Performance Evaluation... of a Laser

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Gelenbe Symposium 2015/ISCIS 2015
Imperial College, London, 23 September 2015
(with typos corrected)

Contents

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Purpose of the talk

Show how techniques commonly used in "performance evaluation" can be used to (re)analyze a model from statistical physics, hopefully providing new insights:

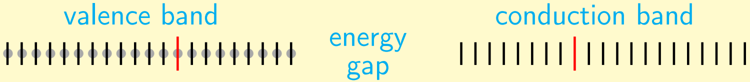
- reversibility, product forms
- combinatorics, generating functions
- QBDs, linear algebra

A work much in progress with intriguing open questions.

Progress

Semi-conductor lasers

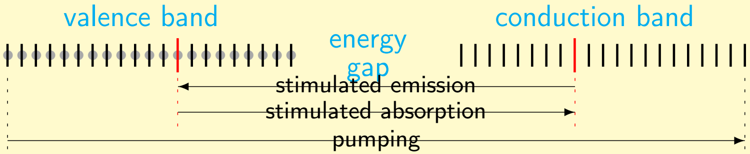
Electrons, two bands of energy levels, 0-1 electron by level



(lasing levels)

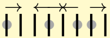
Semi-conductor lasers

Electrons, two bands of energy levels, 0-1 electron by level



(lasing levels)

Other/further moves:

upward and downward thermalization 

(Auger effect, coherent pumping, spontaneous emission, ...)

The model, ctd.

Ingredients:

- $2B$ energy levels, forming two "bands"; at most one electron occupies an energy level;
- two special "lasing levels" L et ℓ in the middle of both bands
- a cavity: supports photons.

Transitions (changes of state):

- electrons may change level in each band (thermalization)
- one electron may emit one photon, and passes simultaneously from L to ℓ
- one electron may absorb a photon, and passes simultaneously from ℓ à L
- photons may exit the cavity
- "pumping" makes an electron pass from the lowest occupied level to the highest unoccupied energy level.

Choice of the state space

What is a state of the system? At least two choices:

electron-centric : each electron has a state, its energy level $\in [0..2B-1]$. The number of photons is *a priori* unbounded.

$$\Longrightarrow \qquad \mathcal{E} \subset [0..2B-1]^N \times \mathbb{N}.$$

But the exclution constraint has to be enforced.

level-centric : each energy level contains at most one electron.

$$\Longrightarrow \qquad \mathcal{E} \subset \{0,1\}^{2B} \times \mathbb{N}.$$

Since the total number of electrons is fixed:

$$\Longrightarrow \qquad \mathcal{E} \subset \left\{ \sigma \in \{0,1\}^{2B} \mid \sum_{i=0}^{2B-1} \sigma_i = N \right\} \times \mathbb{N}.$$

Transition Rates

From state $(\boldsymbol{\sigma}, m)$:

destination	rate	constraints	event
$(\boldsymbol{\sigma}, m-1)$	$m\alpha$	$m > 0$	laser emission
$(\boldsymbol{\sigma}-e_L + e_\ell, m+1)$	$(m+1)/T$	$\sigma_L = 1, \sigma_\ell = 0$	stimulated emission
$(\boldsymbol{\sigma}+e_L - e_\ell, m-1)$	m/T	$\sigma_L = 0, \sigma_\ell = 1$	stimulated absorption
$(\boldsymbol{\sigma}+e_{i+1} - e_i, m)$	p	$\sigma_i = 1, \sigma_{i+1} = 0$	thermalization, down
$(\boldsymbol{\sigma}+e_i - e_{i+1}, m)$	pq	$\sigma_i = 0, \sigma_{i+1} = 1$	thermalization, up
$(\boldsymbol{\sigma}+e_0 - e_{2B}, m)$	J	$\sigma_0 = 0, \sigma_{2B} = 1$	pumping

Physical names: p lattice coupling, $q < 1$ Boltzmann temperature.

Modeling objectives

Metrics of interest: in the stationary regime

- probability of occupancy of energy levels
- distribution of photons in the cavity
- output rate of photons
- spectral density of photon emission process: Poissonian or sub-Poissonian?

Decomposition

The model is fine for simulation although...
The time scales of thermalization and other events (emission, absorption, light, pumping) are very different.

Could the process be seen as:

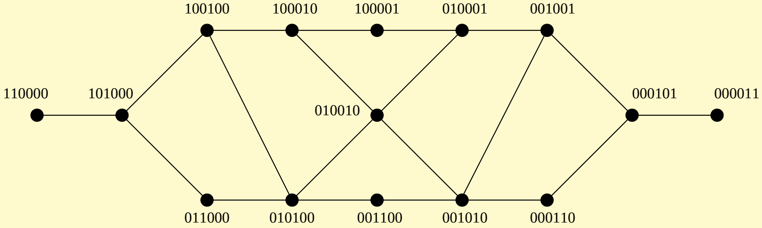
- 2 independent processes in each of the bands
- coupled by *rare* events

$\Longrightarrow$ focus on each band
$\Longrightarrow$ decompose the model by assuming each band stationary

Progress

Generators

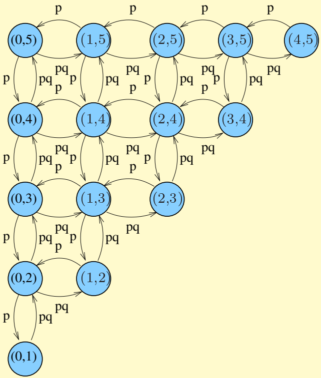
Example of a transition diagram: $B = 6$ levels, $N = 2$ particles.



All transitions represented are two-way. Unwritten transition rates: rate p for transitions to the left (lower energy), pq to the right (higher energy).

And what happens if we adopt the electron-centric model?

$$\mathcal{E} \;=\; \left\{ \mathbf{X} \in [0..2B-1]^N \mid X_1 < X_2 < \ldots < X_N \right\}$$



For instance, still with $B = 6$, $N = 2$, a regular, 2-dimensional grid-like structure

Stationary distribution

The stationary distribution can be computed in *closed form* as:

$$\pi_{B,N}(\boldsymbol{\sigma}) = \frac{1}{Z(B,N)} \prod_{i=0}^{N-1} q^{i\sigma_i} = \frac{1}{Z(B,N)} q^{\sum_{i=0}^{N-1} i\sigma_i} = \frac{q^{U\boldsymbol{\sigma}}}{Z(B,N)} .$$

The *partition function* $Z(B,N)$ is given by:

$$\begin{aligned} Z(B,N) &= \sum_{\boldsymbol{\sigma} \in \mathcal{S}_{B,N}} \prod_{i=0}^{N-1} q^{i\sigma_i} \\ &= \frac{q^{N-1} - q^B}{1 - q^N} \frac{q^{N-2} - q^B}{1 - q^{N-1}} \cdots \frac{1 - q^B}{1 - q} . \end{aligned}$$

Proofs:

- direct check of balance equations
- reversibility and truncation

Occupancies

Computation of $n_i = \mathbb{P}(\sigma_i = 1)$.
Identities:

$$\pi_{B,N}(\sigma_i = 1) \;=\; \sum_{k=0}^{i} q^{(i+1)(N-k)-1} \, \frac{Z(i,k)\; Z(B-1-i, N-k-1)}{Z(B,N)}$$

Recurrences:

$$\pi_{B,N}(\sigma_i = 1) \;=\; q^i \; \frac{1-q^N}{q^{N-1}-q^B} \; (1 - \pi_{B,N-1}(\sigma_i = 1))$$

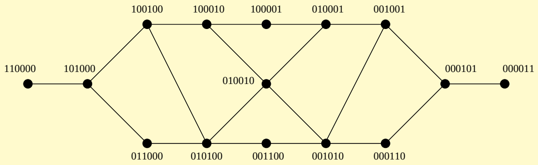
$\Longrightarrow$ fast computations; no enumeration of the state space

Speed of convergence

Speed of convergence $\leftrightarrow$ spectrum of generators.
Structure of the matrices?
Order states lexicographically: matrices can be represented as

$$M_{B,N} = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$M_{B-1,N-1}$}} & \mathbf{0}_{p\times q} & \mathbf{0}_{p\times r} \\ \cline{3-4} \multicolumn{2}{c|}{} & \begin{matrix}\lambda & & \\ & \ddots & \\ & & \lambda\end{matrix} & \mathbf{0}_{q\times r} \\ \hline \mathbf{0}_{q\times p} & \begin{matrix}\mu & & \\ & \ddots & \\ & & \mu\end{matrix} & \multicolumn{2}{c}{\multirow{2}{*}{$M_{B-1,N}$}} \\ \cline{1-2} \mathbf{0}_{r\times p} & \mathbf{0}_{r\times q} & \multicolumn{2}{c}{} \end{array}\right).$$

But if we order by energy level, we get a “quasi-birth-death” process



The matrix is then: $M_{6,2} =$

$$
\left(\begin{array}{c|c|cc|cc|ccc|cc|cc|c|c}
- & \lambda & & & & & & & & & & & & & \\ \hline
\mu & - & \lambda & \lambda & & & & & & & & & & & \\ \hline
 & \mu & - & 0 & \multicolumn{2}{c|}{\lambda} & & & & & & & & & \\
 & \mu & 0 & - & \multicolumn{2}{c|}{\lambda} & & & & & & & & & \\ \hline
 & & \mu & 0 & - & 0 & \lambda & \lambda & 0 & & & & & & \\
 & & \mu & \mu & 0 & - & 0 & \lambda & \lambda & & & & & & \\ \hline
 & & & & \mu & 0 & - & 0 & 0 & \lambda & 0 & & & & \\
 & & & & \mu & \mu & 0 & - & 0 & \lambda & \lambda & & & & \\
 & & & & 0 & \mu & 0 & 0 & - & 0 & \lambda & & & & \\ \hline
 & & & & & & \mu & \mu & 0 & - & 0 & \lambda & 0 & & \\
 & & & & & & 0 & \mu & \mu & 0 & - & \lambda & \lambda & & \\ \hline
 & & & & & & & & & \mu & \mu & - & 0 & \lambda & \\
 & & & & & & & & & 0 & \mu & 0 & - & \lambda & \\ \hline
 & & & & & & & & & & & \mu & \mu & - & \lambda \\ \hline
 & & & & & & & & & & & & & \mu & -
\end{array}\right).
$$

Spectra, end

The basis matrix is:

$$M_{B,1} \; = \; \begin{pmatrix} -\lambda & \lambda & & & \\ \mu & -(\lambda+\mu) & \lambda & & \\ & \ddots & \ddots & \ddots & \\ & & \mu & -(\lambda+\mu) & \lambda \\ & & & \mu & -\mu \end{pmatrix} .$$

The well-known $M/M/1/(B-1)$!!
Its B eigenvalues are 0 and:

$$\omega_k \; = \; -(\lambda+\mu) \; + \; 2\sqrt{\lambda\mu}\cos\frac{k\pi}{B} \; , \qquad k = 1..B-1 \; .$$

Observation: these are also eigenvalues of $M_{B,N}$!

Progress

The simplified model

Band-Cavity interaction model: state space $(n, m) \in \mathcal{E} := [0..B] \times \mathbb{N}$

- n electrons in the conduction band (hence $N - n$ in the valence band)
- m photons in the cavity

Transitions:

origin		destination	rate	constraint
(n, m)	$\rightarrow$	$(n+1, m)$	J	$n < B$
(n, m)	$\rightarrow$	$(n, m-1)$	α	$m > 0$
(n, m)	$\rightarrow$	$(n-1, m+1)$	$(m+1)n_L(1-n_\ell)/T$	$n > 0$
(n, m)	$\rightarrow$	$(n+1, m-1)$	$mn_\ell(1-n_L)/T$	$m > 0$

Stability

QBD representation:

$$Q = \begin{pmatrix} B_0 & C_0 & 0 & 0 & \dots \\ A_1 & B_1 & C_1 & 0 & \dots \\ 0 & A_2 & B_2 & C_2 & \dots \\ & & \ddots & \ddots & \ddots \end{pmatrix} .$$

Flow balance between levels:

$$\pi_u.\mathbf{1}\max\{C_u\mathbf{1}\} \;\geq\; \pi_u C_u\mathbf{1} \;=\; \pi_{u+1}A_{u+1}\mathbf{1} \;\geq\; \pi_{u+1}.\mathbf{1}\min\{A_{u+1}\mathbf{1}\}$$

Hence:

$$\pi_{u+1}.\mathbf{1} \;\leq\; \frac{\max\{C_u\mathbf{1}\}}{\min\{A_{u+1}\mathbf{1}\}}\,\pi_u.\mathbf{1} \;.$$

Cheap way of having bounds on the tail.

Proper QBD representation

Proper representation is with the number of particles.

When $u \geq N$, blocks have constant size and:

$$A_u = \mathrm{diag}(u\alpha, (u-1)\alpha, \ldots, (u-N)\alpha)$$

$$C_u = \begin{pmatrix} 0 & J & 0 & & \\ & \ddots & \ddots & & \\ & & \ddots & J & 0 \\ & & & \ddots & J \\ & & & & 0 \end{pmatrix}.$$

$\Longrightarrow$ stability

QBD solution

Global balance equations $\boldsymbol{\pi} Q = 0$:

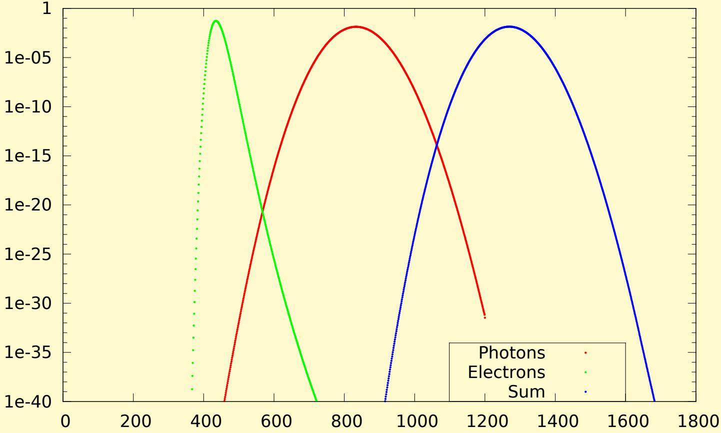
$$\begin{aligned} \pi_0 B_0 + \pi_1 A_1 &= 0, \\ \pi_{u-1} C_{u-1} + \pi_u B_u + \pi_{u+1} A_{u+1} &= 0, \end{aligned}$$

When $u \geq N$, A_u is square and diagonal:

$$\pi_{u+1} = (\pi_u B_u + \pi_{u-1} C_{u-1}) A_{u+1}^{-1}$$

Numerics

Distributions numerically obtained with the MARMOTE software:
$B = 800$, $q = 0.96209$, $\alpha = 0.6$, $J = 500.0$.



Progress

Conclusion

A simplified model of the laser

- dramatic improvement in simulation time
- numerical solution for stationary distributions
- accuracy... to be tested

Conclusion (ctd)

Future work

- focus on spectral density of output process

$$\begin{aligned} \rho^*(\omega) &:= \int_0^\infty e^{-i\omega h}\, \mathbb{E}(X(t+h)X(t))\, \mathrm{d}h \\ &= \Re\left\{ \boldsymbol{\pi}\ \boldsymbol{\Phi}(\omega)\ (\boldsymbol{I} - \boldsymbol{\Phi}(\omega)^{-1})\ \mathbf{1} \right\} \end{aligned}$$

where

$$\Phi_{ij}(\omega) := \mathbb{E}\left(e^{-\omega\tau_{ij}} \mathbf{1}_{\{j\}} | i \right) .$$

Open question: can Poisson pumping generate sub-Poissonian light?