

# Constructive Algebraic Analysis & Algebraic Systems Theory

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  - ODEs, PDEs, difference equations, time-delay equations. . .
  - Determined, overdetermined and underdetermined systems.

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  - Determined, overdetermined and underdetermined systems.
  - **Determined**: integration (closed-forms & numerical analysis).
  - **Overdetermined**: integrability & compatibility conditions  
(Cartan, Riquier, Janet, Spencer. . .)
  - **Underdetermined**: control theory, mathematical physics. . .

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  - We develop a **dictionary** between the properties of linear functional systems and the properties of modules.
  - We develop a **constructive approach to homological algebra** to check the module properties and thus the system properties.
  - We implement the algorithms in **dedicated packages** in **computer algebra systems** (Gröbner/Janet bases).

# Stirred Tank model

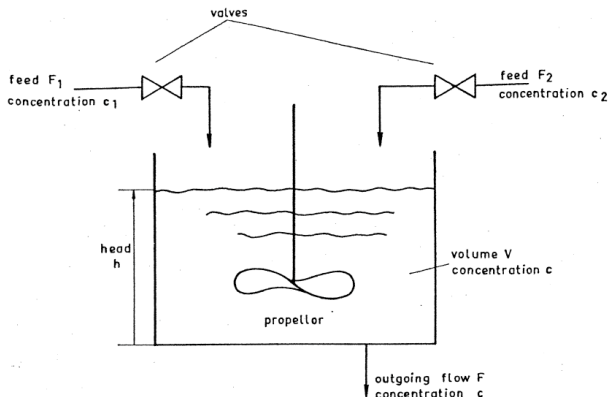


Fig. 1.3. A stirred tank.

H. Kwakernaak, R. Sivan,  
Linear Optimal Control Systems, Wiley, 1972.

# Stirred Tank model

- **Non-linear model** of a stirred tank (p. 7) (mass balance eqs):

$$\begin{cases} \frac{dV(t)}{dt} = -k \sqrt{\frac{V(t)}{S}} + F_1(t) + F_2(t), \\ \frac{d(c(t)V(t))}{dt} = -c(t)k \sqrt{\frac{V(t)}{S}} + c_1 F_1(t) + c_2 F_2(t). \end{cases}$$

- $F_1, F_2$ : flow rates of two incoming flows feeding the tank,
- $c_1, c_2$ : constant concentrations of dissolved materials,
- $c$ : concentration in the tank,
- $V$ : volume,
- $k$ : experimental constant,
- $S$ : constant cross-sectional area.

# Stirred Tank model

- $V_0$ : constant volume,  $c_0$ : constant concentration,

$$F_{10} = \frac{(c_2 - c_0)}{(c_2 - c_1)} k \sqrt{\frac{V_0}{S}}, \quad F_{20} = \frac{(c_0 - c_1)}{(c_2 - c_1)} k \sqrt{\frac{V_0}{S}}.$$

- **Linearized model** around the steady-state situation (p. 8-9):

$$\begin{aligned} V(t) &= V_0 + x_1(t), & c(t) &= c_0 + x_2(t), \\ F_1(t) &= F_{10} + u_1(t), & F_2(t) &= F_{20} + u_2(t). \end{aligned}$$

$$\begin{cases} \dot{x}_1(t) = -\frac{1}{2\theta} x_1(t) + u_1(t) + u_2(t), \\ \dot{x}_2(t) = -\frac{1}{\theta} x_2(t) + \left(\frac{c_1 - c_0}{V_0}\right) u_1(t) + \left(\frac{c_2 - c_0}{V_0}\right) u_2(t), \\ y_1(t) = \frac{1}{2\theta} x_1(t), \quad y_2(t) = x_2(t). \end{cases}$$

where  $F_0 = k \sqrt{(V_0/S)}$  and  $\theta = V_0/F_0$  (holdup time of the tank).

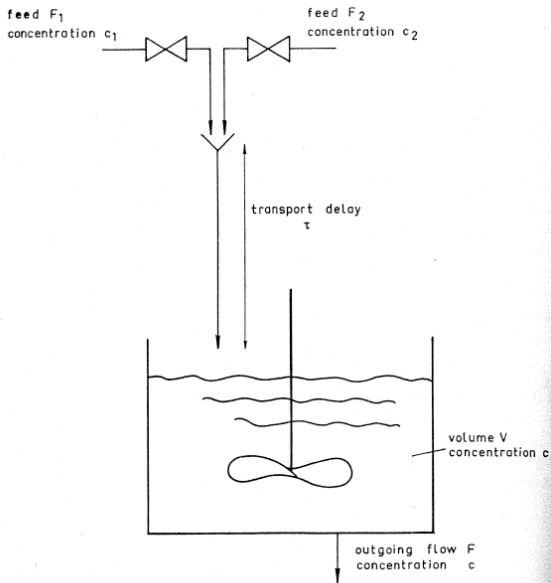
# Stirred Tank model

- **Discrete-time model:** if the stirred tank is commanded by a process control computer, then the valve settings only change at discrete instants, remaining constant in between (p. 449) and

$$\begin{cases} x_1(n+1) = e^{-\frac{\Delta}{2\theta}} x_1(n) + 2\theta(1 - e^{-\frac{\Delta}{2\theta}})(u_1(n) + u_2(n)) \\ x_2(n+1) = e^{-\frac{\Delta}{\theta}} x_2(n) \\ \quad + \frac{\theta(1 - e^{-\frac{\Delta}{\theta}})}{V_0} ((c_1 - c_0)u_1(n) + (c_2 - c_0)u_2(n)) \end{cases}$$

where  $\Delta$  is the constant length of time intervals.

# Stirred Tank model with a transport delay



- **Differential time-delay model**: if there is a transport delay of amplitude  $\tau$  occurring in the pipe (p. 449-451), then

$$\begin{cases} \dot{x}_1(t) = -\frac{1}{2\theta} x_1(t) + u_1(t) + u_2(t), \\ \dot{x}_2(t) = -\frac{1}{\theta} x_2(t) + \left(\frac{c_1 - c_0}{V_0}\right) u_1(t - \tau) + \left(\frac{c_2 - c_0}{V_0}\right) u_2(t - \tau). \end{cases}$$

where  $\tau > 0$  is the amplitude of the delay.

- Other models: PDEs, integro-differential systems...

# Matrices of differential operators

- **Newton**: Fluxion calculus (1666) (“dot-age”)

$$\begin{cases} \ddot{x}_1(t) + \alpha x_1(t) - \alpha u(t) = 0, \\ \ddot{x}_2(t) + \alpha x_2(t) - \alpha u(t) = 0, \end{cases} \quad \alpha = g/l.$$

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- **Boole**: Operational calculus (1859-60)

$$\begin{pmatrix} \frac{d^2}{dt^2} + \alpha & 0 & -\alpha \\ 0 & \frac{d^2}{dt^2} + \alpha & -\alpha \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ u(t) \end{pmatrix} = 0.$$

$\Rightarrow$  ring of differential operators  $D = \mathbb{Q}(\alpha) \left[ \frac{d}{dt} \right]$ :

$$\sum_{i=0}^n a_i \left( \frac{d}{dt} \right)^i \in D, \quad a_i \in \mathbb{Q}(\alpha), \quad \left( \frac{d}{dt} \right)^i = \frac{d}{dt} \circ \dots \circ \frac{d}{dt} = \frac{d^i}{dt^i}.$$

# Functional operators

- Differential operator:  $(\sum_{s=0}^m b_s(t) \partial^s) (\sum_{r=0}^n a_r(t) \partial^r)$

$$\partial: y \mapsto \frac{dy}{dt}, \quad a = a(\cdot), \quad a: y \mapsto ay,$$

$$\begin{aligned} (\partial a)(y) &= \partial(a(y)) = \partial(ay) = \frac{d}{dt}(ay) = a \frac{dy}{dt} + \frac{da}{dt} y \\ &= \left( a \partial + \frac{da}{dt} \right) (y). \end{aligned}$$

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- Shift operator:  $\partial: y_n \mapsto \sigma(y_n) = y_{n+1}, \quad a: y_n \mapsto a_n y_n,$

$$\begin{aligned}(\partial a)(y_n) &= \partial(a(y_n)) = \partial(a_n y_n) = \sigma(a_n y_n) = a_{n+1} y_{n+1} \\ &= (\sigma(a) \partial)(y_n).\end{aligned}$$

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- Time-delay operator:  $\partial: y \mapsto \delta(y) = y(\cdot - \tau),$

$$\begin{aligned}(\partial a)(y) &= \partial(a(y)) = \partial(ay) = \delta(ay) = a(\cdot - \tau) y(\cdot - \tau) \\ &= (\delta(a) \partial)(y).\end{aligned}$$

# Functional operators

- **Other functional operators:** difference, divided difference, Eulerian, Frobenius,  $q$ -dilation,  $q$ -shift,  $q$ -difference. . . operators.
- **Unique expansion:**  $P = \sum_{i=0}^n a_i \partial^i$ ,  $a_i \in A$ : ring of coeffs.
- **Degree condition:**  $\partial a = \alpha \partial + \beta = \alpha(a) \partial + \beta(a)$ ,  $a, b, c \in A$ .

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$$\begin{cases} \partial(a+b) = \alpha(a+b) \partial + \beta(a+b), \\ \partial a = \alpha(a) \partial + \beta(a), \\ \partial b = \alpha(b) \partial + \beta(b), \end{cases}$$

$$\partial(a+b) = \partial a + \partial b \quad \Leftrightarrow \quad \begin{cases} \alpha(a+b) = \alpha(a) + \alpha(b), \\ \beta(a+b) = \beta(a) + \beta(b). \end{cases}$$

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$$\partial(a b) = \alpha(a b) \partial + \beta(a b)$$

$$\begin{aligned} \partial(a b) &= (\partial a) b = (\alpha(a) \partial + \beta(a)) b \\ &= \alpha(a) (\alpha(b) \partial + \beta(b)) + \beta(a) b, \end{aligned}$$

$$\Leftrightarrow \begin{cases} \alpha(a b) = \alpha(a) \alpha(b), \\ \beta(a b) = \alpha(a) \beta(b) + \beta(a) b. \end{cases}$$

- $\alpha$  is an endomorphism of  $A$  and  $\beta$  is a  $\alpha$ -derivation of  $A$ .

# Skew polynomial rings (Ore, 1933)

- **Definition:** A **skew polynomial ring**  $A[\partial; \alpha, \beta]$  is a non-commutative polynomial ring in  $\partial$  with coefficients in  $A$  satisfying

$$\forall a \in A, \quad \partial a = \alpha(a) \partial + \beta(a)$$

where  $\alpha : A \rightarrow A$  and  $\beta : A \rightarrow A$  are such that:

$$\begin{cases} \alpha(1) = 1, \\ \alpha(a + b) = \alpha(a) + \alpha(b), \\ \alpha(ab) = \alpha(a)\alpha(b), \end{cases} \quad \begin{cases} \beta(a + b) = \beta(a) + \beta(b), \\ \beta(ab) = \alpha(a)\beta(b) + \beta(a)b. \end{cases}$$

- $P \in A[\partial; \alpha, \beta]$  has a unique form  $P = \sum_{i=0}^n a_i \partial^i$ ,  $a_i \in A$ .
  - Ring of differential operators:  $A[\partial; \text{id}, \frac{d}{dt}]$ .
  - Ring of shift operators:  $A[\partial; \sigma, 0]$ ,  $A[\partial; \delta, 0]$ .
  - Ring of difference operators:  $A[\partial; \tau, \tau - \text{id}]$ ,  $\tau a(x) = a(x + 1)$ .

# Ore algebras (Chyzak-Salvy, 1996)

- We can iterate skew polynomial rings to get **Ore extensions**:

$$A[\partial_1; \alpha_1, \beta_1] \dots [\partial_n; \alpha_n, \beta_n]$$

- **Definition**: An Ore extension  $A[\partial_1; \alpha_1, \beta_1] \dots [\partial_n; \alpha_n, \beta_n]$  is called an **Ore algebra** if the  $\partial_i$ 's commute, i.e., if we have

$$1 \leq j < i \leq m, \quad \alpha_i(\partial_j) = \partial_j, \quad \beta_i(\partial_j) = 0,$$

and the  $\alpha_{i|_A}$ 's and  $\beta_{j|_A}$ 's commute for  $i \neq j$ .

- Ring of differential operators:  $A \left[ \partial_1; \text{id}, \frac{\partial}{\partial x_1} \right] \dots \left[ \partial_n; \text{id}, \frac{\partial}{\partial x_n} \right]$ .
- Ring of differential delay operators:  $A \left[ \partial_1; \text{id}, \frac{d}{dt} \right] [\partial_2; \delta, 0]$ .
- Ring of shift operators:  $A[\partial_1; \sigma_1, 0] \dots [\partial_n; \sigma_n, 0]$ .

# Matrix of functional operators

- **The stirred tank model** (Kwakernaak-Sivan, 72):

$$\begin{cases} \dot{x}_1(t) + \frac{1}{2\theta} x_1(t) - u_1(t) - u_2(t) = 0, \\ \dot{x}_2(t) + \frac{1}{\theta} x_2(t) - \left(\frac{c_1 - c_0}{V_0}\right) u_1(t - \tau) - \left(\frac{c_2 - c_0}{V_0}\right) u_2(t - \tau) = 0. \end{cases} \quad (\star)$$

- We introduce the **commutative Ore algebra**:

$$D = \mathbb{Q}(\theta, c_0, c_1, c_2, V_0) \left[ \partial_1; \text{id}, \frac{d}{dt} \right] [\partial_2; \delta, 0].$$

- The linear functional system  $(\star)$  can be rewritten as:

$$\begin{pmatrix} \partial_1 + \frac{1}{2\theta} & 0 & -1 & -1 \\ 0 & \partial_1 + \frac{1}{\theta} & -\left(\frac{c_1 - c_0}{V_0}\right) \partial_2 & -\left(\frac{c_2 - c_0}{V_0}\right) \partial_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ u_1(t) \\ u_2(t) \end{pmatrix} = 0.$$

# Matrix of functional operators

- Linearization of the Navier-Stokes  $\sim$  a parabolic Poiseuille profile

$$\begin{cases} \partial_t u_1 + 4y(1-y)\partial_x u_1 - 4(2y-1)u_2 - \frac{1}{Re}(\partial_x^2 + \partial_y^2)u_1 + \partial_x p = 0, \\ \partial_t u_2 + 4y(1-y)\partial_x u_2 - \frac{1}{Re}(\partial_x^2 + \partial_y^2)u_2 + \partial_y p = 0, \\ \partial_x u_1 + \partial_y u_2 = 0. \end{cases} \quad (*)$$

(e.g., Vazquez-Krstic, IEEE 07)

- Let us introduce the so-called Weyl algebra  $A_3(\mathbb{Q}(Re))$

$$D = \mathbb{Q}(Re)[t, x, y] \left[ \partial_t; \text{id}, \frac{\partial}{\partial t} \right] \left[ \partial_x; \text{id}, \frac{\partial}{\partial x} \right] \left[ \partial_y; \text{id}, \frac{\partial}{\partial y} \right].$$

$$(\partial_x y = y \partial_x, \partial_x x = x \partial_x + 1, \partial_x \partial_y = \partial_y \partial_x \dots):$$

- The system (\*) is defined by the matrix of PD operators:

$$\begin{pmatrix} \partial_t + 4y(1-y)\partial_x - \frac{1}{Re}(\partial_x^2 + \partial_y^2) & -4(2y-1) & \partial_x \\ 0 & \partial_t + 4y(1-y)\partial_x - \frac{1}{Re}(\partial_x^2 + \partial_y^2) & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix}.$$

# Noncommutative Gröbner bases

- Let  $D = A[\partial_1; \alpha_1, \beta_1] \dots [\partial_m; \alpha_m, \beta_m]$  be an Ore algebra.
- **Theorem:** (Kredel, 93) Let  $A = k[x_1, \dots, x_n]$  a commutative polynomial ring ( $k = \mathbb{Q}, \mathbb{F}_p$ ) and  $D$  an Ore algebra satisfying

$$\alpha_i(x_j) = a_{ij} x_j + b_{ij}, \quad \beta_i(x_j) = c_{ij},$$

for certain  $0 \neq a_{ij} \in k$ ,  $b_{ij} \in k$ ,  $c_{ij} \in A$  and  $\deg(c_{ij}) \leq 1$ . Then, a non-commutative version of **Buchberger's algorithm** terminates for any term order and its result is a **Gröbner basis**.

- **Implementation** in the Maple package **Ore\_algebra** (Chyzak)  
(Singular:Plural, Macaulay 2, NCAAlgebra, JanetOre...).
- Gröbner bases can be used to **effectively compute over  $D^{1 \times p}/F$** .

# Modules & Linear systems

- Let  $D$  be an Ore algebra,  $R \in D^{q \times p}$  and a left  $D$ -module  $\mathcal{F}$ :

$$\forall d_1, d_2 \in D, \quad \forall f_1, f_2 \in \mathcal{F}, \quad d_1 f_1 + d_2 f_2 \in \mathcal{F}.$$

- Let us consider the system  $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\eta = 0\}$ .
- Let us consider the left  $D$ -homomorphism (left  $D$ -linear map):

$$\begin{array}{ccc} D^{1 \times q} & \xrightarrow{.R} & D^{1 \times p} \\ \lambda = (\lambda_1 \ \dots \ \lambda_q) & \longmapsto & \lambda R. \end{array}$$

- As in number theory or algebraic geometry, we associate with the linear system  $\ker_{\mathcal{F}}(R.)$  the finitely presented left  $D$ -module:

$$M = \operatorname{coker}_D(.R) = D^{1 \times p} / (D^{1 \times q} R).$$

# Examples: algebraic geometry & number theory

- Cauchy's definition of complex numbers:  $\mathbb{C} = \mathbb{R}[x]/(x^2 + 1)$

$$D = \mathbb{R}[x], R = x^2 + 1, M = D/(D R) = D/(x^2 + 1) = \mathbb{C}.$$

- Affine coordinate rings:  $A = k[x, y]/(x^2 + y^2 - 1, x - y)$

$$D = k[x, y], R = \begin{pmatrix} x^2 + y^2 - 1 \\ x - y \end{pmatrix},$$

$$D^{1 \times 2} R = (x^2 + y^2 - 1, x - y), M = D/(D^{1 \times 2} R) = A.$$

## Example: linear system theory

- Let us consider  $D = \mathbb{Q} \left[ \partial_1; \text{id}, \frac{\partial}{\partial x_1} \right] \left[ \partial_2; \text{id}, \frac{\partial}{\partial x_2} \right] \left[ \partial_3; \text{id}, \frac{\partial}{\partial x_3} \right]$  and the **curl operator** defined by:

$$R = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} \in D^{3 \times 3}.$$

- Let us consider the  **$D$ -morphism** ( $D$ -linear map)

$$\begin{aligned} D^{1 \times 3} &\xrightarrow{.R} D^{1 \times 3} \\ \lambda &\longmapsto (\lambda_2 \partial_3 - \lambda_3 \partial_2 \quad -\lambda_1 \partial_3 + \lambda_3 \partial_1 \quad \lambda_2 \partial_2 - \lambda_1 \partial_1), \end{aligned}$$

and the  $D$ -module  $M = \text{coker}_D(.R) = D^{1 \times 3} / (D^{1 \times 3} R)$ .

- If  $\mathcal{F} = C^\infty(\Omega)$ ,  $\mathcal{D}'(\Omega)$ ,  $\mathcal{S}'(\Omega)$ ... is a  $D$ -module, then:

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^3 \mid \vec{\nabla} \wedge \eta = R\eta = 0\} \cong \text{hom}_D(M, \mathcal{F}).$$

# Linear systems of equations

- $M = D^{1 \times p} / (D^{1 \times q} R)$  can be defined by **generators and relations**:
- Let  $\{e_k\}_{k=1, \dots, p}$  be the **standard basis** of  $D^{1 \times p}$ :

$$e_k = (0 \dots 1 \dots 0).$$

- Let  $\pi : D^{1 \times p} \longrightarrow M$  be the **left  $D$ -morphism** sending  $\mu$  to  $\pi(\mu)$ .

$$\forall m \in M, \exists \mu = (\mu_1 \dots \mu_p) \in D^{1 \times p} : m = \pi(\mu) = \sum_{k=1}^p \mu_k \pi(e_k),$$

$\Rightarrow \{y_k = \pi(e_k)\}_{k=1, \dots, p}$  is a **family of generators** of  $M$ .

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$$\pi((R_{l1} \dots R_{lp})) = \pi\left(\sum_{k=1}^p R_{lk} e_k\right) = \sum_{k=1}^p R_{lk} y_k = 0, \quad l = 1, \dots, q,$$

$\Rightarrow y = (y_1 \dots y_p)^T$  satisfies the **relation  $Ry = 0$** .

## Duality modules — systems

- Let  $\mathcal{F}$  be a left  $D$ -module and  $\text{hom}_D(M, \mathcal{F})$  the abelian group:  
 $\text{hom}_D(M, \mathcal{F}) = \{f : M \rightarrow \mathcal{F} \mid f(d_1 m_1 + d_2 m_2) = d_1 f(m_1) + d_2 f(m_2)\}.$
- Applying the contravariant left exact functor  $\text{hom}_D(\cdot, \mathcal{F})$  to

$$D^{1 \times q} \xrightarrow{\cdot R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0,$$

we obtain the following exact sequence of abelian groups:

$$\mathcal{F}^q \xleftarrow{\cdot R} \mathcal{F}^p \xleftarrow{\iota \circ \pi^*} \text{hom}_D(M, \mathcal{F}) \longleftarrow 0.$$

- Theorem** (Malgrange):

$$\text{hom}_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R \cdot) = \{\eta \in \mathcal{F}^p \mid R \eta = 0\}$$

- Remark:**  $\text{hom}_D(M, \mathcal{F})$  intrinsically characterizes  $\ker_{\mathcal{F}}(R \cdot)$  as it does not depend on the embedding of  $\ker_{\mathcal{F}}(R \cdot)$  into  $\mathcal{F}^p$ .

# Linear functional systems

- Let  $\mathcal{F}$  be a left  $D$ -module and  $M = D^{1 \times p} / (D^{1 \times q} R)$ .
- Let  $f : M \longrightarrow \mathcal{F}$  be a left  $D$ -morphism. Then, we have:

$$\begin{aligned} f : M &\longrightarrow \mathcal{F} \\ y_k = \pi(e_k) &\longmapsto \eta_k, \quad k = 1, \dots, p, \quad f(0) = 0. \end{aligned}$$

# Linear functional systems

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$$\pi((R_{l1} \ \dots \ R_{lp})) = \pi\left(\sum_{k=1}^p R_{lk} e_k\right) = \sum_{k=1}^p R_{lk} y_k = 0.$$

$$f\left(\sum_{k=1}^p R_{lk} y_k\right) = \sum_{k=1}^p R_{lk} f(y_k) = \sum_{k=1}^p R_{lk} \eta_k = 0, \quad l = 1, \dots, q.$$

$$\Rightarrow \eta = (\eta_1 \ \dots \ \eta_p)^T \in \mathcal{F}^p : R \eta = 0.$$

# Monge problem (1784)

- Let  $D$  be a **ring of functional operators**.
- Let  $\mathcal{F}$  be a **left  $D$ -module**:

$$\forall P_1, P_2 \in D, \forall y_1, y_2 \in \mathcal{F} : P_1 y_1 + P_2 y_2 \in \mathcal{F}.$$

Let us consider  $R \in D^{q \times p}$  and the **linear functional system**:

$$\ker_{\mathcal{F}}(R.) \triangleq \{\eta \in \mathcal{F}^p \mid R \eta = 0\}.$$

- **Question**: When does  $Q \in D^{p \times m}$  exist such that:

$$\ker_{\mathcal{F}}(R.) = \operatorname{im}_{\mathcal{F}}(Q.) \triangleq Q \mathcal{F}^m?$$

$\Rightarrow Q$  is called a **parametrization** of  $\ker_{\mathcal{F}}(R.)$ .

- **Definition:** 1.  $M$  is **free** if  $\exists r \in \mathbb{Z}_+$  such that  $M \cong D^r$ .
- 2.  $M$  is **stably free** if  $\exists r, s \in \mathbb{Z}_+$  such that  $M \oplus D^s \cong D^r$ .
- 3.  $M$  is **projective** if  $\exists r \in \mathbb{Z}_+$  and a  $D$ -module  $P$  such that:

$$M \oplus P \cong D^r.$$

- 4.  $M$  is **reflexive** if  $\varepsilon : M \longrightarrow \text{hom}_D(\text{hom}_D(M, D), D)$  is an isomorphism, where:

$$\varepsilon(m)(f) = f(m), \quad \forall m \in M, \quad f \in \text{hom}_D(M, D).$$

- 5.  $M$  is **torsion-free** if:

$$t(M) = \{m \in M \mid \exists 0 \neq P \in D : P m = 0\} = 0.$$

- 6.  $M$  is **torsion** if  $t(M) = M$ .

# Classification of modules

- **Theorem:** 1. We have the following implications:

**free  $\Rightarrow$  stably free  $\Rightarrow$  projective  $\Rightarrow$  reflexive  $\Rightarrow$  torsion-free.**

2. If  $D$  is a principal domain (e.g.,  $K[\partial; \text{id}, \frac{d}{dt}]$ ), then:

**torsion-free = free.**

3. If  $D$  is a hereditary ring (e.g.,  $\mathbb{Q}[t][\partial; \text{id}, \frac{d}{dt}]$ ), then:

**torsion-free = projective.**

4. If  $D = k[x_1, \dots, x_n]$  and  $k$  a field, then:

**projective = free** (Quillen-Suslin theorem).

4. If  $D = A_n(k)$  or  $B_n(k)$ ,  $k$  is a field of characteristic 0, then

**projective = free** (Stafford theorem),

for modules of rank at least 2.

# Free resolutions

- **Definition:** A sequence of  $D$ -morphisms  $M' \xrightarrow{f} M \xrightarrow{g} M''$  is called a **complex** if  $g \circ f = 0$ , i.e.,  $\text{im } f \subseteq \ker g$ .

The defect of exactness at  $M$  is  $H(M) = \ker g / \text{im } f$ .

The complex is **exact** at  $M$  if  $\text{im } f = \ker g$ .

- **Definition:** A **free resolution** of a left  $D$ -module  $M$  is an exact sequence of the form

$$\dots \xrightarrow{\cdot R_3} D^{1 \times l_2} \xrightarrow{\cdot R_2} D^{1 \times l_1} \xrightarrow{\cdot R_1} D^{1 \times l_0} \xrightarrow{\pi} M \longrightarrow 0,$$

where  $R_i \in D^{l_i \times l_{i-1}}$  and:

$$\begin{array}{ccc} D^{1 \times l_i} & \xrightarrow{\cdot R_i} & D^{1 \times l_{i-1}} \\ (P_1 \dots P_{l_i}) & \longmapsto & (P_1 \dots P_{l_i}) R_i. \end{array}$$

- **Algorithm:** Find a **basis of the compatibility conditions** of the inhomogeneous system  $R_i y = u$  by **eliminating  $y$**  (e.g., Gb):

$$\forall P \in \ker_D(\cdot R_i), \quad P(R_i y) = P u \Rightarrow P u = 0.$$

# Extension functor $\text{ext}_D^i(\cdot, \mathcal{F})$

- We define the **reduced free resolution** of  $M$  by:

$$\dots \xrightarrow{\cdot R_3} D^{1 \times l_2} \xrightarrow{\cdot R_2} D^{1 \times l_1} \xrightarrow{\cdot R_1} D^{1 \times l_0} \longrightarrow 0 \quad (\star).$$

- Let  $\mathcal{F}$  be a left  $D$ -modules. Applying the functor  $\text{hom}_D(\cdot, \mathcal{F})$  to  $(\star)$ , we obtain the following **complex**:

$$\dots \xleftarrow{R_3 \cdot} \mathcal{F}^{l_2} \xleftarrow{R_2 \cdot} \mathcal{F}^{l_1} \xleftarrow{R_1 \cdot} \mathcal{F}^{l_0} \longleftarrow 0 \quad (\star\star)$$

$$\text{where } \begin{array}{ccc} \mathcal{F}^{l_i} & \xleftarrow{R_i \cdot} & \mathcal{F}^{l_{i-1}} \\ R_i \eta & \longleftarrow & \eta. \end{array}$$

- We denote the **defects of exactness** of  $(\star\star)$  by:

$$\begin{cases} \text{ext}_D^0(M, \mathcal{F}) = \text{hom}_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_1 \cdot), \\ \text{ext}_D^i(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_{i+1} \cdot) / \text{im}_{\mathcal{F}}(R_i \cdot), \quad i \geq 1. \end{cases}$$

- **Theorem:** The abelian group  $\text{ext}_D^i(M, \mathcal{F})$  **only depends on  $M$  and  $\mathcal{F}$**  but not on the resolution  $(\star)$ .

# Auslander transpose

- **Definition:** Let  $R \in D^{q \times p}$ . If  $M = D^{1 \times p} / (D^{1 \times q} R)$  denotes the left  $D$ -module finitely presented by  $R$ , then its **Auslander transpose** is the right  $D$ -module defined by  $N = D^q / (R D^p)$ .
- **Proposition:** The Auslander transpose  $N = D^q / (R D^p)$  **only depends on  $M$**  up to a projective equivalence.

Hence, if we have  $M = D^{1 \times p'} / (D^{1 \times q'} R')$  and  $N' = D^{q'} / (R' D^{p'})$  denotes the corresponding Auslander transpose, then there exist two projective right  $D$ -modules  $P$  and  $P'$  such that:

$$N \oplus P \cong N' \oplus P'.$$

- **Proposition:** If  $P$  is a projective module, then we have:

$$\text{ext}_D^i(P, \mathcal{F}) = 0, \quad \forall i \geq 1.$$

- **Corollary:** The  $\text{ext}_D^i(N, \mathcal{F})$ 's,  $i \geq 1$ , **only depend on  $M$  and  $\mathcal{F}$ .**

Module $M$	Homological algebra	$\mathcal{F}$ injective cogenerator
with torsion	$t(M) \cong \text{ext}_D^1(N, D)$	$\emptyset$
torsion-free	$\text{ext}_D^1(N, D) = 0$	$\ker_{\mathcal{F}}(R.) = Q \mathcal{F}^1$
reflexive	$\text{ext}_D^i(N, D) = 0$ $i = 1, 2$	$\ker_{\mathcal{F}}(R.) = Q_1 \mathcal{F}^1$ $\ker_{\mathcal{F}}(Q_1.) = Q_2 \mathcal{F}^2$
projective = stably free	$\text{ext}_D^i(N, D) = 0$ $1 \leq i \leq n = \text{gld}(D)$	$\ker_{\mathcal{F}}(R.) = Q_1 \mathcal{F}^1$ $\ker_{\mathcal{F}}(Q_1.) = Q_2 \mathcal{F}^2$ ... $\ker_{\mathcal{F}}(Q_{n-1}.) = Q_n \mathcal{F}^n$
free	$\exists Q \in D^{p \times m}, T \in D^{m \times p},$ $\ker_D(.Q) = D^{1 \times q} R, TQ = I_m$	$\ker_{\mathcal{F}}(R.) = Q \mathcal{F}^m,$ $\exists T \in D^{m \times p} : TQ = I_m$

# Parametrizability problem

$$\begin{array}{ccccccc}
 D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 0 \longleftarrow N & \xleftarrow{\kappa} & D^q & \xleftarrow{R \cdot} & D^p & \xleftarrow{\pi^*} & \text{hom}_D(M, D) \longleftarrow 0 \\
 0 \longleftarrow N & \xleftarrow{\kappa} & D^q & \xleftarrow{R \cdot} & D^p & \xleftarrow{Q \cdot} & D^m \\
 D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\cdot Q} & D^{1 \times m} & & 
 \end{array}$$

- Let  $R' \in D^{q' \times p} : \ker_D(\cdot Q) = D^{1 \times q'} R'$ :

$$t(M) \cong \text{ext}_D^1(N, D) \cong \ker_D(\cdot Q) / (D^{1 \times q} R) = (D^{1 \times q'} R') / (D^{1 \times q} R),$$

$$0 \longrightarrow t(M) \xrightarrow{i} M \xrightarrow{p} M/t(M) \longrightarrow 0, \quad M = D^{1 \times p} / (D^{1 \times q} R),$$

$$\Rightarrow M/t(M) \cong D^{1 \times p} / (D^{1 \times q'} R').$$

$$\begin{aligned}\text{ext}_D^1(N, D) &\cong t(M) = (D^{1 \times q'} R') / (D^{1 \times q} R), \\ M/t(M) &\cong D^{1 \times p} / (D^{1 \times q'} R').\end{aligned}$$

- $D^{1 \times q} R \subseteq D^{1 \times q'} R' \Rightarrow \exists R'' \in D^{q \times q'} :$

$$R = R'' R'.$$

- Since  $(D^{1 \times q'} R') / (D^{1 \times q} R)$  is a torsion left  $D$ -module, then:

$$\exists P_i \in D : P_i \pi(R'_{i\bullet}) = 0 \Leftrightarrow \pi(P_i R'_{i\bullet}) = 0$$

$$\Rightarrow \exists \mu_i \in D^{1 \times q'} : P_i R'_{i\bullet} = \mu_i R \Leftrightarrow (P_i \quad - \mu_i) \begin{pmatrix} R'_{i\bullet} \\ R \end{pmatrix} = 0.$$

$\Rightarrow$  Find the compatibility conditions of

$$\begin{cases} R'_{i\bullet} \eta = \tau_i, \\ R \eta = 0. \end{cases} \Rightarrow P_{ik} \tau_i = 0, \quad k = 1, \dots, m_j.$$

# Presentation of $t(M)$

- If  $R'_2 \in D^{r' \times q'}$  is such that  $\ker_D(.R') = D^{1 \times r'} R'_2$ , then:

$$\begin{aligned} t(M) &\cong (D^{1 \times q'} R') / (D^{1 \times q} R) && (R = R'' R') \\ &\cong (D^{1 \times q'} R') / \left( D^{1 \times (q+r')} \begin{pmatrix} R'' & R' \\ R'_2 & R' \end{pmatrix} \right) \\ &\cong D^{1 \times q'} / \left( D^{1 \times (q+r')} \begin{pmatrix} R'' \\ R'_2 \end{pmatrix} \right). \end{aligned}$$

- $t(M)$  admits the following finite presentation

$$D^{1 \times (q+r')} \cdot \begin{pmatrix} R'' \\ R'_2 \end{pmatrix} \longrightarrow D^{1 \times q'} \xrightarrow{\sigma} t(M) \longrightarrow 0,$$

i.e., the torsion elements satisfy the following equations:

$$\begin{cases} R'' \tau = 0, \\ R'_2 \tau = 0. \end{cases} \quad (\tau = R' \eta \text{ \& } R \eta = 0).$$

# Parametrization of torsion free modules

- We have the following commutative exact diagram

$$\begin{array}{ccccccc} D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p} & \xrightarrow{\cdot Q} & D^{1 \times m} & & \\ \parallel & & \parallel & & \uparrow \phi & & \\ D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p} & \xrightarrow{\pi'} & M/t(M) & \longrightarrow & 0, \\ & & & & \uparrow & & \\ & & & & 0 & & \end{array}$$

where  $\phi(\pi'(\lambda)) = \lambda Q$ , for all  $\lambda \in D^{1 \times p}$

$$\Rightarrow M/t(M) \cong \phi(M/t(M)) = D^{1 \times p} Q \subseteq D^{1 \times m}.$$

Hence, every element  $m'$  of  $\phi(M/t(M))$  has the form:

$$m' = \sum_{i=1}^p \nu_i Q_i,$$

for some  $\nu_i \in D^{1 \times p}$  and  $i = 1, \dots, p$ .  $Q$  is a parametrization of  $M$ .

$$\begin{array}{ccccccc}
 D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p} & \xrightarrow{\cdot Q} & D^{1 \times m} & & \\
 \parallel & & \parallel & & \uparrow \phi & & \\
 D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p} & \xrightarrow{\pi'} & M/t(M) & \longrightarrow & 0, \\
 & & & & \uparrow & & \\
 & & & & 0 & & 
 \end{array}$$

where  $\phi(\pi'(\lambda)) = \lambda Q$ , for all  $\lambda \in D^{1 \times p}$

$$\Rightarrow M/t(M) \cong \phi(M/t(M)) = D^{1 \times p} Q \subseteq D^{1 \times m}.$$

•  $D^{1 \times p} \xrightarrow{\cdot Q} D^{1 \times m} \longrightarrow 0 \Leftrightarrow \exists T \in D^{m \times p} : TQ = I_m$ , then:

$$M/t(M) \cong \phi(M/t(M)) = D^{1 \times p} Q = D^{1 \times m}.$$

$\Rightarrow M/t(M)$  is a **free left  $D$ -module of rank  $m$** .

$$\begin{array}{ccccccc}
 & & \xleftarrow{\cdot S} & & \xleftarrow{\cdot T} & & \\
 D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p} & \xrightarrow{\cdot Q} & D^{1 \times m} & \longrightarrow & 0 \\
 \parallel & & \parallel & & \uparrow \phi \downarrow \phi^{-1} & & \\
 D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p} & \xrightarrow{\cdot \pi} & M/t(M) & \longrightarrow & 0,
 \end{array}$$

The **isomorphisms**  $\phi$  and  $\phi^{-1}$  are defined by:

$$\begin{array}{llll}
 \phi : M/t(M) & \longrightarrow & D^{1 \times (p-q)} & \phi^{-1} : D^{1 \times m} \longrightarrow M/t(M) \\
 \pi'(\lambda) & \longmapsto & \lambda Q, & \mu \longmapsto \pi'(\mu T).
 \end{array}$$

- If we denote by  $\{h_k\}_{k=1, \dots, m}$  the standard basis of  $D^{1 \times m}$ , then  $\{\phi^{-1}(h_k) = \pi(h_k T) = \pi(T_{k\bullet})\}_{k=1, \dots, m}$  is a basis of  $M/t(M)$

$\Rightarrow$  the residue classes of the rows of  $T$  in  $M/t(M)$  is a basis..

# Minimal parametrizations

- We generally have  $m \geq \text{rank}_D(M)$  (e.g.,  $\ker_D(Q) \neq 0$ ).
- **Theorem:** Let  $M = D^{1 \times p} / (D^{1 \times q} R)$  be a torsion-free left  $D$ -module. Then, there exists a parametrization  $P \in D^{p \times l}$  of  $M$  such that  $l = \text{rank}_D(M)$ , i.e.,  $L = D^{1 \times l} / (D^{1 \times p} P)$  is torsion.
- $P$  is called a **minimal parametrization** of  $M$ .
- **Algorithm:**  $P$  can be obtained by selecting  $\text{rank}_D(M)$  right  $D$ -linearly independent columns of a parametrization  $Q$  of  $M$ .
- **Heuristic method** for computing basis of a free left  $D$ -modules:
  - 1 Compute  $Q \in D^{p \times m}$  such that  $\ker_D(R) = Q D^m$ .
  - 2 Define  $P \in D^{p \times \text{rank}_D(M)}$  by selecting  $\text{rank}_D(M)$  right  $D$ -linearly independent columns of  $Q$ .
  - 3 Check whether or not  $P$  admits a left-inverse over  $D$ .

## Example

We consider the  $D = \mathbb{Q}[x_1, x_2]$ -module  $M = D^{1 \times 3} / (D R)$ , where:

$$R = (x_1 x_2^2 + 1 \quad 3x_2/2 + x_1 - 1 \quad 2x_1 x_2), \quad \text{rank}_D(M) = 2.$$

- Checking  $\text{ext}_D^1(D/(D^{1 \times 3} R^T), D) = 0$ , we obtain that

$$Q = \begin{pmatrix} -4x_1 - 6x_2 + 4 & 6x_2^2 - 4x_2 & 0 \\ 4 & -4x_2 & 4x_1 x_2 \\ 2x_1 x_2 + 3x_2^2 - 2x_2 & -3x_2^3 + 2x_2^2 + 2 & -2x_1 - 3x_2 + 2 \end{pmatrix},$$

is a **parametrization of  $M$** , i.e.,  $M \cong D^{1 \times 3} Q \subseteq D^{1 \times 3}$ .

- Selecting the first two columns of  $Q$ , we obtain that

$$P = \begin{pmatrix} -4x_1 - 6x_2 + 4 & 6x_2^2 - 4x_2 \\ 4 & -4x_2 \\ 2x_1 x_2 + 3x_2^2 - 2x_2 & -3x_2^3 + 2x_2^2 + 2 \end{pmatrix},$$

is a **minimal parametrization of  $M$** , i.e.:

$$M \cong D^{1 \times 3} Q \subseteq D^{1 \times 2} = D^{1 \times \text{rank}_D(M)}.$$

## Example

- The parametrization  $P$  of  $M$  admits a **left-inverse** defined by:

$$T = \frac{1}{4} \begin{pmatrix} x_2^2 & 1 & 2x_2 \\ x_2 & 0 & 2 \end{pmatrix}.$$

- The set of generators  $\{y_j = \pi(f_j)\}_{j=1,2,3}$  of  $M$  satisfies

$$(x_1 x_2^2 + 1) y_1 + (3 x_2/2 + x_1 - 1) y_2 + 2 x_1 x_2 y_3 = 0,$$

and a basis  $\{z_1, z_2\}$  of  $M$  is defined by:

$$\begin{cases} z_1 = \frac{1}{4} (x_2^2 y_1 + y_2 + 2 x_2 y_3), \\ z_2 = \frac{1}{4} (x_2 y_1 + 2 y_3). \end{cases}$$

- The generators  $\{y_1, y_2, y_3\}$  can be written in the basis  $\{z_1, z_2\}$ :

$$\begin{cases} y_1 = (-4 x_1 - 6 x_2 + 4) z_1 + (6 x_2^2 - 4 x_2) z_2, \\ y_2 = 4 z_1 - 4 x_2 z_2, \\ y_3 = (2 x_1 x_2 + 3 x_2^2 - 2 x_2) z_1 + (-3 x_2^3 + 2 x_2^2 + 2) z_2. \end{cases}$$

# Injective cogenerator modules

- **Definition:** A left  $D$ -module  $\mathcal{F}$  is **injective** if

$$\forall q \geq 1, \quad \forall R \in D^q, \quad \forall \zeta \in \ker_{\mathcal{F}}(S.),$$

where  $\ker_D(\cdot R) = D^{1 \times r} S$ , there **exists**  $\eta \in \mathcal{F}$  **satisfying**  $R\eta = \zeta$ .

- **Definition:** If  $\mathcal{F}$  is a **injective left  $D$ -module**, then we have:

$$\text{ext}_D^i(M, \mathcal{F}) = 0, \quad \forall i \geq 1.$$

- **Definition:** A left  $D$ -module  $\mathcal{F}$  is **cogenerator** if:

$$\text{hom}_D(M, \mathcal{F}) = 0 \Rightarrow M = 0.$$

- **Proposition:** **Injective cogenerator left  $D$ -module** **always exists**.

- **Example:** If  $\Omega$  is an open convex subset of  $\mathbb{R}^n$ , then the  $\mathbb{C} \left[ \partial_1; \text{id}, \frac{\partial}{\partial x_1} \right] \dots \left[ \partial_n; \text{id}, \frac{\partial}{\partial x_n} \right]$ -modules  $C^\infty(\Omega)$ ,  $\mathcal{D}'(\Omega)$ ,  $\mathcal{A}(\Omega)$ ,  $\mathcal{O}(\Omega)$  and  $\mathcal{B}(\Omega)$  are injective cogenerators.

# Duality and parametrizations of systems

$$\begin{array}{ccccccc}
 D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 0 \longleftarrow N & \xleftarrow{\kappa} & D^q & \xleftarrow{R \cdot} & D^p & \xleftarrow{\pi^*} & \text{hom}_D(M, D) \longleftarrow 0 \\
 0 \longleftarrow N & \xleftarrow{\kappa} & D^q & \xleftarrow{R \cdot} & D^p & \xleftarrow{Q \cdot} & D^m \\
 D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\cdot Q} & D^{1 \times m} & & (*)
 \end{array}$$

- $t(M) = 0$  iff  $(*)$  is an exact sequence and  $Q \mathcal{F}^m \subseteq \ker_{\mathcal{F}}(R \cdot)$ .
- If  $t(M) = 0$  and  $\mathcal{F}$  is **injective**, then the exact sequence holds:

$$(**) \quad \mathcal{F}^q \xleftarrow{R \cdot} \mathcal{F}^p \xleftarrow{Q \cdot} \mathcal{F}^m, \quad \text{i.e.,} \quad \ker_{\mathcal{F}}(R \cdot) = Q \mathcal{F}^m.$$

- If  $t(M) = 0$  and  $\mathcal{F}$  is **injective cogenerator**, then:

$(*)$  is exact iff so is  $(**)$ .

# Involutions and adjoints

- **Definition:** A linear map  $\theta : D \longrightarrow D$  is an **involution** of  $D$  if:

$$\forall P, Q \in D : \theta(PQ) = \theta(Q)\theta(P), \quad \theta^2 = \text{id}.$$

- **Example:** 1. If  $D$  is a commutative ring, then  $\theta = \text{id}$ .
- 2. An involution of  $D = A \left[ \partial_1; \text{id}, \frac{\partial}{\partial x_1} \right] \dots \left[ \partial_n; \text{id}, \frac{\partial}{\partial x_n} \right]$  is:

$$\forall a \in A, \quad \theta(a(x)) = a(x), \quad \theta(\partial_i) = -\partial_i, \quad i = 1, \dots, n.$$

- 3. An involution of  $D = A \left[ \partial_1; \text{id}, \frac{d}{dt} \right] \left[ \partial_2; \delta, 0 \right]$  is defined by:

$$\forall a \in A, \quad \theta(a(t)) = a(-t), \quad \theta(\partial_i) = \partial_i, \quad i = 1, 2.$$

- The **adjoint** of  $R \in D^{q \times p}$  is defined by  $\theta(R) = (\theta(R_{ij}))^T \in D^{p \times q}$ .
- $\tilde{N} = D^{1 \times q} / (D^{1 \times p} \theta(R))$  is called the **adjoint** of  $M$ .

# $R$ has full row rank & PD case

Module $M$	Homological algebra	$\dim_D(\tilde{N})$
with torsion	$t(M) \cong \text{ext}_D^1(\tilde{N}, D)_\theta$	$\dim(D) - 1$
torsion-free	$\text{ext}_D^1(\tilde{N}, D)_\theta = 0$	$\dim(D) - 2$
reflexive	$\text{ext}_D^i(\tilde{N}, D)_\theta = 0, i = 1, 2$	$\dim(D) - 3$
projective	$\text{ext}_D^i(\tilde{N}, D)_\theta = 0, i = 1, \dots, \text{gld}(D)$	-1
free	Quillen-Suslin & Stafford	$\emptyset$

# Extension functor $\text{ext}_D^1(N, D)$

$$4. \quad \theta(P)z = y \implies Ry = 0 \quad 1.$$

$$\begin{array}{ccc} \uparrow & & \downarrow \\ \text{involution } \theta & & \text{involution } \theta \\ \uparrow & & \downarrow \end{array}$$

$$3. \quad 0 = P\mu \xleftrightarrow{\text{Gb}} \theta(R)\lambda = \mu \quad 2.$$

$$\begin{aligned} P \circ \theta(R) = 0 \implies \theta(P \circ \theta(R)) &= \theta^2(R) \circ \theta(P) \\ &= R \circ \theta(P) = 0. \end{aligned}$$

$$5. \quad \theta(P)z = y \xleftrightarrow{\text{Gb}} R'y = 0, \quad R' \in D^{q' \times p}.$$

$$\text{ext}_D^1(N, D) \cong (D^{1 \times q'} R') / (D^{1 \times q} R)$$

6. Using Gb, we can test whether or not  $\text{ext}_D^1(N, D) = 0$ .

$$\text{ext}_D^1(N, D) = 0 \implies Ry = 0 \Leftrightarrow y = Qz, \quad Q = \theta(P).$$

# Wind tunnel model (Manitius, IEEE TAC 84)

1. The w.t.m. is defined by the **underdetermined system**:

$$\begin{pmatrix} \partial_1 + a & -k a \partial_2 & 0 & 0 \\ 0 & \partial_1 & -1 & 0 \\ 0 & \omega^2 & \partial_1 + 2\zeta\omega & -\omega^2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ u(t) \end{pmatrix} = 0.$$

2. We compute  $\theta(R) = R^T$  and define  $\theta(R) \lambda = \mu$ :

$$\begin{cases} (\partial_1 + a) \lambda_1 = \mu_1, \\ -k a \partial_2 \lambda_1 + \partial_1 \lambda_2 + \omega^2 \lambda_3 = \mu_2, \\ -\lambda_2 + (\partial_1 + 2\zeta\omega) \lambda_3 = \mu_3, \\ -\omega^2 \lambda_3 = \mu_4. \end{cases} \quad (2)$$

(2) is **over-determined**  $\xrightarrow{\text{Gb}}$  **compatibility conditions**  $P \mu = 0$ .

# Wind tunnel model (Manitius, IEEE TAC 84)

3. We obtain the **compatibility condition**  $P \mu = 0$ :

$$\begin{pmatrix} \omega^2 k a \partial_2 & \omega^2 (\partial_1 - a) & \omega^2 (\partial_1^2 + a \partial_1) \\ (\partial_1^3 + 2 \zeta \omega \partial_1^2 + a \partial_1^2 + \omega^2 \partial_1 + 2 a \zeta \omega \partial_1 + a \omega^2) \end{pmatrix} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_4 \end{pmatrix} = 0.$$

4. We consider the **overdetermined system**  $P^T z = y$ .

$$\begin{cases} \omega^2 k a \partial_2 z = x_1, \\ \omega^2 (\partial_1 - a) z = x_2, \\ \omega^2 (\partial_1^2 + a \partial_1) z = x_3, \\ (\partial_1^3 + (2 \zeta \omega + a) \partial_1^2 + (\omega^2 + 2 a \omega \zeta) \partial_1 + a \omega) z = u. \end{cases} \quad (4)$$

5. The **compatibility conditions** of  $P^T z = y$  are **exactly generated** by  $R y = 0$  and (4) is a **parametrization** of the w.t.m.

1. The **model of a moving tank** is defined by:

$$\begin{pmatrix} \partial_1 & -\partial_1 \partial_2^2 & a \partial_1^2 \partial_2 \\ \partial_1 \partial_2^2 & -\partial_1 & a \partial_1^2 \partial_2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = 0.$$

2. We compute  $\theta(R) = R^T$  and define  $\theta(R) \lambda = \mu$ :

$$\begin{cases} \partial_1 \lambda_1 + \partial_1 \partial_2^2 \lambda_2 = \mu_1, \\ -\partial_1 \partial_2^2 \lambda_1 - \partial_1 \lambda_2 = \mu_2, \\ a \partial_1^2 \partial_2 \lambda_1 + a \partial_1^2 \partial_2 \lambda_2 = \mu_3. \end{cases} \quad (2)$$

- (2) is **overdetermined**  $\xrightarrow{\text{Gb}}$  **compatibility conditions**  $P \mu = 0$ .

# Moving tank (Petit, Rouchon, IEEE TAC 02)

3. We obtain the **compatibility condition**  $P \mu = 0$ :

$$\begin{pmatrix} a \partial_1 \partial_2 & -a \partial_1 \partial_2 & -(1 + \partial_2^2) \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = 0.$$

4. We consider the **overdetermined system**  $P^T z = y$ .

$$\begin{cases} a \partial_1 \partial_2 z = y_1, \\ -a \partial_1 \partial_2 z = y_2, \\ -(1 + \partial_2^2) z = y_3. \end{cases} \quad (4)$$

5. The **compatibility conditions** of  $P^T z = y$  are  $R' y = 0$ :

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & (1 + \partial_2^2) & -a \partial_1 \partial_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0.$$

$$t(M) \cong \text{ext}_D^1(N, D) \cong \left( D^{1 \times 2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 + \partial_2^2 & -a \partial_1 \partial_2 \end{pmatrix} \right) / \left( D^{1 \times 2} \begin{pmatrix} \partial_1 & -\partial_1 \partial_2^2 & a \partial_1^2 \partial_2 \\ \partial_1 \partial_2^2 & -\partial_1 & a \partial_1^2 \partial_2 \end{pmatrix} \right)$$

$$\begin{cases} y_1 + y_2 = z_1, \\ \partial_1 y_1 - \partial_1 \partial_2^2 y_2 + a \partial_1^2 \partial_2 y_3 = 0, \\ \partial_1 \partial_2^2 y_1 - \partial_1 y_2 + a \partial_1^2 \partial_2 y_3 = 0, \end{cases} \xrightarrow{\text{Gb}} \begin{cases} \partial_1 (\partial_2^2 - 1) z_1 = 0. \end{cases}$$

$$\begin{cases} (1 + \partial_2^2) y_2 - a \partial_1 \partial_2 y_3 = z_2, \\ \partial_1 y_1 - \partial_1 \partial_2^2 y_2 + a \partial_1^2 \partial_2 y_3 = 0, \\ \partial_1 \partial_2^2 y_1 - \partial_1 y_2 + a \partial_1^2 \partial_2 y_3 = 0, \end{cases} \xrightarrow{\text{Gb}} \begin{cases} \partial_1 (\partial_2^2 - 1) z_2 = 0. \end{cases}$$

$\Rightarrow z_1(t)$  and  $z_2(t)$  are autonomous elements.

## Examples: reflexive modules

- div-curl-grad:  $\vec{\nabla} \cdot \vec{B} = 0 \Leftrightarrow \vec{B} = \vec{\nabla} \wedge \vec{A}, \vec{\nabla} \wedge \vec{A} = \vec{0} \Leftrightarrow \vec{A} = \vec{\nabla} f.$
- First group of Maxwell equations:

$$\left\{ \begin{array}{l} \frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla} \cdot \vec{B} = 0, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V, \\ \vec{B} = \vec{\nabla} \wedge \vec{A}. \end{array} \right.$$

$$\left\{ \begin{array}{l} -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V = \vec{0}, \\ \vec{\nabla} \wedge \vec{A} = \vec{0}, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \vec{A} = \vec{\nabla} \xi, \\ V = -\frac{\partial \xi}{\partial t}. \end{array} \right.$$

- 3D stress tensor: Maxwell, Morera parametrizations. . .
- Linearized Einstein equations ( $10 \times 10$  system of PDEs)?

$\Rightarrow$  **OREMODULES** (Chyzak, Robertz, Q.)

# Constructive version of Quillen-Suslin theorem

- If  $M$  is a **projective** left  $D$ -module  $M$  which admits a **finite free resolution**, then  $M$  is **stably free** (Serre's theorem).

$\Rightarrow M$  can be rewritten as  $M = D^{1 \times p'} / (D^{1 \times q'} R')$ ,  $R' S = I_{q'}$ .

- $M$  is **free** iff the following **completion problem** can be solved:

$$\begin{pmatrix} R' \\ T \end{pmatrix} \begin{pmatrix} S & Q \end{pmatrix} = \begin{pmatrix} I_{q'} & 0 \\ 0 & I_{p'-q'} \end{pmatrix} = I_{p'}.$$

- A **constructive proof** of the Quillen-Suslin theorem was implemented by Fabiańska in the package **QUILLEN****SUSLIN**.

$$\begin{cases} \dot{x}_1(t) + x_1(t) - u(t) = 0, \\ \dot{x}_2(t) - \dot{x}_2(t-h) - x_1(t) + a x_2(t) = 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1(t) = \dot{z}(t) - z(t-h) + a z(t), \\ x_2(t) = z(t), \\ u(t) = \ddot{z}(t) + \dot{z}(t) - \dot{z}(t-h) - z(t-h) + a \dot{z}(t) + a z(t). \end{cases}$$

# Constructive version of Stafford's theorem

- The time-varying linear control system (Sontag)

$$\begin{cases} \dot{x}_1(t) - t u_1(t) = 0, \\ \dot{x}_2(t) - u_2(t) = 0, \end{cases}$$

is **injectively parametrized** by (STAFFORD (Robertz, Q.))

$$\begin{cases} x_1(t) = t^2 \xi_1(t) - t \dot{\xi}_2(t) + \xi_2(t), \\ x_2(t) = t(t+1) \xi_1(t) - (t+1) \dot{\xi}_2(t) + \xi_2(t), \\ u_1(t) = t \dot{\xi}_1(t) + 2 \xi_1(t) - \ddot{\xi}_2(t), \\ u_2(t) = t(t+1) \dot{\xi}_1(t) + (2t+1) \xi_1(t) - (t+1) \ddot{\xi}_2(t), \end{cases}$$

and  $\{\xi_1, \xi_2\}$  is a **basis** of the **free** left  $A_1(\mathbb{Q})$ -module  $M$  as:

$$\begin{cases} \xi_1(t) = (t+1) u_1(t) - u_2(t), \\ \xi_2(t) = (t+1) x_1(t) - t x_2(t). \end{cases}$$

- Idem for  $\partial_1 y_1 + \partial_2 y_2 + \partial_3 y_3 + x_3 y_1 = 0$ .

# Dictionary systems – modules

<b>Module</b> $M = D^{1 \times p} / (D^{1 \times q} R)$	<b>Structural properties</b> $\ker_{\mathcal{F}}(R \cdot)$ $\mathcal{F}$ injective cogenerator	<b>Stabilization problems</b> <b>Optimal control</b>
<b>Torsion</b>	Autonomous system Poles/zeros classifications	
<b>With torsion</b>	Existence of autonomous elements	
<b>Torsion-free</b>	No autonomous elements, Controllability, Parametrizability, $\pi$ -freeness	Variational problem without constraints (Euler-Lagrange equations)
<b>Projective</b>	Bézout identities, Internal stabilizability	Computation of Lagrange multipliers without integration Existence of a parametrization all stabilizing controllers
<b>Free</b>	Flatness, Poles placement, Doubly coprime factorization	Youla-Kučera parametrization Optimal controller

# Projectiveness, observability and controllability

- **Theorem:** If  $R \in D^{q \times p}$  has full row rank, then the left  $D$ -module  $M = D^{1 \times p} / (D^{1 \times q} R)$  is **projective** iff:

$$N = D^{1 \times q} / (D^{1 \times p} \theta(R)) = 0 \Leftrightarrow \exists S \in D^{p \times q} : RS = I_q.$$

- Let  $D = \mathcal{A}(I) \left[ \partial; \text{id}, \frac{d}{dt} \right]$  and  $R = (\partial I_n - A \quad -B) \in D^{n \times (n+m)}$ .  
 $M = D^{1 \times (n+m)} / (D^{1 \times n} R)$  is **projective** iff  $\theta(R) \lambda = 0 \Leftrightarrow \lambda = 0$ :

$$\begin{cases} -\partial \lambda - A^T \lambda = 0, \\ -B^T \lambda = 0, \end{cases} \Rightarrow \begin{cases} \partial \lambda = -A^T \lambda, \\ B^T \lambda = 0, \\ B^T \partial \lambda + \dot{B}^T \lambda = (-B^T A^T + \dot{B}^T) \lambda = 0. \end{cases}$$

Hence,  $M$  is **projective** iff, for all  $t_0 \in I$ , we have:

$$\text{rank}_{\mathbb{R}}(B \mid AB - \dot{B} \mid A^2 B + \dots \mid A^{n-1} B + \dots \mid \dots)(t_0) = n.$$

- $D^{1 \times p} / (D^{1 \times q} (P(\partial) - Q(\partial)))$  **proj.** iff  $P(\partial) X(\partial) + Q(\partial) Y(\partial) = I_q$ .

# Flatness: two pendula mounted on a car

- We consider two pendula mounted on a car:

$$\begin{cases} m_1 L_1 \ddot{w}_1(t) + m_2 L_2 \ddot{w}_2(t) - w_3(t) + (M + m_1 + m_2) \ddot{w}_4(t) = 0, \\ m_1 L_1^2 \ddot{w}_1(t) - m_1 L_1 g w_1(t) + m_1 L_1 \ddot{w}_4(t) = 0, \\ m_2 L_2^2 \ddot{w}_2(t) - m_2 L_2 g w_2(t) + m_2 L_2 \ddot{w}_4(t) = 0. \end{cases} \quad (\star)$$

- $(\star)$  is **parametrizable** iff  $L_1 \neq L_2$ .
- If  $L_1 \neq L_2$  then a **parametrization** of  $(\star)$  is defined by:

$$\begin{cases} w_1(t) = -L_2 \xi^{(4)}(t) + g \ddot{\xi}(t), \\ w_2(t) = -L_1 \xi^{(4)}(t) + g \ddot{\xi}(t), \\ w_3(t) = L_1 L_2 M \xi^{(6)}(t) - (L_1 m_2 + L_2 m_1 + g(L_1 + L_2) M) \xi^{(4)}(t) \\ \quad + g^2 (m_1 + m_2 + M) \xi^{(2)}(t) \\ w_4(t) = L_1 L_2 \xi^{(4)}(t) - g(L_1 + L_2) \ddot{\xi}(t) + g^2 \xi(t). \end{cases}$$

- The parametrization of  $(\star)$  is **injective** as we have:

$$\xi(t) = \frac{1}{g^2(L_1 - L_2)} (L_1^2 w_1(t) - L_2^2 w_2(t) + (L_1 - L_2) w_4(t)). \quad (\star\star)$$

# Flatness: two pendula mounted on a car

- **Patching problem**  $\Leftrightarrow$  **controllability**:  $T > 0$ .

$w^p = (w_1^p, w_2^p, w_3^p, w_4^p)$  a **past trajectory** of  $(\star)$  on  $] -\infty, 0[$ .

$w^f = (w_1^f, w_2^f, w_3^f, w_4^f)$  a **future trajectory** of  $(\star)$  on  $]T, +\infty[$ .

$\Rightarrow \exists w = (w_1, w_2, w_3, w_4) \in C^\infty(\mathbb{R})^4$  trajectory of  $(\star)$  :

$$\begin{cases} w|_{]-\infty, 0[} = w^p, \\ w|_{]T, +\infty[} = w^f. \end{cases}$$

- Using the **flat output**

$$\xi(t) = \frac{1}{g^2(L_1 - L_2)} (L_1^2 w_1(t) - L_2^2 w_2(t) + (L_1 - L_2) w_4(t))$$

and the parametrization, it is **enough to find**  $\xi \in C^\infty(\mathbb{R})$  s.t.:

$$\xi|_{]-\infty, 0[} = \xi^p \quad \& \quad \xi|_{]T, +\infty[} = \xi^f.$$

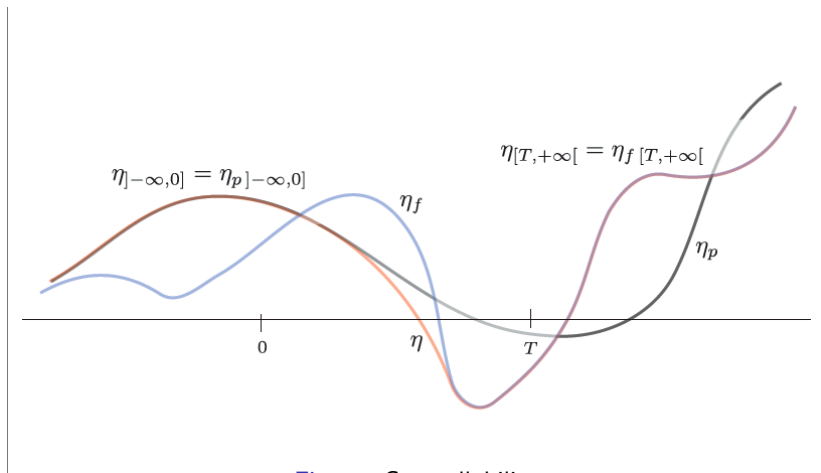


Figure: Controllability

# Autonomous elements & controllability

- Stirred tank model with  $c_2 = c_1 = c_0$

$$\begin{cases} \dot{x}_1(t) = -\frac{1}{2\theta} x_1(t) + u_1(t) + u_2(t), \\ \dot{x}_2(t) = -\frac{1}{\theta} x_2(t) + \left(\frac{c_1 - c_0}{V_0}\right) u_1(t - h) + \left(\frac{c_2 - c_0}{V_0}\right) u_2(t - h), \end{cases}$$

$$\dot{x}_2(t) + \frac{1}{\theta} x_2(t) = 0 \quad \Leftrightarrow \quad \left(\partial + \frac{1}{\theta}\right) x_2 = 0.$$

$\Rightarrow x_2$  **cannot be controlled** using the inputs  $u_1$  and  $u_2$ .

# Autonomous elements & controllability

- Stirred tank model with  $c_2 = c_1 = c_0$

$$\begin{cases} \dot{x}_1(t) = -\frac{1}{2\theta} x_1(t) + u_1(t) + u_2(t), \\ \dot{x}_2(t) = -\frac{1}{\theta} x_2(t) + \left(\frac{c_1 - c_0}{V_0}\right) u_1(t - h) + \left(\frac{c_2 - c_0}{V_0}\right) u_2(t - h), \end{cases}$$
$$\dot{x}_2(t) + \frac{1}{\theta} x_2(t) = 0 \quad \Leftrightarrow \quad \left(\partial + \frac{1}{\theta}\right) x_2 = 0.$$

$\Rightarrow x_2$  **cannot be controlled** using the inputs  $u_1$  and  $u_2$ .

- Moving tank model (Petit, Rouchon, IEEE TAC 02):

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t - 2h) + \alpha \ddot{y}_3(t - h) = 0, \\ \dot{y}_1(t - 2h) - \dot{y}_2(t) + \alpha \ddot{y}_3(t - h) = 0, \end{cases}$$

$$\Rightarrow \begin{cases} z_1(t) = y_1(t) + y_2(t), & \frac{d}{dt} (1 - \delta^2) z_i(t) = 0, \\ z_2(t) = y_2(t) + y_2(t - 2h) - a \dot{y}_3(t - h), & i = 1, 2. \end{cases}$$

# Flat linear functional systems

$$\begin{cases} \dot{x}_1(t) + x_1(t) - u(t) = 0, \\ \dot{x}_2(t) - \dot{x}_2(t-h) - x_1(t) + a x_2(t) = 0. \end{cases} \quad (\star)$$

# Flat linear functional systems

$$\begin{cases} \dot{x}_1(t) + x_1(t) - u(t) = 0, \\ \dot{x}_2(t) - \dot{x}_2(t-h) - x_1(t) + a x_2(t) = 0. \end{cases} \quad (\star)$$

$$M = D^{1 \times 3} / (D^{1 \times 2} R) : \begin{cases} \partial_1 x_1 + x_1 - u = 0, \\ \partial_1 x_2 - \partial_1 \partial_2 x_2 - x_1 + a x_2 = 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} u = (\partial_1 + 1) x_1, \\ x_1 = (\partial_1 - \partial_1 \partial_2 + a) x_2, \end{cases} \Leftrightarrow \begin{cases} u = (\partial_1 + 1) (\partial_1 - \partial_1 \partial_2 + a) x_2, \\ x_1 = (\partial_1 - \partial_1 \partial_2 + a) x_2. \end{cases}$$

$\Rightarrow M$  is a **free**  $D = \mathbb{Q}(a) [\partial_1; \text{id}, \frac{d}{dt}] [\partial_2; \delta, 0]$ -module of basis  $\{z_2\}$ .

# Flat linear functional systems

$$\begin{cases} \dot{x}_1(t) + x_1(t) - u(t) = 0, \\ \dot{x}_2(t) - \dot{x}_2(t-h) - x_1(t) + a x_2(t) = 0. \end{cases} \quad (\star)$$

$$M = D^{1 \times 3} / (D^{1 \times 2} R) : \begin{cases} \partial_1 x_1 + x_1 - u = 0, \\ \partial_1 x_2 - \partial_1 \partial_2 x_2 - x_1 + a x_2 = 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} u = (\partial_1 + 1) x_1, \\ x_1 = (\partial_1 - \partial_1 \partial_2 + a) x_2, \end{cases} \Leftrightarrow \begin{cases} u = (\partial_1 + 1) (\partial_1 - \partial_1 \partial_2 + a) x_2, \\ x_1 = (\partial_1 - \partial_1 \partial_2 + a) x_2. \end{cases}$$

$\Rightarrow M$  is a **free**  $D = \mathbb{Q}(a) [\partial_1; \text{id}, \frac{d}{dt}] [\partial_2; \delta, 0]$ -module of basis  $\{z_2\}$ .

$$(\star) \Leftrightarrow$$

$$\begin{cases} x_1(t) = \dot{x}_2(t) - x_2(t-h) + a x_2(t), \\ x_2(t) = x_2(t), \\ u(t) = \ddot{x}_2(t) + \dot{x}_2(t) - \dot{x}_2(t-h) - x_2(t-h) + a \dot{x}_2(t) + a x_2(t). \end{cases}$$

# Flat linear functional systems

$$\begin{cases} \dot{x}_1(t) + x_1(t) - u(t) = 0, \\ \dot{x}_2(t) - \dot{x}_2(t-h) - x_1(t) + a x_2(t) = 0. \end{cases} \quad (\star)$$

$$(\star) \Leftrightarrow$$

$$\begin{cases} x_1(t) = \dot{x}_2(t) - x_2(t-h) + a x_2(t), \\ x_2(t) = x_2(t), \\ u(t) = \ddot{x}_2(t) + \dot{x}_2(t) - \dot{x}_2(t-h) - x_2(t-h) + a \dot{x}_2(t) + a x_2(t). \end{cases}$$

- $y = x_2$ : output  $\Rightarrow u(t) = \phi(y(t), \dot{y}(t), \ddot{y}(t), \dot{y}(t-h))$ .

- **Flexible rod with a torque** (Mounier 95):

$$\begin{cases} \sigma^2 \frac{\partial^2 q(\tau, x)}{\partial \tau^2} - \frac{\partial^2 q(\tau, x)}{\partial x^2} = 0, \\ \frac{\partial q}{\partial x}(\tau, 0) = -u(\tau), \\ \frac{\partial q}{\partial x}(\tau, L) = -J \frac{\partial^2 q}{\partial \tau^2}(\tau, L), \\ y(\tau) = q(\tau, L). \end{cases} \quad (\star)$$

- $q(\tau, x) = \phi(\tau + \sigma x) + \psi(\tau - \sigma x)$ ,  $t = (\sigma/J)\tau$ ,  $v = (2J/\sigma^2)u$ ,

$$(\star) \Rightarrow \ddot{y}(t+1) + \ddot{y}(t-1) + \dot{y}(t+1) - \dot{y}(t-1) = v(t)$$

$$\Leftrightarrow \begin{cases} y(t) = \xi(t-1), \\ v(t) = \ddot{\xi}(t) + \ddot{\xi}(t-2) + \dot{\xi}(t) - \dot{\xi}(t-2). \end{cases}$$

- If  $y_r$  is a **desired trajectory** then  $\xi_r(t) = y_r(t+1)$  and we obtain the **open-loop control law**:

$$v_r(t) = \ddot{y}_r(t+1) + \ddot{y}_r(t-1) + \dot{y}_r(t+1) - \dot{y}_r(t-1).$$

# Optimal control

- Let us **minimize**  $\frac{1}{2} \int_0^T (x(t)^2 + u(t)^2) dt$  (1) under:

$$\dot{x}(t) + x(t) - u(t) = 0, \quad x(0) = x_0. \quad (2)$$

- (2) is **parametrized** by 
$$\begin{cases} x(t) = \xi(t), \\ u(t) = \dot{\xi}(t) + \xi(t). \end{cases} \quad (3)$$

- (1) & (3)  $\Rightarrow \min \frac{1}{2} \int_0^T (\xi(t)^2 + (\dot{\xi}(t) + \xi(t))^2) dt,$

$$\Rightarrow \text{Euler-Lagrange equations} \quad \begin{cases} \xi(t) = x(t), \\ \dot{\xi}(t) + \xi(t) = u(t), \\ \ddot{\xi}(t) - 2\xi(t) = 0, \\ \dot{\xi}(T) + \xi(T) = 0, \\ \xi(0) = x_0, \end{cases}$$

$$\Rightarrow u(t) = \frac{-e^{\sqrt{2}(t-T)} + e^{-\sqrt{2}(t-T)}}{(1 - \sqrt{2})e^{\sqrt{2}(t-T)} - (1 + \sqrt{2})e^{-\sqrt{2}(t-T)}} x(t).$$

# Variational problems

- Let us extremize **the electromagnetic action**

$$\int \left( \frac{1}{2\mu_0} \|\vec{B}\|^2 - \frac{\epsilon_0}{2} \|\vec{E}\|^2 \right) dx_1 dx_2 dx_3 dt, \quad (1)$$

where  $\vec{B}$  and  $\vec{E}$  satisfy:

$$\begin{cases} \frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla} \cdot \vec{B} = 0, \end{cases} \Leftrightarrow \begin{cases} \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}, \\ \vec{B} = \vec{\nabla} \wedge \vec{A}. \end{cases} \quad (3)$$

- Substituting (3) in (1) and using **Lorentz gauge**

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0, \quad c^2 = 1/(\epsilon_0 \mu_0),$$

$$\Rightarrow \begin{cases} \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \Delta \vec{A} = 0, \\ \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} - \Delta V = 0. \end{cases} \quad (\text{electromagnetic waves}).$$

# General Monge problem

- If  $\mathcal{F}$  be a left  $D$ -module then we have the following equivalence:

$$R\eta = 0 \Leftrightarrow \begin{cases} R''\tau = 0, \\ R'_2\tau = 0, \\ R'\eta = \tau. \end{cases}$$

$\Rightarrow$  integration of  $R\eta = 0$  in cascade:

- 1 Find the general solution  $\bar{\tau} \in \mathcal{F}^p$  of the autonomous system:

$$\begin{cases} R''\tau = 0, \\ R'_2\tau = 0. \end{cases}$$

- 2 Find a particular solution  $\eta^* \in \mathcal{F}^p$  of  $R'\eta = \bar{\tau}$ .
- 3 If  $\mathcal{F}$  is injective then  $\ker_{\mathcal{F}}(R'.) = Q\mathcal{F}^m$  and:

$$\forall \eta \in \ker_{\mathcal{F}}(R.), \exists \xi \in \mathcal{F}^m : \eta = \eta^* + Q\xi.$$

# General Monge problem

- The following canonical short exact sequence **splits**,

$$0 \longrightarrow t(M) \xrightarrow{i} M \xrightarrow{\rho} M/t(M) \longrightarrow 0, \text{ i.e., } \Leftrightarrow M \cong t(M) \oplus M/t(M),$$

iff there exist  $X \in D^{p \times q'}$ ,  $Y \in D^{q' \times q}$  and  $Z \in D^{q' \times r'}$  such that:

$$R' X + (Y \quad Z) \begin{pmatrix} R'' \\ R'_2 \end{pmatrix} = I_{q'}. \quad (*)$$

- **Remark:**  $(*) \Leftrightarrow R' - R' X R' = Y R$ , i.e., iff  $R'$  admits a generalized inverse modulo  $D^{q' \times q} R$ .

- **Remark:** Applying  $(*)$  to  $\bar{\tau}$ , we get:

$$\bar{\tau} = R'(X \bar{\tau}), \quad \text{i.e., } \eta^* = X \bar{\tau}.$$

- $M/t(M)$  is **projective** iff  $R'$  admits a **generalized inverse**.
- If  $(*)$  or  $(**)$  is satisfied and  $\mathcal{F}$  is injective, then:

$$\forall \eta \in \ker_{\mathcal{F}}(R.), \exists \xi \in \mathcal{F}^m : \eta = X \bar{\tau} + Q \xi.$$

# Kronecker product

- Let  $D$  be a **commutative polynomial ring**.
- The **Kronecker product** of  $E \in D^{q \times p}$  and  $F \in D^{r \times s}$  is:

$$E \otimes F = \begin{pmatrix} E_{11} F & \dots & E_{1p} F \\ \vdots & \vdots & \vdots \\ E_{q1} F & \dots & E_{qp} F \end{pmatrix} \in D^{(qr) \times (ps)}.$$

**Lemma:** If  $U \in D^{a \times b}$ ,  $V \in D^{b \times c}$  and  $W \in D^{c \times d}$ , then we have:

$$\text{row}(U V W) = \text{row}(V) (U^T \otimes W),$$

- $\text{row}(R' X) = \text{row}(R' X I_{q'}) = \text{row}(X) (R'^T \otimes I_{q'})$ .

- $\text{row} \left( I_{q'} (Y \ Z) \begin{pmatrix} R'' \\ R'_2 \end{pmatrix} \right) = \text{row}(Y \ Z) \left( I_{q'} \otimes \begin{pmatrix} R'' \\ R'_2 \end{pmatrix} \right)$ .

$$\Rightarrow \text{row}(I_{q'}) = (\text{row}(X) \ \text{row}(Y \ Z)) \begin{pmatrix} R'^T \otimes I_{q'} \\ I_{q'} \otimes \begin{pmatrix} R'' \\ R'_2 \end{pmatrix} \end{pmatrix} \text{ (GB)}.$$

# OREMODULES (Chyzak, Q., Robertz)

- **OREMODULES** is a tool-box developed in *Maple*.
- **OREMODULES** uses *Ore\_algebra* developed by Chyzak.
- **OREMODULES** handles linear systems of ODEs, PDEs, discrete equations, differential time-delay equations. . .
- **OREMODULES** computes:
  1. free resolutions,  $\text{ext}_D^i(\cdot, D)$ , projective dim., Hilbert series,
  2. torsion elements, autonomous elements,
  3. parametrizations of under-determined systems,
  4. left-/right-/generalized inverses,
  5. bases, flat outputs,  $\pi$ -polynomials,
  6. first integrals of motion, Euler-Lagrange equations. . .

<http://wwwb.math.rwth-aachen.de/OreModules/>

# Conclusion

- Based on algebraic analysis, constructive homological algebra and Ore algebras, we have developed a general **non-commutative polynomial approach to functional linear systems**.
- The different results are implemented in the packages:

OREMODULES, STAFFORD, QUILLENUSULIN.

This approach allowed us to:

- 1 Develop an intrinsic approach (independent of the form).
- 2 Develop generic algorithms and generic implementations.
- 3 Constructively study the parametrizability problem.
- 4 Solve conjectures in mathematical systems theory.