Performance Evaluation of a Laser

Alain Jean-Marie\textsuperscript{1,2,3}  Fabrice Philippe\textsuperscript{2,3}

\textsuperscript{1}Inria/MAESTRO

\textsuperscript{2}LIRMM, University of Montpellier

\textsuperscript{3}Funded by ANR, project MARMOTE, grant MONU-12-0019

Gelenbe Symposium 2015/ISCIS 2015
Imperial College, London, 23 September 2015
(with typos corrected)
Contents

1. Purpose
2. The model
   - State space
     - Lasers as Markov chains
3. Analysis of one band
   - Generators for one band
   - Stationary distributions
   - Speed of convergence
4. Simplified model
   - QBD and stability
   - Numerical results
5. Conclusion
Performance Evaluation... of a Laser

1. Purpose

2. The model
   - State space
     - Lasers as Markov chains

3. Analysis of one band
   - Generators for one band
   - Stationary distributions
   - Speed of convergence

4. Simplified model
   - QBD and stability
   - Numerical results

5. Conclusion
Purpose of the talk

Show how techniques commonly used in “performance evaluation” can be used to (re)analyze a model from statistical physics, hopefully providing new insights:

- reversibility, product forms
- combinatorics, generating functions
- QBDs, linear algebra

A work much in progress with intriguing open questions.
Progress

1. Purpose

2. The model
   - State space
     - Lasers as Markov chains

3. Analysis of one band
   - Generators for one band
   - Stationary distributions
   - Speed of convergence

4. Simplified model
   - QBD and stability
   - Numerical results

5. Conclusion
Semi-conductor lasers

Electrons, two bands of energy levels, 0-1 electron by level

valence band

| energy gap |

conduction band

(lasing levels)
Semi-conductor lasers

Electrons, two bands of energy levels, 0-1 electron by level

(valence band) ─── energy gap ─── (conduction band) (lasing levels)

stimulated absorption ─── pumped ─── stimulated emission

Other/further moves:
upward and downward thermalization
(Auger effect, coherent pumping, spontaneous emission, ...
Ingredients:

- $2B$ energy levels, forming two “bands”; at most one electron occupies an energy level;
- two special “lasing levels” $L$ et $\ell$ in the middle of both bands
- a cavity: supports photons.

Transitions (changes of state):

- electrons may change level in each band (thermalization)
- one electron may emit one photon, and passes simultaneously from $L$ to $\ell$
- one electron may absorb a photon, and passes simultaneously from $\ell$ à $L$
- photons may exit the cavity
- “pumping” makes an electron pass from the lowest occupied level to the highest unoccupied energy level.
Choice of the state space

What is a state of the system? At least two choices:

**electron-centric**: each electron has a state, its energy level \( \in [0..2B - 1] \). The number of photons is *a priori* unbounded.

\[
\Rightarrow \quad \mathcal{E} \subset [0..2B - 1]^N \times \mathbb{N}.
\]

But the exclusion constraint has to be enforced.

**level-centric**: each energy level contains at most one electron.

\[
\Rightarrow \quad \mathcal{E} \subset \{0, 1\}^{2B} \times \mathbb{N}.
\]

Since the total number of electrons is fixed:

\[
\Rightarrow \quad \mathcal{E} \subset \left\{ \sigma \in \{0, 1\}^{2B} \mid \sum_{i=0}^{2B-1} \sigma_i = N \right\} \times \mathbb{N}.
\]
# Transition Rates

From state \((\sigma, m)\):

<table>
<thead>
<tr>
<th>destination</th>
<th>rate</th>
<th>constraints</th>
<th>event</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\sigma, m - 1))</td>
<td>(m\alpha)</td>
<td>(m &gt; 0)</td>
<td>laser emission</td>
</tr>
<tr>
<td>((\sigma - e_L + e_\ell, m + 1))</td>
<td>((m + 1)/(T))</td>
<td>(\sigma_L = 1, \sigma_\ell = 0)</td>
<td>stimulated emission</td>
</tr>
<tr>
<td>((\sigma + e_L - e_\ell, m - 1))</td>
<td>(m/(T))</td>
<td>(\sigma_L = 0, \sigma_\ell = 1)</td>
<td>stimulated absorption</td>
</tr>
<tr>
<td>((\sigma + e_i + 1 - e_i, m))</td>
<td>(p)</td>
<td>(\sigma_i = 1, \sigma_i + 1 = 0)</td>
<td>thermalization, down</td>
</tr>
<tr>
<td>((\sigma + e_i - e_i + 1, m))</td>
<td>(pq)</td>
<td>(\sigma_i = 0, \sigma_i + 1 = 1)</td>
<td>thermalization, up</td>
</tr>
<tr>
<td>((\sigma + e_0 - e_{2B}, m))</td>
<td>(J)</td>
<td>(\sigma_0 = 0, \sigma_{2B} = 1)</td>
<td>pumping</td>
</tr>
</tbody>
</table>

Physical names: \(p\) lattice coupling, \(q < 1\) Boltzmann temperature.
Metrics of interest: in the stationary regime

- probability of occupancy of energy levels
- distribution of photons in the cavity
- output rate of photons
- spectral density of photon emission process: Poissonian or sub-Poissonian?
The model is fine for simulation although...
The time scales of thermalization and other events (emission, absorption, light, pumping) are very different.

Could the process be seen as:
- 2 independent processes in each of the bands
- coupled by rare events

⇒ focus on each band
⇒ decompose the model by assuming each band stationary
Progress

1 Purpose
2 The model
   - State space
     - Lasers as Markov chains
3 Analysis of one band
   - Generators for one band
   - Stationary distributions
   - Speed of convergence
4 Simplified model
   - QBD and stability
   - Numerical results
5 Conclusion
Example of a transition diagram: $B = 6$ levels, $N = 2$ particles.

All transitions represented are two-way. Unwritten transition rates: rate $p$ for transitions to the left (lower energy), $pq$ to the right (higher energy).
And what happens if we adopt the electron-centric model?

\[ \mathcal{E} = \left\{ \mathbf{X} \in [0..2B - 1]^N \mid X_1 < X_2 < \ldots < X_N \right\} \]

For instance, still with \( B = 6, N = 2 \), a regular, 2-dimensional grid-like structure.
Stationary distribution

The stationary distribution can be computed in \textit{closed form} as:

\[
\pi_{B,N}(\sigma) = \frac{1}{Z(B, N)} \prod_{i=0}^{N-1} q^{i\sigma_i} = \frac{1}{Z(B, N)} q^{\sum_{i=0}^{N-1} i\sigma_i} = \frac{q^U \sigma}{Z(B, N)}.
\]

The \textit{partition function} $Z(B, N)$ is given by:

\[
Z(B, N) = \sum_{\sigma \in S_{B,N}} \prod_{i=0}^{N-1} q^{i\sigma_i}
= \frac{q^{N-1} - q^B}{1 - q^N} \frac{q^{N-2} - q^B}{1 - q^{N-1}} \cdots \frac{1 - q^B}{1 - q}.
\]

Proofs:
- direct check of balance equations
- reversibility and truncation
Performance Evaluation... of a Laser

Analysis of one band

Stationary distributions

Occupancies

Computation of \( n_i = \mathbb{P}(\sigma_i = 1) \).

Identities:

\[
\pi_{B,N}(\sigma_i = 1) = \sum_{k=0}^{i} q^{i+1} (N-k-1) \frac{Z(i, k) Z(B - 1 - i, N - k - 1)}{Z(B, N)}
\]

Recurrences:

\[
\pi_{B,N}(\sigma_i = 1) = q^i \frac{1 - q^N}{q^{N-1} - q_B} (1 - \pi_{B,N-1}(\sigma_i = 1))
\]

\[\implies\] fast computations; no enumeration of the state space
Speed of convergence}

Speed of convergence \(\leftrightarrow\) spectrum of generators.
Structure of the matrices?
Order states lexicographically: matrices can be represented as

\[
M_{B,N} = \begin{pmatrix}
M_{B-1,N-1} & \begin{pmatrix}
\mathbf{0}_{p \times q} & \mathbf{0}_{p \times r} \\
\lambda & \cdots & 0_{q \times r}
\end{pmatrix} \\
\mathbf{0}_{q \times p} & \mathbf{\mu} & \mathbf{0}_{r \times q} \\
0_{r \times p} & \mathbf{0}_{r \times q} & M_{B-1,N}
\end{pmatrix}
\]
But if we order by energy level, we get a “quasi-birth-death” process

The matrix is then: 

$$M_{6,2} = \begin{pmatrix} - & \lambda & & & & \\ \mu & - & \lambda & \lambda & & \\ - & 0 & \lambda \\ 0 & - & \lambda \\ - & 0 & \lambda & \lambda & 0 \\ \mu & 0 & - & 0 & 0 & \lambda & 0 \\ \mu & 0 & 0 & - & 0 & \lambda & \lambda \\ 0 & \mu & 0 & 0 & - & 0 & \lambda \\ \mu & \mu & 0 & - & 0 & \lambda & 0 \\ 0 & \mu & \mu & 0 & - & \lambda & \lambda \\ \mu & \mu & 0 & - & \lambda \\ 0 & \mu & \mu & 0 & - & \lambda \\ \mu & \mu & - & \lambda \\ \mu & - & \lambda \\ \mu & - & \lambda \\ \mu & - \end{pmatrix}$$
The basis matrix is:

$$M_{B,1} = \begin{pmatrix}
-\lambda & \lambda \\
\mu & -(\lambda + \mu) & \lambda \\
& \ddots & \ddots & \ddots \\
& & \mu & -(\lambda + \mu) & \lambda \\
& & & \mu & -\mu \\
\end{pmatrix}.$$ 

The well-known $M/M/1/(B - 1)$!!

Its $B$ eigenvalues are 0 and:

$$\omega_k = -(\lambda + \mu) + 2\sqrt{\lambda\mu}\cos\frac{k\pi}{B}, \quad k = 1..B - 1.$$ 

Observation: these are also eigenvalues of $M_{B,N}$!
Progress

1. Purpose

2. The model
   - State space
     - Lasers as Markov chains

3. Analysis of one band
   - Generators for one band
   - Stationary distributions
   - Speed of convergence

4. Simplified model
   - QBD and stability
   - Numerical results

5. Conclusion
The simplified model

Band-Cavity interaction model: state space

$(n, m) \in \mathcal{E} := [0..B] \times \mathbb{N}$

- $n$ electrons in the conduction band
  (hence $N - n$ in the valence band)
- $m$ photons in the cavity

Transitions:

<table>
<thead>
<tr>
<th>origin</th>
<th>destination</th>
<th>rate</th>
<th>constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(n, m)$</td>
<td>$(n + 1, m)$</td>
<td>$J$</td>
<td>$n &lt; B$</td>
</tr>
<tr>
<td>$(n, m)$</td>
<td>$(n, m - 1)$</td>
<td>$\alpha$</td>
<td>$m &gt; 0$</td>
</tr>
<tr>
<td>$(n, m)$</td>
<td>$(n - 1, m + 1)$</td>
<td>$(m + 1)n_L(1 - n_\ell)/T$</td>
<td>$n &gt; 0$</td>
</tr>
<tr>
<td>$(n, m)$</td>
<td>$(n + 1, m - 1)$</td>
<td>$mn_\ell(1 - n_L)/T$</td>
<td>$m &gt; 0$</td>
</tr>
</tbody>
</table>
Stability

QBD representation:

\[ Q = \begin{pmatrix} B_0 & C_0 & 0 & 0 & \ldots \\ A_1 & B_1 & C_1 & 0 & \ldots \\ 0 & A_2 & B_2 & C_2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

Flow balance between levels:

\[ \pi_u.1 \max\{C_u.1\} \geq \pi_u C_u.1 = \pi_{u+1} A_{u+1}.1 \geq \pi_{u+1}.1 \min\{A_{u+1}.1\} \]

Hence:

\[ \pi_{u+1}.1 \leq \frac{\max\{C_u.1\}}{\min\{A_{u+1}.1\}} \pi_u.1. \]

Cheap way of having bounds on the tail.
Proper QBD representation

Proper representation is with the **number of particles**.

When \( u \geq N \), blocks have constant size and:

\[
A_u = \text{diag}(u\alpha, (u-1)\alpha, \ldots, (u-N)\alpha)
\]

\[
C_u = \begin{pmatrix}
0 & J & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & J
\end{pmatrix}
\]

\[\Rightarrow \text{stability}\]
Global balance equations $\pi Q = 0$:

\[
\begin{align*}
\pi_0 B_0 + \pi_1 A_1 &= 0, \\
\pi_{u-1} C_{u-1} + \pi_u B_u + \pi_{u+1} A_{u+1} &= 0,
\end{align*}
\]

When $u \geq N$, $A_u$ is square and diagonal:

\[
\pi_{u+1} = (\pi_u B_u + \pi_{u-1} C_{u-1}) A_{u+1}^{-1}
\]
Distributions numerically obtained with the MARMOTE software: 

\[ B = 800, \quad q = 0.96209, \quad \alpha = 0.6, \quad J = 500.0. \]
Performance Evaluation... of a Laser

Conclusion

Progress

1. Purpose

2. The model
   - State space
     - Lasers as Markov chains

3. Analysis of one band
   - Generators for one band
   - Stationary distributions
   - Speed of convergence

4. Simplified model
   - QBD and stability
   - Numerical results

5. Conclusion
A simplified model of the laser

- dramatic improvement in simulation time
- numerical solution for stationary distributions
- accuracy... to be tested
Future work

- focus on spectral density of output process

\[
\rho^*(\omega) := \int_0^\infty e^{-i\omega h} \mathbb{E}(X(t + h)X(t)) \, dh
\]

\[
= \Re \left\{ \pi \Phi(\omega) (I - \Phi(\omega)^{-1}) \mathbf{1} \right\}
\]

where

\[
\Phi_{ij}(\omega) := \mathbb{E}\left( e^{-\omega \tau_{ij}} 1_{\{j\}|i} \right).
\]

Open question: can Poisson pumping generate sub-Poissonian light?