On the existence of incentive equilibria

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Introduction

Principle of incentive equilibria: developed for dynamic games by Ehtamo and Hämäläinen *Journal of Economic Dynamics and Control* (1993) and *Group Decision and Negotiation* (1995), inspired from the work of Osborne (1976) about the definition of a "quota rule" able to explain the stability of a Cartel.
Used in several papers by those authors, Jørgensen and Zaccour, Martín-Herrán and Zaccour, in discrete-time and continuous-time games.

Results

We consider:

- static games
- dynamic games with open-loop strategies

We find:

- Necessary conditions for $\exists$ of credible incentive equilibria
- credible incentive equilibria with differentiable incentive do not exist without very strong conditions on the payoff (Nash+Pareto!)
- for piecewise-differentiable incentive functions, an infinity of credible incentive equilibria can be chosen.

We illustrate with examples:

- the stability of a Cartel
- an environmental problem

Plan

1. Introduction
2. Definitions: incentives and credibility
3. Analysis of the static case
4. Analysis of a dynamic case: Nash open-loop
5. Examples

Definitions

Consider a two-player game: strategies $E_i \in \Sigma_i$, payoff functions:

$$J_i : \Sigma_1 \times \Sigma_2 \to \mathbb{R} .$$

Players agree, before playing the game, that a certain Pareto optimum E^* is a desired output of the game

Definition (Incentive equilibrium)

Consider a Pareto optimum (E_1^*, E_2^*) of the game. An incentive equilibrium strategy at this optimum is a pair of mappings (Ψ_1, Ψ_2), with $\Psi_1 : \Sigma_2 \to \Sigma_1$, $\Psi_2 : \Sigma_1 \to \Sigma_2$, and such that:

$$\begin{array}{rclrcl} J_1(E_1, \Psi_2(E_1)) & \leq & J_1(E_1^*, \Psi_2(E_1^*)) & & & \forall E_1 \in \Sigma_1 \\ J_2(\Psi_1(E_2), E_2) & \leq & J_2(\Psi_1(E_2^*), E_2^*) & & & \forall E_2 \in \Sigma_2 \\ \Psi_1(E_2^*) & = & E_1^* & \Psi_2(E_1^*) & = & E_2^* . \end{array}$$

Incentive design and Credibility

The incentive problem:

- construct a game in which players are induced to play a cooperative desired outcome E^* by defining an incentive rule
- E^* equilibrium of the game
- the incentive rule is credible

Credibility holds if every player, if faced with a deviation from her opponent, would prefer to follow the incentive rather than sticking to her equilibrium value.

Credibility

Definition (Credible incentive equilibrium)

The pair (Ψ_1, Ψ_2) is a credible incentive equilibrium at (E_1^*, E_2^*) if it is an incentive equilibrium, and if there exists a subset $\Sigma_1' \times \Sigma_2'$ of $\Sigma_1 \times \Sigma_2$ such that:

$$\begin{aligned} J_1(\Psi_1(E_2), E_2) &\geq J_1(E_1^*, E_2) \ , & \forall E_2 \in \Sigma_2' \\ J_2(E_1, \Psi_2(E_1)) &\geq J_2(E_1, E_2^*) \ , & \forall E_1 \in \Sigma_1' \ . \end{aligned}$$

An equilibrium is credible if: whatever the *admissible deviation* E_2 of Player 2 with respect to the equilibrium E_2^*, Player 1 is better off by following the incentive $\Psi_1(E_2)$ rather than sticking to the equilibrium E_1^*.

Weak credible incentive equilibrium

A weaker concept: dropping the requirement that (E_1^*, E_2^*) be a Pareto solution.

Definition (Weak credible incentive equilibrium)

A weak credible incentive equilibrium is a couple of states (E_1^*, E_2^*) and of incentives (Ψ_1, Ψ_2) such that

$$\begin{array}{rclr} J_1(E_1, \Psi_2(E_1)) & \leq & J_1(E_1^*, \Psi_2(E_1^*)) & \forall E_1 \in \Sigma_1 \\ J_2(\Psi_1(E_2), E_2) & \leq & J_2(\Psi_1(E_2^*), E_2^*) & \forall E_2 \in \Sigma_2 \\ J_1(\Psi_1(E_2), E_2) & \geq & J_1(E_1^*, E_2) & \forall E_2 \in \Sigma_2 \\ J_2(E_1, \Psi_2(E_1)) & \geq & J_1(E_2, E_1^*) & \forall E_1 \in \Sigma_1 \\ \Psi_1(E_2^*) = E_1^* & & \Psi_2(E_1^*) = E_2^* \,. & \end{array}$$

Conjectural variations

Conjectural variations equilibria have been introduced for static games by Bowley (1924), Frish (1933) and others.
Both definitions are formally similar.

Definition (Conjectural variations equilibrium)

A conjectural variations equilibrium is a pair of functions (r_1, r_2) and a pair of states (E_1^*, E_2^*) such that:

$$\frac{\partial J_1}{\partial E_1}(E_1^*, E_2^*) \; + \; r_2(E_1^*) \, \frac{\partial J_1}{\partial E_2}(E_1^*, E_2^*) \; = \; 0 \; ,$$

and symmetrically for J_2.

$r_2(e_1)$ is the anticipated variation of Player 2: Player 1 thinks that if he deviates from δe_1, Player 2 will deviate from $\delta e_2 = r_2(e_1)\delta e_1$.

Conjectural variations as an optimization problem

Theorem

A conjectural variations equilibrium is such that:

$$J_1(E_1, \Psi_2(E_1)) \;\leq\; J_1(E_1^*, \Psi_2(E_1^*)) \qquad \forall E_1 \in \Sigma_1$$

(symmetrically for J_2), with:

$$\Psi_2(E_2) \;=\; E_2^* \;+\; \int_{E_2^*}^{E_2} r_2(u)\, du \;.$$

Comparisons

- In incentive equilibria, incentive functions Ψ_i are chosen and imposed/recommended by an authority. The equilibrium (E_1^*, E_2^*) is a given goal to be reached.
- In conjectural equilibria, conjectures are resulting from a private representation of players, their utilities and their intentions. The equilibrium (E_1^*, E_2^*) is a result.
- Both lead to optimization problems, where the value of the payoff of the opponent is unknown.
- Both lead to *rationality* problems:
  - why wouldn't a player cheat=deviate from the incentive?
  - how does a player form its conjectures?

The static case

Incentive functions:

$$\Psi_i(E_j) \;=\; \begin{cases} \Psi_i^+(E_j) & \text{if } E_j \geq E_j^* \\ \Psi_i^-(E_j) & \text{if } E_j \leq E_j^* \end{cases},$$

where Ψ_i^+ and Ψ_i^- are differentiable, including at $E_j = E_j^*$.

$$a_i^+ \;=\; (\Psi_i^+)'(E_j^*)\;, \quad \text{and} \quad a_i^- \;=\; (\Psi_i^-)'(E_j^*)$$

$$A_i \;=\; -\;\frac{\partial J_i/\partial E_i}{\partial J_i/\partial E_j}(E_1^*, E_2^*)\;.$$

Necessary conditions for equilibria

E^* NOT NECESSARILY A PARETO OPTIMUM

Theorem

Necessary conditions for the existence of weak incentive equilibrium function Ψ_1 are as follows:

$\partial J_1/\partial E_1$	$\partial J_2/\partial E_1$	*Conditions*
-	-	$A_2 = 0$ *and* $a_1^+ = a_1^- = 0$
$+$	$+$	$A_2 = 0$ *and* $a_1^+ = a_1^- = 0$
-	$+$	$a_1^+ \leq \min(A_2, 0) \leq \max(A_2, 0) \leq a_1^-$
$+$	-	$a_1^- \leq \min(A_2, 0) \leq \max(A_2, 0) \leq a_1^+$
0	-	$a_1^- \leq A_2 \leq a_1^+$
0	$+$	$a_1^+ \leq A_2 \leq a_1^-$
$+$	*0*	$\partial J_2/\partial E_2 = 0$ *and* $a_1^- \leq 0 \leq a_1^+$
-	*0*	$\partial J_2/\partial E_2 = 0$ *and* $a_1^+ \leq 0 \leq a_1^-$
0	*0*	$\partial J_2/\partial E_2 = 0$

Necessary conditions for equilibria (ctd)

Corollary

Let E^* be a Pareto optimum,

a/ $\partial J_1/\partial E_1 > 0$, $\partial J_1/\partial E_2 < 0$, $\partial J_2/\partial E_1 < 0$, $\partial J_2/\partial E_2 > 0$.

$$\rightarrow \quad a_1^- \leq 0 \leq A_2 \leq a_1^+, \quad \textit{and} \quad a_2^- \leq 0 \leq A_1 \leq a_2^+.$$

b/ $\partial J_1/\partial E_1 < 0$, $\partial J_1/\partial E_2 > 0$, $\partial J_2/\partial E_1 > 0$, $\partial J_2/\partial E_2 < 0$.

$$\rightarrow \quad a_1^+ \leq 0 \leq A_2 \leq a_1^-, \quad \textit{and} \quad a_2^+ \leq 0 \leq A_1 \leq a_2^-.$$

c/ $\partial J_1/\partial E_1 > 0$, $\partial J_1/\partial E_2 > 0$, $\partial J_2/\partial E_1 < 0$, $\partial J_2/\partial E_2 < 0$.

$$\rightarrow \quad a_1^- \leq A_2 \leq 0 \leq a_1^+, \quad \textit{and} \quad a_2^+ \leq A_1 \leq 0 \leq a_2^-.$$

....

Differentiable incentive equilibrium

Incentives functions are assumed differentiable: $a_i^+ = a_i^- = a_i$.

E^* NOT NECESSARILY A PARETO OPTIMUM

a/ if $a_1 = 0$ and $a_2 = 0$, then necessarily

$$\frac{\partial J_i}{\partial E_i}(E_1^*, E_2^*) \;=\; 0 \;, \qquad i = 1,2 \;;$$

b/ if $a_1 \neq 0$ and $a_2 \neq 0$, then necessarily

$$\frac{\partial J_i}{\partial E_j}(E_1^*, E_2^*) \;=\; 0 \;, \qquad i,j = 1,2 \;;$$

c/ if $a_1 = 0$ and $a_2 \neq 0$, then necessarily

$$\frac{\partial J_2}{\partial E_2}(E_1^*, E_2^*) = 0 \;, \quad \frac{\partial J_1}{\partial E_1}(E_1^*, E_2^*) \;+\; a_2 \,\frac{\partial J_1}{\partial E_2}(E_1^*, E_2^*) = 0$$

Differentiable credible incentive equilibria

A credible incentive equilibrium requires strong properties on the payoff functions:

Theorem

Let (Ψ_1, Ψ_2) be a credible incentive equilibrium at a Pareto optimum, where the incentive functions Ψ_i are differentiable. Then, necessarily:

$$\frac{\partial J_i}{\partial E_j}(E_1^*, E_2^*) \; = \; 0 \; , \qquad i, j = 1, 2 \; .$$

The converse property holds: a simultaneous maximum (E_1^*, E_2^*) is a credible incentive equilibrium, for any incentive functions Ψ_i.

Nash equilibria

Among candidate weak credible incentive equilibria, one finds Nash equilibria:

Theorem

A Nash equilibrium (E_1^*, E_2^*) is a weak credible incentive equilibrium for the (constant) incentive functions: $\Psi_i(E_j) = E_i^*$.

One-sided incentives

One-sided incentives:

$$\Psi_i^+(E_j) \;=\; E_i^* \qquad \text{or} \qquad \Psi_i^-(E_j) \;=\; E_i^* \;,$$

or in other words,

$$a_i^+ \;=\; 0 \qquad \text{or} \qquad a_i^- \;=\; 0 \;.$$

We can use the table to find on which side of the incentive equilibrium E^* it can be credible not to react, depending on the sign of the partial derivatives of the payoff functions.

Osborne's example

The topic of Osborne's paper is the stability of a Cartel (for instance, the OPEC).

- Strategies E_i: level of production of the firms, $J_i(\cdot)$, their profit functions. With $\partial J_i/\partial E_i > 0$ and $\partial J_i/\partial E_j < 0$.
- In this context, the "incentive" function is actually a threat function, with which members of the Cartel would retaliate to potential cheaters.

$$\Psi_i(E_j) \;=\; \max\left\{ E_i^*, E_i^* + \frac{E_i^*}{E_j^*}(E_j - E_j^*) \right\} \,,$$

This is a one-sided credible incentive equilibrium, with:

$$a_i^- \;=\; 0, \qquad a_i^+ \;=\; \frac{E_i^*}{E_j^*} \,.$$

The case of Nash Open Loop equilibria

The state of the system evolves according to the differential equation

$$\dot{x}(t) \ = \ f(E_1(t), E_2(t), x(t)) \ , \quad x(0) = x_0 \ ,$$

where $E_i(t)$ is the action of player i at time t according to her strategy E_i. Payoff of player i:

$$J_i(E_1, E_2; x_0) \ = \ \int_0^T e^{-\rho t} F_i(E_1(t), E_2(t), x(t)) \ dt \ ,$$

with a time horizon $T < +\infty$ and a discount factor $\rho \geq 0$. Consider *affine* incentive equilibria of the form:

$$\Psi_i(E_j)(t) \ = \ E_i^*(t) + v_i(t)(E_j(t) - E_j^*(t)) \ .$$

for some scalar functions $v_1(t)$ and $v_2(t)$.

Necessary conditions for the Open-Loop case

A credible affine incentive equilibrium at a Pareto optimum is a solution of the following system of equations, for some $\alpha_1 > 0$ and $\alpha_2 > 0$:

Conditions for being Pareto

$$\begin{cases} 0 = \alpha_1 \dfrac{\partial F_1}{\partial E_i} + \alpha_2 \dfrac{\partial F_2}{\partial E_i} + \lambda^* \dfrac{\partial f}{\partial E_i} & i = 1, 2 \\ \dot{\lambda}^* = -\alpha_1 \dfrac{\partial F_1}{\partial x} - \alpha_2 \dfrac{\partial F_2}{\partial x} - \lambda^* \dfrac{\partial f}{\partial x} + \rho \lambda^* \ ; & \lambda^*(T) = 0 \\ \dot{x}^* = f \ ; \qquad x(0) = x_0 \end{cases}$$

Necessary conditions for the Open-Loop case

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Conditions for being an incentive equilibria

$$\left\{\begin{array}{rcl} 0 & = & \dfrac{\partial F_1}{\partial E_1} + v_2 \dfrac{\partial F_1}{\partial E_2} + \lambda^1 \left(\dfrac{\partial f}{\partial E_1} + v_2 \dfrac{\partial f}{\partial E_2} \right) \\ \dot{\lambda}^1 & = & -\dfrac{\partial F_1}{\partial x} - \lambda^1 \dfrac{\partial f}{\partial x} + \rho \lambda^1 \ ; \quad \lambda^1(T) = 0 \\ 0 & = & v_1 \dfrac{\partial F_2}{\partial E_1} + \dfrac{\partial F_2}{\partial E_2} + \lambda^2 \left(v_1 \dfrac{\partial f}{\partial E_1} + \dfrac{\partial f}{\partial E_2} \right) \\ \dot{\lambda}^2 & = & -\dfrac{\partial F_2}{\partial x} - \lambda^2 \dfrac{\partial f}{\partial x} + \rho \lambda^2 \ ; \quad \lambda^2(T) = 0 \end{array}\right.$$

Necessary conditions for the Open-Loop case

Conditions for being credible

$$\begin{cases} 0 &= -v_1 \dfrac{\partial F_1}{\partial E_1} + \lambda^1 \dfrac{\partial f}{\partial E_2} + \lambda^{1c} \left(v_1 \dfrac{\partial f}{\partial E_1} + \dfrac{\partial f}{\partial E_2} \right) \\ \dot{\lambda}^{1c} &= \dfrac{\partial F_1}{\partial x} - \lambda^{1c} \dfrac{\partial f}{\partial x} + \rho \lambda^{1c} \ ; \qquad \lambda^{1c}(T) = 0 \\ 0 &= -v_2 \dfrac{\partial F_2}{\partial E_2} + \lambda^2 \dfrac{\partial f}{\partial E_1} + \lambda^{2c} \left(\dfrac{\partial f}{\partial E_1} + v_2 \dfrac{\partial f}{\partial E_2} \right) \\ \dot{\lambda}^{2c} &= \dfrac{\partial F_2}{\partial x} - \lambda^{2c} \dfrac{\partial f}{\partial x} + \rho \lambda^{2c} \ ; \qquad \lambda^{2c}(T) = 0 \end{cases}$$

Elements of the proof

Pareto. First, E_i^* is required to be the solution for the cooperative problem:

$$\max_{E_1,E_2} [J_1(E_1,E_2,x_0) + J_2(E_1,E_2,x_0)] = \max_{E_1,E_2} \sum_{i=1}^{2} \int_0^T F_i(E_1(t),E_2(t),x(t)$$

such that

$$\dot{x}(t) = f(E_1(t),E_2(t),x(t)), \quad x(0) = x_0.$$

The Hamiltonian for this problem is

$$H^*(E_1,E_2,x,\lambda^*) = F_1(E_1,E_2,x) + F_2(E_1,E_2,x) + \lambda^* f(E_1,E_2,x)$$

and the first order conditions give the first group of equations.

Elements of the proof (ctd.)

Incentives. $v_i(t)$, $i = 1, 2$, must be the solutions of the following optimization problems:

$$\max_{E_1} J_1(E_1, \Psi_2(E_1), x_0) \;=\; \max_{E_1} \int_0^T F_1(E_1(t), \Psi_2(E_1), x^1(t))\; dt\;,$$

such that

$$\dot{x}^1(t) \;=\; f(E_1(t), \Psi_2(E_1)(t), x^1(t)), \quad x^1(0) = x_0.$$

and symmetrically for $x^2(t)$. For this problems the corresponding Hamiltonians are:

$$\begin{aligned} H^1(E_1, x, \lambda^1) &= F_1(E_1, \Psi_2(E_1), x) + \lambda^1 f(E_1, \Psi_2(E_1), x) \\ H^2(E_2, x, \lambda^2) &= F_2(\Psi_1(E_2), E_2, x) + \lambda^2 f(\Psi_1(E_2), E_2, x)\;. \end{aligned}$$

and the first order conditions give the second group of equations because $x^1 \equiv x^2 \equiv x^*$.

Elements of the proof (end)

Credibility. Consider the point of view of Player 1. The following conditions must hold:

$$J_1(\Psi_1(E_2), E_2; x_0) \geq J_1(E_1^*, E_2; x_0), \quad \forall E_2$$

where

$$J_1(\Psi_1(E_2), E_2; x_0) = \int_0^T F_1(\Psi_1(E_2), E_2, x^2)\ dt\ ,$$

$$\dot{x}^2(t) \ = \ f(\Psi_1(E_2), E_2, x^2(t)), \quad x^2(0) = x_0\ ,$$

and

$$J_1(E_1^*, E_2, x_0) = \int_0^T F_1(E_1^*, E_2, x^{2c}(t))\ dt\ ,$$

$$\dot{x}^{2c}(t) \ = \ f(E_1^*, E_2, x^{2c}(t)), \quad x^{2c}(0) = x_0 .$$

Consider the difference:

$$D_1(E_2) \ = \ J_1(E_1^*, E_2; x_0) \ - \ J_1(\Psi_1(E_2), E_2; x_0)\ .$$

The equilibrium is credible iff $D_1(E_2) \leq 0$ for all $E_2 \in \Sigma_2 \iff$

Counting the equations

Why do we face existence problems?
For a (strong) credible incentive equilibrium, there are: 12 equations with 10 unknowns:

- the cooperative solution E_1^* and E_2^*;
- the incentive coefficients $v_1(t)$ and $v_2(t)$;
- the state variable $x^*(t)$;
- the adjoint variables $\lambda^*(t), \lambda^1(t), \lambda^2(t), \lambda^{1c}(t)$ and $\lambda^{2c}(t)$.

For a weak credible incentive equilibrium, 8 equations and 8 unknowns.

Weak credible equilibria

Theorem

A (strong) credible affine incentive equilibrium may hold only if one of the following conditions is met:

- $v_1 = v_2 = 0$ *and*

$$\frac{\partial F_i}{\partial E_i} + \lambda^i \frac{\partial f}{\partial E_i} = 0 , \qquad i = 1, 2 .$$

- $v_1 \neq 0$ *and* $v_2 \neq 0$ *and*

$$\frac{\partial F_i}{\partial E_j} + \lambda^i \frac{\partial f}{\partial E_j} = 0 , \qquad i, j = 1, 2 .$$

- $v_1 = 0$ *and* $v_2 \neq 0$ *(and symetrically if* $v_1 \neq 0$ *and* $v_2 = 0$*), and*

$$\begin{aligned} 0 &= \frac{\partial F_2}{\partial E_2} + \lambda^2 \frac{\partial f}{\partial E_2} \\ 0 &= \frac{\partial F_1}{\partial E_1} + v_2 \frac{\partial F_1}{\partial E_2} + \lambda^1 \left(\frac{\partial f}{\partial E_1} + v_2 \frac{\partial f}{\partial E_2} \right) \end{aligned}$$

Properties

- Necessary conditions stated $\rightarrow E^*$ must be a simultaneous maximum for both payoff functions J_i.
- Piecewise-differentiable incentive functions.
  $V_i(t, E_j(t)) = V_i^+(t, E_j(t))$ if $E_j(t) \geq E_j^*(t)$ and
  $V_i(t, E_j(t)) = V_i^-(t, E_j(t))$ if $E_j(t) \leq E_j^*(t)$.

  Left and right-derivatives: $v_i^{\pm}(t) = \partial V_i^{\pm} / \partial E_j(t, E_j^*(t))$.

  Transposition of the results of the static case: replace "$\partial J_i / \partial E_j$" by "$\partial F_i / \partial E_j + \lambda^i \partial f / \partial E_j$".

Examples for dynamic games

The examples illustrate

- The computation of (one-sided) credible incentives
- The problem of guaranteeing credibility with respect to deviations in a dynamic setting
- That incentives may be credible over a limited domain

Case study: the model of Jørgensen and Zaccour

We consider in this section a dynamic model proposed by Jørgensen and Zaccour in Jørgensen, S. and Zaccour, G., "Channel coordination over time: Incentive equilibria and credibility", 2002.

$$\begin{aligned} \dot{x} &= E_1 + E_2 - \delta x \ , \\ F_i(E_1, E_2, x) &= \phi_i \ x \ - \ \frac{w_i}{2} \ E_i^2 \ . \end{aligned}$$

Checking the necessary conditions of Corollary 11, we have the result:

There does not exist a credible incentive equilibrium in this game.

Weak credible incentive equilibria

According to the Theorem, we have:

- If $v_1 \neq 0$ and $v_2 \neq 0$, no solution exists.
- If $v_1 = v_2 = 0$, this is the Nash-credible incentive equilibrium.
- If $v_1 = 0$ and $v_2 \neq 0$, then the candidate solutions

$$\begin{aligned} E_1^*(t) &= \frac{\phi_1(1+v_2)}{w_1\delta}\,(1-e^{\delta(t-T)}) \\ E_2^*(t) &= \frac{\phi_2}{w_2\delta}\,(1-e^{\delta(t-T)}) \end{aligned}$$

satisfy the necessary conditions, but not the credibility equations.

Deviations from reference strategies

Computations can be performed in the following framework: consider that the players have strategies of the form:

$$E_i(t) \;=\; A_i\; g(t) \;-\; \Delta_i \;, \qquad g(t) \;=\; 1 - e^{\delta(t-T)} \;, \qquad (1)$$

The calculations lead to the conclusion:

> *For the current example, there does not exist credible incentive equilibria, even if the deviations are constrained to be of the form* (1).

Credibility with respect to small deviations

Ehtamo and Hämäläinen, Jørgensen and Zaccour, have studied the credibility of incentive equilibria with respect to *small impulsional* deviations. Define Δ_τ to be the control such that:

$$\Delta_\tau(t) = \begin{cases} \Delta & \text{if } t \leq \tau \\ 0 & \text{if } t > \tau. \end{cases}$$

Definition (Locally Credible incentive equilibrium)

The pair (Ψ_1, Ψ_2) is a locally credible incentive equilibrium at (E_1^*, E_2^*) if

$$\begin{aligned} J_1(\Psi_1(E_2), E_2) &\geq J_1(E_1^*, E_2) \ , \\ J_2(E_1, \Psi_2(E_1)) &\geq J_1(E_2, E_1^*) \ , \end{aligned}$$

for all $E_1 = E_1^* - \Delta_\tau$ and $E_2 = E_2^* - \Delta_\tau$, for some Δ and τ small.

Difficulty of the analysis

Find the sign of:
$\phi(\tau, \Delta) = J_1(E_1^* - v_1\Delta_\tau, E_2^* - \Delta_\tau) - J_1(E_1^*, E_2^* - \Delta_\tau)$ with

$$J_1(E_1^* - v_1\Delta_\tau, E_2^* - \Delta_\tau)$$
$$= \int_0^\tau F_1(E_1^* - v_1\Delta, E_2^* - \Delta, x)\, dt + \int_\tau^T F_1(E_1^*, E_2^*$$

for x sol. of
$$\dot{x} = \begin{cases} f_1(E_1^* - v_1\Delta, E_2^* - \Delta, x) & 0 \le t \\ f_1(E_1^*, E_2^*, x) & \tau < t \end{cases}$$

$$J_1(E_1^*, E_2^* - \Delta_\tau) = \int_0^\tau F_1(E_1^*, E_2^* - \Delta, \tilde{x})\, dt + \int_\tau^T F_1(E_1^*, E_2^*, \tilde{x})\, d$$

for $\tilde{x}$ sol. of
$$\dot{\tilde{x}} = \begin{cases} f_1(E_1^*, E_2^* - \Delta, \tilde{x}) & 0 \le t \le \tau \\ f_1(E_1^*, E_2^*, \tilde{x}) & \tau < t \le T\ . \end{cases}$$

Environmental example

An exemple of pollution accumulation due to production activities.

$$J_i(E_1(\cdot), E_2(\cdot); x_0) \;=\; \int_0^\infty e^{-\rho t}\left(\log(E_i(t)) - \phi_i x(t)\right)\, dt \;,$$

$$\dot{x}(t) \;=\; E_1(t) + E_2(t) - \delta x(t), \quad x(0) = x_0 \;.$$

Pareto solution, the maximization of $\sum_i \alpha_i J_i$, is:

$$E_i^* \;=\; \frac{\alpha_i(\delta + \rho)}{\alpha_1\phi_1 + \alpha_2\phi_2} \;.$$

The Pareto-optimal control does not depend on time

Environmental example. Static credibility

Consider only time-invariant strategies. The total payoff of player i is given by:

$$J_i(e_1, e_2; x_0) \;=\; \frac{1}{\rho}\log(e_i) \;-\; \frac{\phi_i}{\rho(\rho+\delta)}(e_1+e_2) \;-\; \frac{\phi_i x_0}{\rho+\delta} \;.$$

$$A_i \;=\; \frac{\alpha_j \phi_j}{\alpha_i \phi_i}\;, \qquad A_j \;=\; \frac{1}{A_i} \;=\; \frac{\alpha_i \phi_i}{\alpha_j \phi_j}\;.$$

$$a_i^- \leq 0 \leq \frac{\alpha_i \phi_i}{\alpha_j \phi_j} \leq a_i^+ .$$

Environmental example. Static credibility

We select the piecewise affine function:

$$\Psi_i(e_j) \;=\; \max\left\{e_i^*, e_i^* + \frac{\alpha_i\phi_i}{\alpha_j\phi_j}(e_j - e_j^*)\right\} \;.$$

For Player 1, the credibility condition becomes: for $e_2 \geq e_2^*$:

$$0 \;\leq\; \log\frac{e_1^* + \alpha_1\phi_1/\alpha_2\phi_2(e_2 - e_2^*)}{e_1^*} \;-\; \phi_1\,\frac{\alpha_1\phi_1}{\alpha_2\phi_2}\,(e_2 - e_2^*) \;.$$

$\exists$ interval $[e_2^*, \bar{e}_2]$ where the condition is satisfied.

$$\bar{e}_2 \;\geq\; e_2^*\left\{1 + 2\,\frac{\alpha_2}{\alpha_1}\left(\frac{\phi_2}{\phi_1}\right)^2\right\} \;.$$

Environmental example. Dynamic credibility

$\alpha_1 = \alpha_2 = 1$.

This implies that $e_1^* = e_2^* = e^* = (\delta + \rho)/(\phi_1 + \phi_2)$.

We select the incentive function:

$$\Psi_i(E_j)(t) \;=\; e^* \;+\; \max\left\{\frac{\phi_i}{\phi_j}(E_j(t) - e^*), 0\right\} \;.$$

Environmental example. Dynamic credibility

Condition of credibility for player 1

$$\int_0^{+\infty} \left[\log(\frac{\Psi_1(E_2(t))}{e^*}) - \phi_1 x^\Psi(t) + \phi_1 x^*(t)\right] \; e^{-\rho t} \; dt \;\; \geq \;\; 0 \; ,$$

where the two trajectories $x^\Psi(\cdot)$ and $x^*(\cdot)$ are the respective solutions of

$$\begin{aligned} \dot{x} &= e^* \; + \; \frac{\phi_1}{\phi_2} \max(0, E_2(t) - e^*) \; + \; E_2(t) - \delta x(t) \\ \dot{x} &= e^* \; + \; E_2(t) - \delta x(t) \end{aligned}$$

Environmental example. Dynamic credibility

Assume that there exists $M \geq 1$ such that for all t,

$$E_2(t) \;\leq\; M\; e^* \;.$$

Credibility implies that $E_2(\cdot)$ verifies

$$\int_0^{+\infty} \left[\frac{\phi_1}{\phi_2} + \frac{2\phi_1^2}{\phi_2^2} \right] \frac{E_2(t)}{e^*} e^{-\rho t} dt \;\geq$$

$$\frac{1}{\rho} \left[\frac{\phi_1^2}{\phi_2^2}(M^2 - 1) + \frac{\phi_1}{\phi_2} + \frac{\phi_1^2}{\phi_2^2} \frac{M-1}{\rho + \delta} e^* \right] \;.$$

Environmental example. Dynamic credibility

For instance, it can be checked that the equilibrium is credible with respect to strategies of the form

$$E_2(t) \;=\; e^N + (e^* - e^N)e^{-\alpha t} \;,$$

or

$$E_2(t) \;=\; e^* + (e^N - e^*)e^{-\alpha t} \;,$$

where $e^N = (\rho + \delta)/2$ is the Nash equilibrium of the game (a time-invariant strategy as well).

Conclusion

- Credibility is difficult to obtain in static and continuous-time games: at a Pareto solution as well as elsewhere, if the incentive function is required to be differentiable. A credible incentive equilibria may happen only at critical points of both payoff functions simultaneously.
- With piecewise-differentiable incentive functions, (local) credibility is rather easy to obtain, and many slopes are generally allowed for these incentive functions. The actual challenge is to find incentive functions that provide a "domain of credibility" as large as possible.

Extensions

As logical continuations of this work, we mention:

- Study whether credibility of open-loop strategies may hold in a neighborhood of the equilibrium, not only in a particular subset of deviations.
- Extend the analysis to discrete-time problems.
- Investigate incentives defined on Nash-Feedback strategies.
- Extend study to $T = +\infty$, discounted.
- Alternate "rationality" condition: consistency of conjectural equilibria?