quad M' = D^{1 \times p'} / (D^{1 \times q'} R').$$

- $\text{hom}_D(M, M')$: abelian group of D -morphisms from M to M' :

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & & & \downarrow f & & \\ D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0. \end{array}$$

Homomorphisms of finitely presented modules

- Let D be an Ore algebra of functional operators.
- Let $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$ be two matrices.
- We have the following commutative exact diagram:

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & \downarrow \cdot Q & & \downarrow \cdot P & & \downarrow f & \\ D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0. \end{array}$$

$\exists f : M \rightarrow M' \Leftrightarrow \exists P \in D^{p \times p'}, Q \in D^{q \times q'}$ such that:

$$R P = Q R'.$$

Moreover, we have $f(\pi(\lambda)) = \pi'(\lambda P)$, for all $\lambda \in D^{1 \times p}$.

Eigenring: $\partial y = E y$ & $\partial z = F z$

- $D = A[\partial; \alpha, \beta]$, $E, F \in A^{p \times p}$, $R = \partial I_p - E$, $R' = \partial I_p - F$.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & D^{1 \times p} & \xrightarrow{.(\partial_p I - E)} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \downarrow .Q & & \downarrow .P & & \downarrow f & & \\
 0 & \longrightarrow & D^{1 \times p} & \xrightarrow{.(\partial I_p - F)} & D^{1 \times p} & \xrightarrow{\pi'} & M' & \longrightarrow & 0.
 \end{array}$$

$$(\partial I_p - E) P = Q (\partial I_p - F) \Leftrightarrow \begin{cases} \alpha(P) = Q \in A^{p \times p}, \\ \beta(P) = E P - \alpha(P) F. \end{cases}$$

If $P \in A^{p \times p}$ is **invertible**, we then have:

$$F = -\alpha(P)^{-1}(\beta(P) - E P).$$

Eigenring: $\partial y = E y$ & $\partial z = F z$

- $D = A[\partial; \alpha, \beta]$, $E, F \in A^{p \times p}$, $R = \partial I_p - E$, $R' = \partial I_p - F$.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & D^{1 \times p} & \xrightarrow{\cdot(\partial_p I - E)} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \downarrow \cdot Q & & \downarrow \cdot P & & \downarrow f & & \\
 0 & \longrightarrow & D^{1 \times p} & \xrightarrow{\cdot(\partial I_p - F)} & D^{1 \times p} & \xrightarrow{\pi'} & M' & \longrightarrow & 0.
 \end{array}$$

$$(\partial I_p - E) P = Q (\partial I_p - F) \Leftrightarrow \begin{cases} \alpha(P) = Q \in A^{p \times p}, \\ \beta(P) = E P - \alpha(P) F. \end{cases}$$

If $P \in A^{p \times p}$ is **invertible**, we then have:

$$F = -\alpha(P)^{-1}(\beta(P) - E P).$$

- **Differential case:** $\beta = \frac{d}{dt}$, $\alpha = \text{id}$:

$$\dot{P} = E P - P F, \quad F = -P^{-1}(\dot{P} - E P).$$

Eigenring: $\partial y = E y$ & $\partial z = F z$

- $D = A[\partial; \alpha, \beta]$, $E, F \in A^{p \times p}$, $R = \partial I_p - E$, $R' = \partial I_p - F$.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & D^{1 \times p} & \xrightarrow{.(\partial_p I - E)} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \downarrow .Q & & \downarrow .P & & \downarrow f & & \\
 0 & \longrightarrow & D^{1 \times p} & \xrightarrow{.(\partial I_p - F)} & D^{1 \times p} & \xrightarrow{\pi'} & M' & \longrightarrow & 0.
 \end{array}$$

$$(\partial I_p - E) P = Q (\partial I_p - F) \Leftrightarrow \begin{cases} \alpha(P) = Q \in A^{p \times p}, \\ \beta(P) = E P - \alpha(P) F. \end{cases}$$

If $P \in A^{p \times p}$ is **invertible**, we then have:

$$F = -\alpha(P)^{-1}(\beta(P) - E P).$$

- **Discrete case:** $\beta = 0$, $\alpha(k) = k + 1$:

$$E_k P_k - P_{k+1} F_k = 0, \quad F_k = P_{k+1}^{-1} E_k P_k.$$

Example: Lax pairs for the KdV equation

- Let us consider the differential ring $\mathbb{Q}\{u\}$ formed by differential polynomials in u , the **prime differential ideal** of $\mathbb{Q}\{u\}$ defined by

$$\mathfrak{p} = \left\{ \frac{\partial u}{\partial t} - 6u \left(\frac{\partial u}{\partial x} \right) + \frac{\partial^3 u}{\partial x^3} \right\},$$

the **differential ring** $L = \mathbb{Q}\{u\}/\mathfrak{p}$ and $K = \{n/d \mid 0 \neq d, n \in L\}$ the **differential field** defined by the **KdV equation**.

- Let us consider the rings $A = K[\partial_x; \text{id}, \frac{\partial}{\partial x}]$, $D = A[\partial_t; \text{id}, \frac{\partial}{\partial t}]$,

$$\begin{cases} E = -4\partial_x^3 + 6u\partial_x + 3 \left(\frac{\partial u}{\partial x} \right) \in D, \\ R = \partial_t - E \in D, \end{cases} \quad M = D/(DR).$$

- The **Schrödinger operator** $P = -\partial_x^2 + u$ with **potential** u satisfies:

$$RP - PR = \partial_t P - EP + PE = \frac{\partial u}{\partial t} - 6u \left(\frac{\partial u}{\partial x} \right) - \frac{\partial^3 u}{\partial x^3} = 0.$$

Example: Lax pairs for the KdV equation

In the **inverse scattering theory**, a key point is that the smooth one-parameter family of differential operators

$$t \longmapsto -\partial_x^2 + u(x, t)$$

defines an **isospectral flow** on the solutions of $\partial_t \eta = E \eta$:

$$(-\partial_x^2 + u(x, 0)) \psi(x) = \lambda \psi(x),$$

$$\begin{cases} \partial_t \eta(x, t) = E \eta(x, t), & E = -4 \partial_x^3 + 6 u \partial_x + 3 \left(\frac{\partial u}{\partial x} \right), \\ \eta(x, 0) = \psi(x), \end{cases}$$

$$\Rightarrow (-\partial_x^2 + u(x, t)) \eta(x, t) = \lambda \eta(x, t),$$

\Rightarrow the inverse scattering method proves that the KdV equation is **completely integrable**.

Computation of $\text{hom}_D(M, M')$

- **Problem:** Given $R \in D^{q \times p}$ and $R' \in D^{q' \times p'}$, find $P \in D^{p \times p'}$ and $Q \in D^{q \times q'}$ satisfying the relation $R P = Q R'$.
- If D is a **commutative ring**, then $\text{hom}_D(M, M')$ is a **D -module**.
- The **Kronecker product** of $E \in D^{q \times p}$ and $F \in D^{r \times s}$ is:

$$E \otimes F = \begin{pmatrix} E_{11} F & \dots & E_{1p} F \\ \vdots & \vdots & \vdots \\ E_{q1} F & \dots & E_{qp} F \end{pmatrix} \in D^{(qr) \times (ps)}.$$

Lemma: If $U \in D^{a \times b}$, $V \in D^{b \times c}$ and $W \in D^{c \times d}$, then we have:

$$\text{row}(U V W) = \text{row}(V) (U^T \otimes W).$$

$$\text{row}(R P I_{p'}) = \text{row}(P) (R^T \otimes I_{p'}), \quad \text{row}(I_q Q R') = \text{row}(Q) (I_q \otimes R'),$$

$$\Rightarrow (\text{row}(P) - \text{row}(Q)) \begin{pmatrix} R^T \otimes I_{p'} \\ -I_q \otimes R' \end{pmatrix} = 0.$$

Computation of $\text{hom}_D(M, M')$

- If D is a **non-commutative ring**, then $\text{hom}_D(M, M')$ is an **abelian group** and generally an **infinite-dimensional k -vector space**.

⇒ Find a k -basis of morphisms with **given degrees in x_i and in ∂_j** :

- 1 Take an ansatz for P of fixed degrees.
- 2 Compute RP and a Gröbner basis G of the rows of R' .
- 3 Reduce the rows of RP w.r.t. G .
- 4 Solve the system on the coefficients of the ansatz so that all the normal forms vanish.
- 5 Substitute the solutions in P and compute Q by means of a factorization.

Example: OD system

- Let $D = \mathbb{Q}[t] \left[\partial; \text{id}, \frac{d}{dt} \right]$ and $M = D^{1 \times 4} / (D^{1 \times 4} R)$, where:

$$R = \begin{pmatrix} \partial & -t & t & \partial \\ \partial & t\partial - t & \partial & -1 \\ \partial & -t & \partial + t & \partial - 1 \\ \partial & \partial - t & t & \partial \end{pmatrix} \in D^{4 \times 4}.$$

- $f \in \text{end}_D(M)$ is defined by (P, P) where $P \in \mathbb{Q}[t]^{4 \times 4}$ satisfies

$$P = \begin{pmatrix} a_4 - 2 a_2 t^2 & a_1 + a_5 t^2 + a_3 t^4 & 0 & 0 \\ -4 a_3 & a_4 + 2 a_5 + 2 a_3 t^2 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix},$$

where $a_1, a_2, a_3, a_4, a_5 \in \mathbb{Q}$, i.e., $RP = PR$.

Euler-Tricomi equation

- Let us consider the **Euler-Tricomi equation** (transonic flow):

$$\partial_1^2 u(x_1, x_2) - x_1 \partial_2^2 u(x_1, x_2) = 0.$$

- Let $D = A_2(\mathbb{Q})$, $R = (\partial_1^2 - x_1 \partial_2^2) \in D$ and $M = D/(D R)$.

- $\text{end}_D(M)_{1,1}$ is defined by:

$$\begin{cases} P = a_1 + a_2 \partial_2 + \frac{3}{2} a_3 x_2 \partial_2 + a_3 x_1 \partial_1, \\ Q = (a_1 + 2 a_3) + a_2 \partial_2 + \frac{3}{2} a_3 x_2 \partial_2 + a_3 x_1 \partial_1, \end{cases}$$

- $\text{end}_D(M)_{2,0}$ is defined by $P = Q = a_1 + a_2 \partial_2 + a_3 \partial_2^2$.
- $\text{end}_D(M)_{2,1}$ is defined by:

$$\begin{cases} P = a_1 + a_2 \partial_2 + \frac{3}{2} a_3 x_2 \partial_2 + a_3 x_1 \partial_1 \\ \quad + a_4 \partial_2^2 + \frac{3}{2} a_5 x_2 \partial_2^2 + a_5 x_1 \partial_1 \partial_2, \\ Q = (a_1 + 2 a_3) + a_2 \partial_2 + \frac{3}{2} a_3 x_2 \partial_2 + a_3 x_1 \partial_1 \\ \quad + a_4 \partial_2^2 + a_5 x_1 \partial_1 \partial_2 + 2 a_5 \partial_2 + \frac{3}{2} a_5 x_2 \partial_2^2. \end{cases}$$

Galois-like transformations

We have the following **commutative exact diagram**:

$$\begin{array}{ccccccc}
 D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 \downarrow \cdot Q & & \downarrow \cdot P & & \downarrow f & & \\
 D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0.
 \end{array} \quad (*)$$

If \mathcal{F} is a left D -module, then, applying the functor $\text{hom}_D(\cdot, \mathcal{F})$ to $(*)$, we obtain the following **commutative exact diagram**:

$$\begin{array}{ccccccc}
 0 = Q(R' y) = R(P y) & \longleftarrow & P y & & & & \\
 \mathcal{F}^q & & \xleftarrow{\cdot R} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(R.) & \longleftarrow 0 \\
 \uparrow Q. & & & \uparrow P. & & \uparrow f^* & \\
 \mathcal{F}^{q'} & & \xleftarrow{\cdot R'} & \mathcal{F}^{p'} & \longleftarrow & \ker_{\mathcal{F}}(R'.) & \longleftarrow 0. \\
 0 = R' y & & \longleftarrow & y & & &
 \end{array}$$

$\Rightarrow f^*$ sends $\ker_{\mathcal{F}}(R'.)$ to $\ker_{\mathcal{F}}(R.)$.

$(R' = R$: Galois-like transformations).

Example: Linear elasticity

- Consider the **Killing operator for the euclidian metric** defined by:

$$R = \begin{pmatrix} \partial_1 & 0 \\ \partial_2/2 & \partial_1/2 \\ 0 & \partial_2 \end{pmatrix}.$$

- The system $Ry = 0$ admits the following **general solution**:

$$y = \begin{pmatrix} c_1 x_2 + c_2 \\ -c_1 x_1 + c_3 \end{pmatrix}, \quad c_1, c_2, c_3 \in \mathbb{R}. \quad (\star)$$

- $\text{end}_D(M)$, where $M = D^{1 \times 2} / (D^{1 \times 3} R)$, is defined by:

$$P = \begin{pmatrix} \alpha_1 & \alpha_2 \partial_1 \\ 0 & 2\alpha_3 \partial_1 + \alpha_1 \end{pmatrix}, \quad \alpha_1, \alpha_2, \alpha_3 \in D.$$

- Applying P to (\star) , we get the **new solution**:

$$\bar{y} = Py = \begin{pmatrix} \alpha_1 c_1 x_2 + \alpha_1 c_2 - \alpha_2 c_1 \\ -\alpha_1 c_1 x_1 + \alpha_1 c_3 - 2\alpha_3 c_1 \end{pmatrix}, \quad \text{i.e., } R\bar{y} = 0.$$

Formal adjoint

- Let $D = A \left[\partial_1; \text{id}, \frac{\partial}{\partial x_1} \right] \dots \left[\partial_n; \text{id}, \frac{\partial}{\partial x_n} \right]$ be the **ring of differential operators** with coefficients in A (e.g., $k[x_1, \dots, x_n]$, $k(x_1, \dots, x_n)$).
- The **formal adjoint** $\tilde{R} \in D^{p \times q}$ of $R \in D^{q \times p}$ is defined by:

$$(\lambda, R \eta) = (\tilde{R} \lambda, \eta) + \sum_{i=1}^n \partial_i \Phi_i(\lambda, \eta).$$

- The **formal adjoint** \tilde{R} can be defined by $\tilde{R} = (\theta(R_{ij}))^T \in D^{p \times q}$, where $\theta : D \rightarrow D$ is the **involution** defined by:

- 1 $\forall a \in A, \quad \theta(a) = a.$
- 2 $\theta(\partial_i) = -\partial_i, \quad i = 1, \dots, n.$

Involution: $\theta^2 = \text{id}_D, \quad \forall P_1, P_2 \in D: \quad \theta(P_1 P_2) = \theta(P_2) \theta(P_1).$

Quadratic conservation laws

- Let us consider the left D -modules:

$$M = D^{1 \times p} / (D^{1 \times q} R), \quad \tilde{N} = D^{1 \times q} / (D^{1 \times p} \tilde{R}).$$

- Let $f : \tilde{N} \rightarrow M$ be a **homomorphism** defined by P and Q .
- Let \mathcal{F} be a left D -module and the **commutative exact diagram**:

$$\begin{array}{ccccccc} \mathcal{F}^p & \xleftarrow{\tilde{R}.} & \mathcal{F}^q & \longleftarrow & \ker_{\mathcal{F}}(\tilde{R}.) & \longleftarrow & 0 \\ \uparrow Q. & & \uparrow P. & & \uparrow f^* & & \\ \mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(R.) & \longleftarrow & 0. \end{array}$$

- $\eta \in \mathcal{F}^p$ **solution** of $R\eta = 0 \Rightarrow \lambda = P\eta$ is a **solution** of $\tilde{R}\lambda = 0$.

$$\Rightarrow (P\eta, R\eta) - (\tilde{R}(P\eta), \eta) = \sum_{i=1}^n \partial_i \Phi_i(P\eta, \eta) = 0,$$

i.e., $\Phi = (\Phi_1(P\eta, \eta), \dots, \Phi_n(P\eta, \eta))^T$ satisfies **div $\Phi = 0$** .

Example: Hydrodynamics

- The movement of an **incompressible rotating fluid with a rotation axis lies along the x_3 axis and a small velocity** is defined by:

$$\begin{cases} \rho_0 \partial_t u_1 - 2 \rho_0 \Omega_0 u_2 + \partial_1 p = 0, \\ \rho_0 \partial_t u_2 + 2 \rho_0 \Omega_0 u_1 + \partial_2 p = 0, \\ \rho_0 \partial_t u_3 + \partial_3 p = 0, \\ \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0, \end{cases}$$

$u = (u_1 \ u_2 \ u_3)^T$: local rate of velocity, p : pressure, ρ_0 : constant fluid density, Ω_0 : constant angle speed.

- We have: $R = \begin{pmatrix} \rho_0 \partial_t & -2 \rho_0 \Omega_0 & 0 & \partial_1 \\ 2 \rho_0 \Omega_0 & \rho_0 \partial_t & 0 & \partial_2 \\ 0 & 0 & \rho_0 \partial_t & \partial_3 \\ \partial_1 & \partial_2 & \partial_3 & 0 \end{pmatrix} = -\tilde{R}.$

Example: Hydrodynamics

- $\tilde{R} = -R$ implies that if (\vec{u}, p) is a **solution of the system, so is:**

$$\lambda_1 = u_1, \quad \lambda_2 = u_2, \quad \lambda_3 = u_3, \quad \lambda_4 = p.$$

- Denote by $\xi = (u_1 \quad u_2 \quad u_2 \quad p)^T$. We have the **identity:**

$$(\lambda, R \xi) = (\xi, \tilde{R} \lambda) + (\partial_t \quad \partial_1 \quad \partial_2 \quad \partial_3) \begin{pmatrix} \rho_0 (\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3) \\ \lambda_1 p + \lambda_4 u_1 \\ \lambda_2 p + \lambda_4 u_2 \\ \lambda_3 p + \lambda_4 u_3 \end{pmatrix}$$

- If we take $\lambda = \xi$, then we get $\tilde{R} \lambda = 0$ and

$$\partial_t (\rho_0 (u_1^2 + u_2^2 + u_3^2)) + \partial_1 (2 p u_1) + \partial_2 (2 p u_2) + \partial_3 (2 p u_3) = 0,$$

i.e., we obtain the **quadratic conservation of law:**

$$\partial_t \left(\frac{1}{2} \rho_0 \|\vec{u}\|^2 \right) + \operatorname{div} (p \vec{u}) = 0.$$

Example: Electromagnetism

- Let us consider the **Maxwell equations in the vacuum**:

$$\begin{cases} \partial_t \vec{B} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \frac{1}{\mu_0} \vec{\nabla} \wedge \vec{B} - \epsilon_0 \partial_t \vec{E} = \vec{0}, \end{cases}$$

where \vec{B} (resp., \vec{E}): **magnetic** (resp., **electric**) **field**, μ_0 (resp., ϵ_0): **magnetic** (resp., **electric**) **constant**.

- Let us consider $D = \mathbb{Q}(\mu_0, \epsilon_0)[\partial_t, \partial_1, \partial_2, \partial_3]$ and the **matrix**:

$$R = \begin{pmatrix} \partial_t & 0 & 0 & 0 & -\partial_3 & \partial_2 \\ 0 & \partial_t & 0 & \partial_3 & 0 & -\partial_1 \\ 0 & 0 & \partial_t & -\partial_2 & \partial_1 & 0 \\ 0 & -\partial_3/\mu_0 & \partial_2/\mu_0 & -\epsilon_0 \partial_t & 0 & 0 \\ \partial_3/\mu_0 & 0 & -\partial_1/\mu_0 & 0 & -\epsilon_0 \partial_t & 0 \\ -\partial_2/\mu_0 & \partial_1/\mu_0 & 0 & 0 & 0 & -\epsilon_0 \partial_t \end{pmatrix}.$$

Example: Electromagnetism

$$\tilde{R} = \begin{pmatrix} -\partial_t & 0 & 0 & 0 & -\partial_3/\mu_0 & \partial_2/\mu_0 \\ 0 & -\partial_t & 0 & \partial_3/\mu_0 & 0 & -\partial_1/\mu_0 \\ 0 & 0 & -\partial_t & -\partial_2/\mu_0 & \partial_1/\mu_0 & 0 \\ 0 & -\partial_3 & \partial_2 & \epsilon_0 \partial_t & 0 & 0 \\ \partial_3 & 0 & -\partial_1 & 0 & \epsilon_0 \partial_t & 0 \\ -\partial_2 & \partial_1 & 0 & 0 & 0 & \epsilon_0 \partial_t \end{pmatrix}.$$

- $\xi = (B_1 \ B_2 \ B_3 \ E_1 \ E_2 \ E_3)^T$, $\lambda = (C_1 \ C_2 \ C_3 \ F_1 \ F_2 \ F_3)^T$.
- We have the **differential relation**:

$$(\lambda, R \xi) = (\xi, \tilde{R} \lambda) + \partial_t \left(\sum_{i=1}^3 C_i B_i - \epsilon_0 \sum_{i=1}^3 F_i E_i \right) + \vec{\nabla} \cdot \begin{pmatrix} C_3 E_2 - C_2 E_3 + (F_3 B_2 - F_2 B_3)/\mu_0 \\ C_1 E_3 - C_3 E_1 + (F_1 B_3 - F_3 B_1)/\mu_0 \\ C_2 E_1 - C_1 E_2 + (F_2 B_1 - F_1 B_2)/\mu_0 \end{pmatrix}.$$

Example: Electromagnetism

- Let us consider $M = D^{1 \times 6} / (D^{1 \times 6} R)$ and $\tilde{N} = D^{1 \times 6} / (D^{1 \times 6} \tilde{R})$.
- A homomorphism $f \in \text{hom}_D(\tilde{N}, M)$ is defined by:

$$P = \begin{pmatrix} 1/\mu_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\mu_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\mu_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad Q = -P.$$

- If ξ is a solution of the system, then $\lambda = P \xi$, i.e.,

$$C_i = B_i/\mu_0, \quad F_i = -E_i, \quad i = 1, 2, 3,$$

is a solution of $\tilde{R} \lambda = 0$. Then, we obtain the **conservation law**:

$$\underbrace{\partial_t \left(\frac{1}{\mu_0} \|\vec{B}\|^2 + \epsilon_0 \|\vec{E}\|^2 \right)}_{\text{electromagnetic energy}} + \text{div} \underbrace{\left(\frac{1}{\mu_0} (\vec{E} \wedge \vec{B}) \right)}_{\text{Poynting vector}} = 0.$$

Kernel and factorization

$$\begin{array}{ccccccc}
 & & \lambda & \longmapsto & y & & \\
 D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 \downarrow \cdot Q & & \downarrow \cdot P & & \downarrow f & & \\
 D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0 \\
 \exists \mu & \longmapsto & \mu R' = \lambda P & \longmapsto & 0 & &
 \end{array}$$

- $\ker_D \left(\cdot \begin{pmatrix} P \\ R' \end{pmatrix} \right) = D^{1 \times r} (S \quad - T)$

$$\Rightarrow \{ \lambda \in D^{1 \times p} \mid \lambda P \in D^{1 \times q} R' \} = D^{1 \times r} S$$

$$\Rightarrow \ker f = (D^{1 \times r} S) / (D^{1 \times q} R).$$

- $(D^{1 \times q} (R \quad - Q)) \in \ker_D \left(\cdot \begin{pmatrix} P \\ R' \end{pmatrix} \right) \Rightarrow (D^{1 \times q} R) \subseteq (D^{1 \times r} S).$

$$\exists V \in D^{q \times r} : R = VS.$$

Example: Linearized Euler equations

- Let $R = \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 & 0 \\ \partial_t & 0 & 0 & \partial_1 \\ 0 & \partial_t & 0 & \partial_2 \\ 0 & 0 & \partial_t & \partial_3 \end{pmatrix}$ over $D = \mathbb{Q}[\partial_t, \partial_1, \partial_2, \partial_3]$.
- Let us consider $f \in \text{end}_D(M)$ defined by:

$$P = \begin{pmatrix} 0 & \partial_3 & -\partial_2 & 0 \\ -\partial_3 & 0 & \partial_1 & 0 \\ \partial_2 & -\partial_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_3 & -\partial_2 \\ 0 & -\partial_3 & 0 & \partial_1 \\ 0 & \partial_2 & -\partial_1 & 0 \end{pmatrix}.$$

- Computing $\ker_D \left(\begin{pmatrix} P \\ -R \end{pmatrix} \right)$ and factorizing R by S , we get:

$$V = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & \partial_1 \\ 0 & 0 & 1 & 0 & \partial_2 \\ 0 & 0 & 0 & -1 & \partial_3 \end{pmatrix}, \quad S = \begin{pmatrix} -\partial_t & 0 & 0 & 0 \\ \partial_1 & \partial_2 & \partial_3 & 0 \\ 0 & \partial_t & 0 & 0 \\ 0 & 0 & -\partial_t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Example: Linearized Euler equations

- We have $R = VS$ where:

$$\begin{pmatrix} \partial_1 & \partial_2 & \partial_3 & 0 \\ \partial_t & 0 & 0 & \partial_1 \\ 0 & \partial_t & 0 & \partial_2 \\ 0 & 0 & \partial_t & \partial_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & \partial_1 \\ 0 & 0 & 1 & 0 & \partial_2 \\ 0 & 0 & 0 & -1 & \partial_3 \end{pmatrix} \begin{pmatrix} -\partial_t & 0 & 0 & 0 \\ \partial_1 & \partial_2 & \partial_3 & 0 \\ 0 & \partial_t & 0 & 0 \\ 0 & 0 & -\partial_t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- The solutions of $Sy = 0$ are particular solutions of $Ry = 0$.
- Integrating S , we obtain the following solutions of $Ry = 0$:

$$\begin{cases} \vec{v}(x, t) = \operatorname{curl} \vec{\psi}(x), \\ p(x, t) = 0, \end{cases} \quad \forall \vec{\psi} = (\psi_1, \psi_2, \psi_3)^T \in C^\infty(\Omega)^3.$$

Ker f , im f , coim f and coker f

- **Proposition:** Let $M = D^{1 \times p} / (D^{1 \times q} R)$, $M' = D^{1 \times p'} / (D^{1 \times q'} R')$ and $f : M \rightarrow M'$ be a **homomorphism** defined by $R P = Q R'$.

Let us consider the matrices $S \in D^{r \times p}$, $T \in D^{r \times q'}$, $U \in D^{s \times r}$ and $V \in D^{q \times r}$ satisfying $R = V S$, $\ker_D(\cdot S) = D^{1 \times s} U$ and:

$$\ker_D \left(\cdot \begin{pmatrix} P \\ R' \end{pmatrix} \right) = D^{1 \times r} (S \quad -T).$$

Then, we have:

- $\ker f = (D^{1 \times r} S) / (D^{1 \times q} R) \cong D^{1 \times r} / \left(D^{1 \times (q+s)} \begin{pmatrix} U \\ V \end{pmatrix} \right)$,
- $\text{coim } f = M / \ker f = D^{1 \times p} / (D^{1 \times r} S)$,
- $\text{im } f = D^{1 \times (p+q')} \begin{pmatrix} P \\ R' \end{pmatrix} / (D^{1 \times q} R) \cong D^{1 \times p} / (D^{1 \times r} S)$,
- $\text{coker } f = M' / \text{im } f = D^{1 \times p'} / \left(D^{1 \times (p+q')} \begin{pmatrix} P \\ R' \end{pmatrix} \right)$.

Equivalence of linear systems

- **Corollary:** Let us consider $f \in \text{hom}_D(M, M')$. Then:
 - 1 f is **injective** iff one of the assertions holds:
 - There exists $L \in D^{r \times q}$ such that $S = LR$.
 - $\begin{pmatrix} U \\ V \end{pmatrix}$ admits a **left-inverse** over D .
 - 2 f is **surjective** iff $\begin{pmatrix} P \\ R' \end{pmatrix}$ admits a **left-inverse** over D .
 - 3 f is an **isomorphism**, i.e., $M \cong M'$, iff 1 and 2 are satisfied.

Example

- Equivalence of the systems defined by the following R and R' ?

$$R = \begin{pmatrix} \partial_1^2 \partial_2^2 - 1 & -\partial_1 \partial_2^3 - \partial_2^2 \\ \partial_1^3 \partial_2 - \partial_1^2 & -\partial_1^2 \partial_2^2 \end{pmatrix}, \quad R' = (\partial_1 \partial_2 - 1 \quad -\partial_2^2).$$

- We find a **homomorphism** $f \in \text{hom}_D(M, M')$ defined by:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 + \partial_1 \partial_2 \\ \partial_1^2 \end{pmatrix}.$$

- $\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 + \partial_1 \partial_2 \\ \partial_1^2 \end{pmatrix}$ admits the **left-inverse** $(1 - \partial_1 \partial_2 \quad \partial_2^2)$.

- $\begin{pmatrix} P \\ R' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_1 \partial_2 - 1 & -\partial_2^2 \end{pmatrix}$ admits the **left-inverse** $(I_2 \quad 0)$.

$$\Rightarrow M = D^{1 \times 2} / (D^{1 \times 2} R) \cong M' = D^{1 \times 2} / (D R').$$

Block triangular decomposition

- **Theorem:** Let $R \in D^{q \times p}$, $M = D^{1 \times p} / (D^{1 \times q} R)$ and $f \in \text{end}_D(M)$ defined by P and Q satisfying $R P = Q R$.

If the left D -modules

$$\ker_D(.P), \quad \text{coim}_D(.P) = D^{1 \times p} / \ker_D(.P),$$

$$\ker_D(.Q), \quad \text{coim}_D(.Q) = D^{1 \times q} / \ker_D(.Q),$$

are free of rank m , $p - m$, l , $q - l$, then there exist two matrices

$$U = (U_1^T \quad U_2^T)^T \in \text{GL}_p(D), \quad V = (V_1^T \quad V_2^T)^T \in \text{GL}_q(D),$$

such that

$$\bar{R} = V R U^{-1} = \begin{pmatrix} V_1 R W_1 & 0 \\ V_2 R W_1 & V_2 R W_2 \end{pmatrix} \in D^{q \times p},$$

where $U^{-1} = (W_1 \quad W_2)$, $W_1 \in D^{p \times m}$, $W_2 \in D^{p \times (p-m)}$ and:

$$U_1 \in D^{m \times p}, \quad U_2 \in D^{(p-m) \times p}, \quad V_1 \in D^{l \times q}, \quad V_2 \in D^{(q-l) \times q}.$$

Exemple: Electromagnetism

$$\sigma \partial_t \vec{A} + \frac{1}{\mu} \vec{\nabla} \wedge \vec{\nabla} \vec{A} - \sigma \vec{\nabla} V = 0$$

$$\Rightarrow R = \begin{pmatrix} \sigma \partial_t - \frac{1}{\mu} (\partial_2^2 + \partial_3^2) & \frac{1}{\mu} \partial_1 \partial_2 & \frac{1}{\mu} \partial_1 \partial_3 & -\sigma \partial_1 \\ \frac{1}{\mu} \partial_1 \partial_2 & \sigma \partial_t - \frac{1}{\mu} (\partial_1^2 + \partial_3^2) & \frac{1}{\mu} \partial_2 \partial_3 & -\sigma \partial_2 \\ \frac{1}{\mu} \partial_1 \partial_3 & \frac{1}{\mu} \partial_2 \partial_3 & \sigma \partial_t - \frac{1}{\mu} (\partial_1^2 + \partial_2^2) & -\sigma \partial_3 \end{pmatrix}.$$

- Let $D = \mathbb{Q}[\partial_t, \partial_1, \partial_2, \partial_3]$ and $M = D^{1 \times 4} / (D^{1 \times 3} R)$.

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma \mu \partial_t & 0 & -\sigma \mu \partial_2 \\ 0 & 0 & \sigma \mu \partial_t & -\sigma \mu \partial_3 \\ 0 & \partial_t \partial_2 & \partial_t \partial_3 & -(\partial_2^2 + \partial_3^2) \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ -\partial_1 \partial_2 & \sigma \mu \partial_t - \partial_2^2 & -\partial_2 \partial_3 \\ -\partial_1 \partial_3 & -\partial_2 \partial_3 & \sigma \mu \partial_t - \partial_3^2 \end{pmatrix},$$

satisfy $RP = QR$ and define a **endomorphism** $f \in \text{end}_D(M)$.

Exemple: Electromagnetism

- The modules $\ker_D(.P)$, $\text{coim}_D(.P)$, $\ker_D(.Q)$, $\text{coim}_D(.Q)$ are **free D -modules** (Quillen-Suslin theorem) and:

$$\left\{ \begin{array}{l} \ker_D(.P) = D^{1 \times 2} U_1, \quad U_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \partial_2 & \partial_3 & -\sigma \mu \end{pmatrix}, \\ \text{coim}_D(.P) = D^{1 \times 2} U_2, \quad U_2 = \frac{1}{\sigma \mu} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ \ker_D(.Q) = D^{1 \times 2} V_1, \quad V_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \\ \text{coim}_D(.Q) = D^{1 \times 2} V_2, \quad V_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{array} \right.$$

- The matrix R is then **equivalent** to $\bar{R} = V R U^{-1}$ defined by:

$$\bar{R} = \begin{pmatrix} \sigma \partial_t - \frac{1}{\mu} (\partial_2^2 + \partial_3^2) & \frac{1}{\mu} \partial_1 & 0 & 0 \\ \frac{1}{\mu} \partial_1 \partial_2 & \frac{1}{\mu} \partial_2 & \sigma (\sigma \mu \partial_t - (\partial_1^2 + \partial_2^2 + \partial_3^2)) & 0 \\ \frac{1}{\mu} \partial_1 \partial_3 & \frac{1}{\mu} \partial_3 & 0 & \sigma (\sigma \mu \partial_t - (\partial_1^2 + \partial_2^2 + \partial_3^2)) \end{pmatrix}.$$

Idempotents of $\text{end}_D(M)$

- **Lemma:** An endomorphism f of $M = D^{1 \times p} / (D^{1 \times q} R)$, defined by the matrices P and Q , is a **idempotent**, i.e., $f^2 = f$, iff there exist $Z \in D^{p \times q}$ and $Z' \in D^{q \times t}$ such that

$$\begin{cases} P^2 = P + Z R, \\ Q^2 = Q + R Z + Z' R_2, \end{cases}$$

where $R_2 \in D^{t \times q}$ satisfies $\ker_D(\cdot R) = D^{1 \times t} R_2$.

- **Example:** $D = A_1(\mathbb{Q})$, $R = (\partial^2 \quad -t\partial - 1)$, $M = D^{1 \times 2} / (D R)$.

$$P = \begin{pmatrix} -(t+a)\partial + 1 & t^2 + at \\ 0 & 1 \end{pmatrix}, \quad P^2 = P + \begin{pmatrix} (t+a)^2 \\ 0 \end{pmatrix} R.$$

Idempotents of $\text{end}_D(M)$

- **Proposition:** f is a **idempotents of $\text{end}_D(M)$** , i.e., $f^2 = f$, iff there exists a matrix $X \in D^{p \times s}$ such that $P = I_p - X S$ and we have the following **commutative exact diagram**:

$$\begin{array}{ccccccc}
 & & & & & 0 & \\
 & & & & & \downarrow & \\
 & & & & & \ker f & \\
 & & & & & \downarrow i & \\
 & & D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \cdot T \uparrow \downarrow \cdot V & & \cdot P \uparrow \downarrow \cdot I_p & & f \uparrow \downarrow \kappa & & \\
 D^{1 \times s} & \xrightarrow{\cdot U} & D^{1 \times r} & \xrightarrow{\cdot S} & D^{1 \times p} & \xrightarrow{\pi'} & M / \ker f & \longrightarrow & 0. \\
 & & & \xleftarrow{\cdot X} & & & \downarrow & & \\
 & & & & & & 0 & &
 \end{array}$$

$$\Rightarrow M \cong \ker f \oplus \text{im } f \quad \& \quad S - S X S = T R. \quad (\star)$$

- **Corollary:** If $\ker_D(\cdot S) = 0$, then $R = V S$ satisfies:

$$S X - T V = I_r.$$

Decomposition of solutions

- **Corollary:** Let us suppose that \mathcal{F} is an **injective left D -module**. Then, we have the following **commutative exact diagram**:

$$\begin{array}{ccccccc}
 & & Vz = 0 = Ry & \longleftarrow & y & & \\
 & & \mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(R.) \longleftarrow 0 \\
 & & \uparrow v. & & \parallel & & \uparrow f^* \\
 \mathcal{F}^s & \xleftarrow{U.} & \mathcal{F}^r & \xleftarrow{S.} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(S.) \longleftarrow 0. \\
 & & & \xrightarrow{X.} & & & \\
 0 = Uz & \longleftarrow & z = Sy & \longleftarrow & y & &
 \end{array}$$

Moreover, we have: $Ry = 0 \Leftrightarrow \begin{pmatrix} U \\ V \end{pmatrix} z = 0, \quad Sy = z.$

General solution: $y = u + Xz$ where $Su = 0$ and $\begin{pmatrix} U \\ V \end{pmatrix} z = 0.$

Example: OD system

- Let $D = \mathbb{Q}[t][\partial; \text{id}, \partial]$ and $M = D^{1 \times 4} / (D^{1 \times 4} R)$, where:

$$R = \begin{pmatrix} \partial & -t & t & \partial \\ \partial & t\partial - t & \partial & -1 \\ \partial & -t & \partial + t & \partial - 1 \\ \partial & \partial - t & t & \partial \end{pmatrix} \in D^{4 \times 4}.$$

- An **idempotent** of $\text{end}_D(M)$ is defined by:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{Q}^{4 \times 4} : \quad P^2 = P.$$

- We obtain the **factorization** $R = VS$, where:

$$S = \begin{pmatrix} \partial & -t & 0 & 0 \\ 0 & \partial & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & t & \partial \\ 1 & t & \partial & -1 \\ 1 & 0 & \partial + t & \partial - 1 \\ 1 & 1 & t & \partial \end{pmatrix}.$$

Example

- Using the identity $I_p - P = X S$, we obtain:

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$R y = 0 \Leftrightarrow y = u + X z : V z = 0, S u = 0.$$

- The **solution of $S u = 0$** is defined by:

$$u_1 = \frac{1}{2} C_1 t^2 + C_2, \quad u_2 = C_1, \quad u_3 = 0, \quad u_4 = 0.$$

- The **solution of $V z = 0$** is defined by: $z_1 = 0, z_2 = 0$ and
 $z_3(t) = C_3 \text{Ai}(t) + C_4 \text{Bi}(t), z_4(t) = C_3 \partial \text{Ai}(t) + C_4 \partial \text{Bi}(t).$
- The **general solution** of $R y = 0$ is then defined by:

$$y = u + X z = \left(\frac{1}{2} C_1 t^2 + C_2 \quad C_1 \quad z_3(t) \quad z_4(t) \right)^T.$$

Idempotents of $\text{end}_D(M)$ and $D^{p \times p}$

- **Lemma:** Let us suppose that $\ker_D(.R) = 0$ and $P^2 = P + ZR$. If there exists a solution $\Lambda \in D^{p \times q}$ of the **algebraic Riccati equation**

$$\Lambda R \Lambda + (P - I_p) \Lambda + \Lambda Q + Z = 0, \quad (\star)$$

then the matrices $\bar{P} = P + \Lambda R$ and $\bar{Q} = Q + R \Lambda$ satisfy:

$$R \bar{P} = \bar{Q} R, \quad \bar{P}^2 = \bar{P}, \quad \bar{Q}^2 = \bar{Q}.$$

- **Example:** $\Lambda = (a t \quad a \partial - 1)^T$ is a solution of (\star)

$$\Rightarrow \bar{P} = \begin{pmatrix} a t \partial^2 - (t + a) \partial + 1 & t^2 (1 - a \partial) \\ (a \partial - 1) \partial^2 & -a t \partial^2 + (t - 2a) \partial + 2 \end{pmatrix}, \bar{Q} = 0,$$

then satisfy $\bar{P}^2 = \bar{P}$ and $\bar{Q}^2 = \bar{Q}$.

Block diagonal decomposition

- **Theorem:** Let $R \in D^{q \times p}$, $M = D^{1 \times p} / (D^{1 \times q} R)$ and $f \in \text{end}_D(M)$ defined by P and Q satisfying:

$$P^2 = P, \quad Q^2 = Q \quad (\text{idempotents}) \quad \Rightarrow \quad f^2 = f.$$

If the **left** D -modules

$$\ker_D(.P), \quad \text{im}_D(.P) = \ker_D(. (I_p - P)),$$

$$\ker_D(.Q), \quad \text{im}_D(.Q) = \ker_D(. (I_q - Q)),$$

are **free** of rank m , $p - m$, l , $q - l$, then there exist two matrices

$$U = (U_1^T \quad U_2^T)^T \in \text{GL}_p(D), \quad V = (V_1^T \quad V_2^T)^T \in \text{GL}_q(D),$$

such that

$$\bar{R} = V R U^{-1} = \begin{pmatrix} V_1 R W_1 & 0 \\ 0 & V_2 R W_2 \end{pmatrix} \in D^{q \times p},$$

where $U^{-1} = (W_1 \quad W_2)$, $W_1 \in D^{p \times m}$, $W_2 \in D^{p \times (p-m)}$ and:

$$U_1 \in D^{m \times p}, \quad U_2 \in D^{(p-m) \times p}, \quad V_1 \in D^{l \times q}, \quad V_2 \in D^{(q-l) \times q}.$$

Example: OD system

- Let us consider the matrix again:

$$R = \begin{pmatrix} \partial & -t & t & \partial \\ \partial & t\partial - t & \partial & -1 \\ \partial & -t & \partial + t & \partial - 1 \\ \partial & \partial - t & t & \partial \end{pmatrix}.$$

- An idempotent $f \in \text{end}_D(M)$ is defined by the matrices

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} t+1 & 1 & -1 & -t \\ 1 & 1 & -1 & 0 \\ t+1 & 1 & -1 & -t \\ t & 1 & -1 & -t+1 \end{pmatrix}.$$

where P and Q satisfy:

$$RP = QR, \quad P^2 = P, \quad Q^2 = Q.$$

Example: OD system

- **Computing bases** of the left D -modules

$$\ker_D(.P), \quad \text{im}_D(.P), \quad \ker_D(.Q), \quad \text{im}_D(.Q),$$

we obtain the **unimodular matrices**:

$$U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} -1 & 0 & 1 & 0 \\ -t & -1 & 1 & t \\ t+1 & 1 & -1 & -t \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

- R is then equivalent to the following **block diagonal matrix**:

$$\bar{R} = V R U^{-1} = \begin{pmatrix} \partial & -1 & 0 & 0 \\ t & \partial & 0 & 0 \\ 0 & 0 & \partial & -t \\ 0 & 0 & 0 & \partial \end{pmatrix}.$$

Example: Cauchy-Riemann equations

- Let us consider the **Cauchy-Riemann equations**:

$$\begin{cases} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0. \end{cases}$$

- $D = \mathbb{Q}(i)[\partial_x, \partial_y]$, $R = \begin{pmatrix} \partial_x & -\partial_y \\ \partial_y & \partial_x \end{pmatrix}$, $M = D^{1 \times 2} / (D^{1 \times 2} R)$.

- The matrices P and Q defined by $P = Q = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

satisfy $RP = PR$ and $P^2 = P$, i.e., define an **idempotent**.

$$\begin{cases} \ker_{\mathbb{Q}(i)}(.P) = \mathbb{Q}(i) \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}, \\ \text{im}_{\mathbb{Q}(i)}(.P) = \mathbb{Q}(i) \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \end{cases} \Rightarrow U = V = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \in \text{GL}_2(D).$$

$$\Rightarrow \bar{R} = URU^{-1} = \begin{pmatrix} \partial_x - i\partial_y & 0 \\ 0 & \partial_x + i\partial_y \end{pmatrix} = 2 \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix}.$$

Example: Wave equation

- Let us consider the following **system of PDEs**:

$$\begin{cases} \frac{\partial y_1}{\partial x} + a \frac{\partial y_2}{\partial t} = 0, \\ \frac{\partial y_1}{\partial t} + b \frac{\partial y_2}{\partial x} = 0. \end{cases}$$

- Acoustic wave**: $y_1 = u$, $y_2 = p$, $a = 1/\rho$, $b = \rho c^2$.
- LC transmission line**: $y_1 = v$, $y_2 = i$, $a = L$, $b = 1/C$.
- $D = \mathbb{Q}(a, b)[\partial_x, \partial_t]$, $R = \begin{pmatrix} \partial_x & a \partial_t \\ \partial_t & b \partial_x \end{pmatrix}$, $M = D^{1 \times 2} / (D^{1 \times 2} R)$.
- An **idempotent** $f \in \text{end}_D(M)$ is defined by the **idempotents**

$$P = \frac{1}{2} \begin{pmatrix} 1 & 2ab\alpha \\ 2\alpha & 1 \end{pmatrix}, \quad Q = \frac{1}{2} \begin{pmatrix} 1 & 2a\alpha \\ 2b\alpha & 1 \end{pmatrix},$$

where α satisfies $4ab\alpha^2 - 1 = 0$.

Example: Wave equation

- Let us denote by $D' = \mathbb{Q}(a, b, \alpha)/(4ab\alpha^2 - 1)[\partial_x, \partial_t]$.
- $\ker_{D'}(.P)$, $\text{im}_{D'}(.P)$, $\ker_{D'}(.Q)$ and $\text{im}_{D'}(.Q)$ are **free with bases**:

$$\begin{cases} \ker_{D'}(.P) = D' U_1, & U_1 = (-2\alpha \ 1), \\ \text{im}_{D'}(.P) = D' U_2, & U_2 = (2\alpha \ 1). \end{cases}$$

$$\begin{cases} \ker_{D'}(.Q) = D' V_1, & V_1 = (2b\alpha \ -1), \\ \text{im}_{D'}(.Q) = D' V_2, & V_2 = (2b\alpha \ 1). \end{cases}$$

- $U = (U_1^T \ U_2^T)^T \in \text{GL}_2(D')$, $V = (V_1^T \ V_2^T)^T \in \text{GL}_2(D')$.
- The matrix R is **equivalent** to $(1/(2\alpha) = \sqrt{ab})$:

$$\bar{R} = V R U^{-1} = \begin{pmatrix} -b\partial_x + \frac{1}{2\alpha}\partial_t & 0 \\ 0 & b\partial_x + \frac{1}{2\alpha}\partial_t \end{pmatrix}.$$

Example: Dirac equation

- Let us consider the following **complex matrices**:

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

- The **Dirac equation** has the form $\sum_{i=1}^4 \gamma^i \partial y / \partial x_i = 0$:

$$\begin{cases} \partial_4 y_1 - i \partial_3 y_3 - (i \partial_1 + \partial_2) y_4 = 0, \\ \partial_4 y_2 - (i \partial_1 - \partial_2) y_3 + i \partial_3 y_4 = 0, \\ i \partial_3 y_1 + (i \partial_1 + \partial_2) y_2 - \partial_4 y_3 = 0, \\ (i \partial_1 - \partial_2) y_1 - i \partial_3 y_2 - \partial_4 y_4 = 0, \end{cases}$$

Example: Dirac equation

- Let us consider $D = \mathbb{Q}(i)[\partial_1, \partial_2, \partial_3, \partial_4]$, the matrix

$$R = \begin{pmatrix} \partial_4 & 0 & -i\partial_3 & -(i\partial_1 + \partial_2) \\ 0 & \partial_4 & -i\partial_1 + \partial_2 & i\partial_3 \\ i\partial_3 & i\partial_1 + \partial_2 & -\partial_4 & 0 \\ i\partial_1 - \partial_2 & -i\partial_3 & 0 & -\partial_4 \end{pmatrix} \in D^{4 \times 4},$$

and the finitely presented D -module $M = D^{1 \times 4} / (D^{1 \times 4} R)$.

- Computing idempotents of $\text{end}_D(M)$, we obtain a **idempotent** f defined by the pair of matrices:

$$P = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad Q = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

- We have $P^2 = P$ and $Q^2 = Q$, i.e., the D -modules $\ker_D(.P)$, $\text{im}(.P)$, $\ker_D(.Q)$ and $\text{im}(.Q)$ are **free**.

Example: Dirac equation

- Computing **bases** for these modules, we then get:

$$\left\{ \begin{array}{l} \ker_D(.P) = D^{1 \times 2} U_1, \quad U_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}, \\ \operatorname{im}(.P) = D^{1 \times 2} U_2, \quad U_2 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \\ \ker_D(.Q) = D^{1 \times 2} V_1, \quad V_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \\ \operatorname{im}(.Q) = D^{1 \times 2} V_2, \quad V_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}. \end{array} \right.$$

- Let us form the **unimodular matrices**:

$$U = (U_1^T \quad U_2^T)^T \in \operatorname{GL}_4(D), \quad V = (V_1^T \quad V_2^T)^T \in \operatorname{GL}_4(D).$$

Example: Dirac equation

- The matrix R is then **equivalent** to the **block-diagonal** one:

$$V R U^{-1} = \begin{pmatrix} i \partial_3 - \partial_4 & -i \partial_1 - \partial_2 & 0 & 0 \\ i \partial_1 - \partial_2 & i \partial_3 + \partial_4 & 0 & 0 \\ 0 & 0 & i \partial_3 + \partial_4 & i \partial_1 + \partial_2 \\ 0 & 0 & i \partial_1 - \partial_2 & -i \partial_3 + \partial_4 \end{pmatrix}.$$

- If we denote by $z = U y$, we obtain that the Dirac equation is then equivalent to the **decoupled system** of PDEs:

$$\begin{cases} (i \partial_3 - \partial_4) z_1 - (i \partial_1 + \partial_2) z_2 = 0, \\ (i \partial_1 - \partial_2) z_1 + (i \partial_3 + \partial_4) z_2 = 0, \\ (i \partial_3 + \partial_4) z_3 + (i \partial_1 + \partial_2) z_4 = 0, \\ (i \partial_1 - \partial_2) z_3 + (-i \partial_3 + \partial_4) z_4 = 0. \end{cases}$$

Example: 2-D rotational isentropic flow

- We consider the **linearized approximation of the steady two-dimensional rotational isentropic flow** (Courant-Hilbert)

$$\begin{cases} u\rho\frac{\partial\omega}{\partial x} + c^2\frac{\partial\sigma}{\partial x} = 0, \\ u\rho\frac{\partial\lambda}{\partial x} + c^2\frac{\partial\sigma}{\partial y} = 0, \\ \rho\frac{\partial\omega}{\partial x} + \rho\frac{\partial\lambda}{\partial y} + u\frac{\partial\sigma}{\partial x} = 0, \end{cases}$$

where u is a **constant velocity** parallel to the x -axis, ρ a **constant density** and c the **sound speed**.

- Let us consider $D = \mathbb{Q}(u, \rho, c)[\partial_x, \partial_y]$, the system matrix

$$R = \begin{pmatrix} u\rho\partial_x & c^2\partial_x & 0 \\ 0 & c^2\partial_y & u\rho\partial_x \\ \rho\partial_x & u\partial_x & \rho\partial_y \end{pmatrix} \in D^{3 \times 3},$$

and the D -module $M = D^{1 \times 3} / (D^{1 \times 3} R)$.

Example: 2-D rotational isentropic flow

- If α satisfies $1 + 4(c^2 - u^2)\alpha^2 = 0$ and we denote by

$$D' = (\mathbb{Q}(u, \rho, c, \alpha) / (1 + 4(c^2 - u^2)\alpha^2))[\partial_x, \partial_y],$$

$$U = \begin{pmatrix} 0 & 2\alpha c(c^2 - u^2) & u\rho \\ 0 & 2\alpha c(c^2 - u^2) & -u\rho \\ u\rho & c^2 & 0 \end{pmatrix} \in \text{GL}_3(D'),$$

$$V = \begin{pmatrix} 2\alpha c & 1 & -2\alpha c u \\ 2\alpha c & -1 & -2\alpha c u \\ 1 & 0 & 0 \end{pmatrix} \in \text{GL}_3(D'),$$

$$\Rightarrow \bar{R} = V R U^{-1} = \begin{pmatrix} \partial_x - 2\alpha c \partial_y & 0 & 0 \\ 0 & \partial_x + 2\alpha c \partial_y & 0 \\ 0 & 0 & \partial_x \end{pmatrix}.$$

- We have $M \cong M_1 \oplus M_2 \oplus M_3$, where $M_3 = D' / (D' \partial_x)$ and:

$$M_1 = D' / (D' (\partial_x - 2\alpha c \partial_y)), \quad M_2 = D' / (D' (\partial_x + 2\alpha c \partial_y)).$$

Example: Tank model I

- We consider $D = \mathbb{Q}[\partial, \delta]$ and the **system matrix**

$$R = \begin{pmatrix} \delta^2 & 1 & -2\partial\delta \\ 1 & \delta^2 & -2\partial\delta \end{pmatrix}$$

considered in Dubois, Petit, Rouchon, ECC99.

- An **idempotent** $f \in \text{end}_D(M)$ is defined by the matrices

$$P = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad Q = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

i.e., P and Q satisfy:

$$RP = QR, \quad P^2 = P, \quad Q^2 = Q.$$

Example: Tank model I

$$\begin{cases} U_1 = \ker_D(.P) = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}, \\ U_2 = \operatorname{im}_D(.P) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ V_1 = \ker_D(.Q) = \begin{pmatrix} 1 & -1 \end{pmatrix}, \\ V_2 = \operatorname{im}_D(.Q) = \begin{pmatrix} 1 & 1 \end{pmatrix}, \end{cases}$$

and we obtain the following two **unimodular matrices**:

$$U = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

- We easily check that we have the following **block diagonal matrix**:

$$\bar{R} = V R U^{-1} = \begin{pmatrix} \delta^2 - 1 & 0 & 0 \\ 0 & 1 + \delta^2 & -4\partial\delta \end{pmatrix}.$$

Example: Tank model I

$$\bar{R} = \begin{pmatrix} \delta^2 - 1 & 0 & 0 \\ 0 & 1 + \delta^2 & -4\partial\delta \end{pmatrix}.$$

- If $\mathcal{F} = C^\infty(\mathbb{R})$ and ψ is any **smooth 2h-periodic function**, then

$$\forall \xi \in \mathcal{F}, \quad \begin{cases} z_1(t) = \psi(t), \\ z_2(t) = 4\partial\delta\xi(t) = 4\dot{\xi}(t-h), \\ v(t) = (\delta^2 + 1)\xi(t) = \xi(t-2h) + \xi(t), \end{cases}$$

is a **solution of $\bar{R}z = 0$** . Hence, the solution of $Ry = 0$ are:

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ u(t) \end{pmatrix} = U^{-1} \begin{pmatrix} z_1(t) \\ z_2(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\psi(t) + 2\dot{\xi}(t-h) \\ -\frac{1}{2}\psi(t) + 2\dot{\xi}(t-h) \\ \xi(t-2h) + \xi(t) \end{pmatrix}$$

Example: Tank model II

- **Model of a one-dimensional tank** containing a fluid subjected to an horizontal move (Petit, Rouchon, IEEE TAC, 2002):

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t - 2h) + \alpha \ddot{y}_3(t - h) = 0, \\ \dot{y}_1(t - 2h) - \dot{y}_2(t) + \alpha \ddot{y}_3(t - h) = 0, \end{cases} \quad \alpha \in \mathbb{R}, \quad h \in \mathbb{R}_+.$$

- Let us consider $D = \mathbb{Q}(\alpha) [\partial, \delta]$, the system matrix

$$R = \begin{pmatrix} \partial & -\partial \delta^2 & \alpha \partial^2 \delta \\ \partial \delta^2 & -\partial & \alpha \partial^2 \delta \end{pmatrix} \in D^{2 \times 3},$$

and the D -module $M = D^{1 \times 3} / (D^{1 \times 2} R)$.

- The matrices $P = \begin{pmatrix} 1 & 0 & 0 \\ \delta^2 & 0 & \alpha \partial \delta \\ 0 & 0 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & -\delta^2 \\ 0 & 0 \end{pmatrix}$

satisfy $RP = QR$, $P^2 = P$, $Q^2 = Q$.

Example: Tank model II

- $\ker_D(.P)$, $\text{im}_D(.P)$, $\ker_D(.Q)$ and $\text{im}_D(.Q)$ are **free with bases**:

$$\begin{cases} \ker_D(.P) = D \begin{pmatrix} \delta^2 & -1 & \alpha \partial \delta \end{pmatrix}, & \ker_D(.Q) = D \begin{pmatrix} 0 & 1 \end{pmatrix}, \\ \text{im}_D(.P) = D^{1 \times 2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{im}_D(.Q) = D \begin{pmatrix} -1 & \delta^2 \end{pmatrix}. \end{cases}$$

- If we denote by

$$U = \begin{pmatrix} \delta^2 & -1 & \alpha \partial \delta \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \text{GL}_3(D), \quad V = \begin{pmatrix} 0 & 1 \\ -1 & \delta^2 \end{pmatrix} \in \text{GL}_2(D),$$

then R is **equivalent** to the following **block-diagonal matrix**:

$$V R U^{-1} = \begin{pmatrix} \partial & & 0 \\ 0 & \partial(\delta-1)(\delta+1)(\delta^2+1) & \alpha \partial^2 \delta(\delta-1)(\delta+1) \end{pmatrix}.$$

Example: Tank model II

- Another **idempotent** of $\text{end}_D(M)$ is defined by the idempotent matrices P' and Q' defined by:

$$P' = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad Q' = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

- Using **linear algebraic techniques**, we obtain

$$U' = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \text{GL}_3(D), \quad V' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \text{GL}_2(D),$$

and R is **equivalent** to the following **block-diagonal matrix**:

$$V' R U'^{-1} = \begin{pmatrix} \partial(1-\delta)(\delta+1) & 0 & 0 \\ 0 & \partial(\delta^2+1) & 2\alpha\partial^2\delta \end{pmatrix}.$$

Example: Flexible rod

- **Flexible rod** (Mounier, Rudolph, Petitot, Fliess ECC95):

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) = 0. \end{cases}$$

$$\Rightarrow R = \begin{pmatrix} \partial & -\partial\delta & -1 \\ 2\partial\delta & -\partial\delta^2 - \partial & 0 \end{pmatrix}.$$

$$P = \begin{pmatrix} 1 + \delta^2 & -\frac{1}{2}\delta(1 + \delta^2) & 0 \\ 2\delta & -\delta^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & -\frac{1}{2}\delta \\ 0 & 0 \end{pmatrix},$$

$$\Rightarrow U = \begin{pmatrix} -2\delta & \delta^2 + 1 & 0 \\ 2\partial(1 - \delta^2) & \partial\delta(\delta^2 - 1) & -2 \\ -1 & \frac{1}{2}\delta & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -1 \\ 2 & -\delta \end{pmatrix},$$

$$\Rightarrow \bar{R} = VRU^{-1} = \begin{pmatrix} \partial & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Example: Flexible rod

$$\bar{R} = \begin{pmatrix} \partial & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

- All the smooth solutions of the differential time-delay system

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) = 0, \end{cases}$$

are of the form

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ u(t) \end{pmatrix} = U^{-1} \begin{pmatrix} c \\ 0 \\ z_3(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}c - z_3(t-2) - z_3(t) \\ c - 2z_3(t-1) \\ \dot{z}_3(t-2) - \dot{z}_3(t) \end{pmatrix},$$

where c (resp., z_3) is an arbitrary constant (resp., smooth function).

Corollary

• **Corollary:** Let $R \in D^{q \times p}$, $M = D^{1 \times p} / (D^{1 \times q} R)$ and $f \in \text{end}_D(M)$ be defined by P and Q and satisfying $P^2 = P$ and $Q^2 = Q$. Let us suppose that one of the conditions holds:

- 1 $D = A[\partial]$, where A is a field,
- 2 $D = k[\partial_1, \dots, \partial_n]$ is a commutative Ore algebra,
- 3 $D = A[\partial_1, \dots, \partial_n]$, where $A = k[x_1, \dots, x_n]$ or $k(x_1, \dots, x_n)$ and k is a field of characteristic 0, and:

$$\begin{aligned} \text{rank}_D(\ker_D(.P)) &\geq 2, & \text{rank}_D(\text{im}_D(.P)) &\geq 2, \\ \text{rank}_D(\ker_D(.Q)) &\geq 2, & \text{rank}_D(\text{im}_D(.Q)) &\geq 2. \end{aligned}$$

Then, there exist $U \in \text{GL}_p(D)$ and $V \in \text{GL}_q(D)$ such that $\bar{R} = V R U^{-1}$ is a block diagonal matrix.

The OREMORPHISMS package

- The algorithms have been implemented in a **Maple package** called **OREMORPHISMS** based on the library OREMODULES:

`http://www-sop.inria.fr/members/Alban.Quadrat/OreMorphisms/index.html`

- List of the main functions:
 - Morphisms, MorphismsConstCoeff, MorphismsRat.
 - Idempotents, IdempotentsConstCoeff, IdempotentsRat.
 - KerMorphism, ImMorphism, CokerMorphism, CoimMorphism.
 - TestSurj, TestInj, TestIso.
 - QuadraticFirstIntegralConst. . .

The ORE Morphisms package

- Computation of bases of free left D -modules:
 - If D is a left principal ideal domain, then we can use **Smith** or **Jacobson normal forms** (Culianez-Q.).
 - If D is the Weyl algebra $A_n(\mathbb{Q})$ or $B_n(\mathbb{Q})$, then we use the implementation of the **Stafford theorem** (Q.-Robertz).
 - If D is a commutative polynomial ring, then we use the implementation of the **Quillen-Suslin theorem** (Fabiańska-Q.).
- ORE Morphisms uses the following packages:

JACOBSON, QUILLEN SUSLIN & STAFFORD.