Optimality of Impulse Harvesting Policies

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Plan

Progress

Context

Optimal exploitation of a “stock” which has a natural increase rate and can be consumed to produce utility.

- Analysis of harvesting behavior in renewable resource economics (“Mathematical BioEconomics”).

  Describe the harvesting process (fishery, forestry), emphasizing different aspects of harvesting behavior and resulting in different harvesting policies.

- Optimal capital accumulation and consumption
  Describe the balance between capital accumulation and consumption in one-sector economies.

Modeling

Different types of mathematical models are used for this.
In the context of optimal control problems:

- stochastic or deterministic
- finite or infinite horizon
- discrete or continuous time

There exists also a literature on (not necessary optimal) continuous/impulse intervention, *e.g.* biological pest control.

Main Questions

The main structural questions are:

- does there exist stationary values for stock and consumption?
- do optimal paths converge smoothly (turnpike property) or do they oscillate?
- can the system be chaotic?
- how does this depend on the properties of the utility and production functions, and the discount rate?

General answers

- there are stationary points
- in discrete-time: there can be turnpike, cycles and even chaos for 1-dimensional models;
- in continuous-time: continuous consumption in 1-D, cycles possible if several dimensions

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A singular control model: Clark's model.

Continuous time, infinite horizon control problem

$$\max_{h(\cdot)} \int_0^\infty e^{-rt} \left[p - c(x(t))\right] h(t) dt$$

$$\dot{x}(t) = F(x(t)) - h(t) \quad x(0) = x_0, \quad 0 \leq h(t) \leq h_{max}$$

- $x(t)$ is the level of the resource stock at time t, $F(x)$ is the natural growth function,
- p represents the resource price,
- $c(x)$ the unit harvest costs and r the discount rate.

A singular control model: Results

The solution is found as follows:

- The profit maximizing stock level leads to a steady state x^* solution of:

$$\boxed{F'(x^*) \; - \; \frac{c'(x^*)F(x^*)}{p - c(x^*)} \; = \; r \; .}$$

- If $x_0 < x^*$, the optimal control is $h(t) = 0$ as long as $x(t) < x^*$.
- If $x_0 > x^*$, the optimal control is $h(t) = h_{max}$ as long as $x(t) > x^*$.
- If $x(t) = x^*$, the optimal control keeps $x(t)$ constant.

$\Longrightarrow$ A turnpike and most rapid approach trajectory

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Discrete-time modeling

Two versions of the principal model:

- the primitive form:

$$\max_{\{c_t\}} \sum_t \delta^t U(k_t, c_t)$$

  with $k_{t+1} = f(k_t) - c_t$ the production function.

- the reduced form:

$$\max_{\{k_t\}} \sum_t \delta^t V(k_t, k_{t+1})$$

Classical assumption: U depends only on c_t and is concave, production function is concave.
Non-classical: "wealth effects", non-concavities.

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The model dynamics

We consider a renewable resource, the dynamics of which, absent any harvest is given by: $\dot{x}(t) \;=\; F(x(t))\;, \quad t \geq 0$,
$x(t)$ is the size of the population at any time t
F, stationary through time, is the growth rate function.

- $\exists\ x_{sup}$ and x_{ns}, $0 < x_{ns} < x_{sup} < +\infty$.
- $F : (0, x_{sup}) \to \mathbb{R}$ is of class C^2
- positive over the interval $(0, x_{ns})$ and negative over the interval (x_{ns}, x_{sup}),
- $F(0) = F(x_{ns}) = 0$, where $\lim_{x\downarrow 0} F(x) = F(0)$, and $\lim_{x\uparrow x_{sup}} F(x) = -\infty$.

Impulse policies

An impulse policy $IP := \{(t_i, l_i), i = 1, 2, \ldots\}$ as a sequence of harvesting dates t_i and instantaneous harvests l_i, one for each date.

- $0 \leq t_1$, and $t_i < t_{i+1}$ for each $i = 1, 2, \ldots$.
- if the sequence is finite with $n \geq 0$ values, then $t_i = +\infty$ for all $i > n$.
- $l_i \leq 0$ and $x_i - l_i \geq 0$, where x_i is the size of the population just before the harvesting date t_i.

The model: The profit function

We assume that the profit function is stationary through time so that whatever t_i, l_i and x_i, the current profits at time t_i amount to $\pi(x_i, l_i)$.

- the domain is $\mathcal{D} := \{(x, l), x \in (0, x_{sup}),\ l \in [0, x)\}$.
- It is of class C^2 and bounded above by $\bar{\pi} < +\infty$,
- $\pi(x, 0) = 0,\ \forall x \in (0, x_{sup})$.
- $(\partial\pi/\partial l)(x, l)$ admits a limit when $l \downarrow 0$ for all $x \in (0, x_{sup})$.

An impulse optimal control problem

(P): Maximize over t_i, I_i for $i = 1 \ldots \infty$

$$\mathcal{G}(\{t_i, I_i\}_{i=1}^{\infty}, x_0) \;=\; \sum_{i=1}^{\infty} e^{-rt_i}\pi(x_i, I_i)$$

s.t.

$$\dot{x}(t) = F(x(t)) \quad \text{if } t \geq 0, \quad t \notin t_i \;, \quad i = 1, 2, .. \quad x(0) = x_0$$

$$\lim_{t \to t_i^+} x(t_i) = \lim_{t \to t_i^-} x(t_i) - I_i, \quad x_i = \lim_{t \to t_i^-} x(t_i),$$

$$I_i \leq x_i, \qquad x(t) \in [0, x_{sup}].$$

Boundedness of the profit

Does the objective function have a finite supremum?

Denote $\pi^+(x, I) = \max(\pi(x, I), 0)$.

Property

Assume that for all x and some constant ℓ.

$$\pi^+(x, I) \ \leq \ \ell I \ .$$

Then for any $\Delta t \leq (F_{\max})^{-1}$,

$$\Pi \ \leq \ \ell \, \frac{x_{ns} + F_{\max}\Delta t}{1 - \exp(-r\Delta t)} \ .$$

Dynamic optimization from the book

Usual approach: Maximum Principle + Lagrangian/Hamiltonian. See Termansen, Léonard and Long, Seierstad and Sydsaeter.

The Hamiltonian:

$$H(x, \lambda) = \lambda(t) F(x(t)),$$

the discounted instantaneous cost $\pi(x, I, t) = e^{-rt} g(x, x - I)$.

At the points without jumps ($t \neq t_j$):

$$\dot{\lambda}(t) = -\lambda(t) \frac{\partial F}{\partial x}(x(t)), \qquad \lambda(t) \geq \frac{\partial \pi(x(t), 0, t)}{\partial I}.$$

Dynamic optimization from the book (ctd.)

At the jump points:

$$\lambda_j^+ = \frac{\partial \pi(x_j^-, I_j^*, t_j)}{\partial I},$$

$$\lambda_j^+ - \lambda_j^- = -\frac{\partial \pi(x_j^-, I_j^*, t_j)}{\partial x},$$

$$H(x_j^+, \lambda_j^+) - H(x_j^-, \lambda_j^-) - \frac{\partial \pi(x_j^-, I_j^*, t_j)}{\partial t} = 0.$$

Notation: $x_j^- = \lim_{x \to t_i^-} x(t_i)$, and $x_j^+ = \lim_{x \to t_i^+} x(t_i)$.
Likewise, for λ_j^- and λ_j^+.

The dynamic programming principle

Value function approach (original in this context).

Theorem

The value function

$$v(x) \; = \; \sup_{\mathrm{IP} \in \mathcal{F}_x} \Pi(\mathrm{IP})$$

is the unique solution of the following variational equation:

$$v(x) \; = \; \sup_{\substack{y \in [0, x_{sup}) \\ t \geq 0}} \; e^{-rt} \left[\pi(\phi(t,x), \phi(t,x) - y) + v(y) \right] \; ,$$

where $\phi(t,x)$ is the trajectory of the system at time t, solution of the dynamics with $x(0) = x$.

Cyclical policies

Consider $x_0 \leq \bar{x}$ and the family of policies

Cyclical Policy

A cyclical policy consists in:

- let the resource x_t grow until $\bar{x}$,
- harvest until $\underline{x}$

and repeat.

Define $\tau(x, y)$ as the time necessary for the dynamics to go from value x to y:

$$\tau(x, y) = \int_x^y \frac{1}{F(u)} \mathrm{d}u.$$

Special cases: $x = 0$ and $y = x_{ns}$.

Value of cyclical policies

Consider $x_0 \leq \bar{x}$. A cyclical policy has two parameters, $\underline{x}$ and $\bar{x}$, $l = \bar{x} - \underline{x}$. Define:

$$G(\underline{x}, \bar{x}, x_0) \ := \ \pi(\bar{x}, \bar{x} - \underline{x}) \frac{e^{-r\tau(x_0, \bar{x})}}{1 - e^{-r\tau(\underline{x}, \bar{x})}} \ .$$

G corresponds to $\mathcal{G}$ valued at: $t_1 = \tau(x_0, \bar{x})$,
$t_i = t_1 + (i-1)\tau(\underline{x}, \bar{x})$, $i = 2\ldots$, $x_i = \bar{x}, x_i - l_i = \underline{x}$, $i = 1\ldots$.

Limiting case: $\underline{x} = \bar{x}$. Then:

$$G_d(x) := G(x, x, x_0) \ = \ \pi_I(x, 0) \ \frac{F(x)}{r} \ e^{-r\tau(x_0, x)} \ .$$

Auxiliary problems

We define now:

Auxiliary problem (AP)

$$(AP): \qquad \max_{\underline{x},\, \bar{x};\ \underline{x} \leq \bar{x}} G(\underline{x}, \bar{x}, x_0).$$

Under the assumption that (AP) has a unique solution $(\underline{x}^*, \bar{x}^*)$:

Auxiliary problem (TP)

$$(TP): \qquad \max_{\substack{x,y;\\ 0 \leq y \leq x \leq x_{ns}\\ x_0 \leq x;\ y \leq \bar{x}^*}} e^{-r\tau(x_0,x)} \left[\pi(x, x-y) + G(\underline{x}^*, \bar{x}^*, y)\right] .$$

Relations between problems (P), (AP) and (TP)

Characterization of the solution to (P):

Theorem

Under the assumptions on $F(\cdot)$ and $\pi(\cdot,\cdot)$, if:

- for every $a > b \geq c > d$,

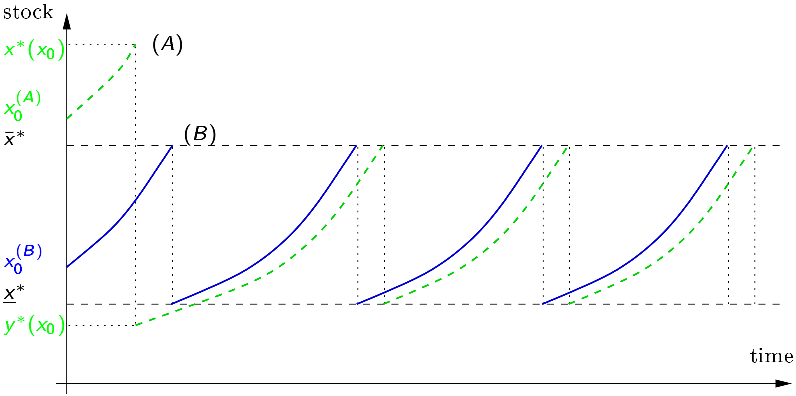
$$\pi(a, a-c) + \pi(b, b-d) \;\leq\; \pi(a, a-d) + \pi(b, b-c)$$

- Problem (AP) has a unique solution, $\underline{x}^*, \bar{x}^*$,

let $(x^*(x_0), y^*(x_0))$ solve the maximization problem (TP). Then the value function of (P) is: $v(x_0) =$

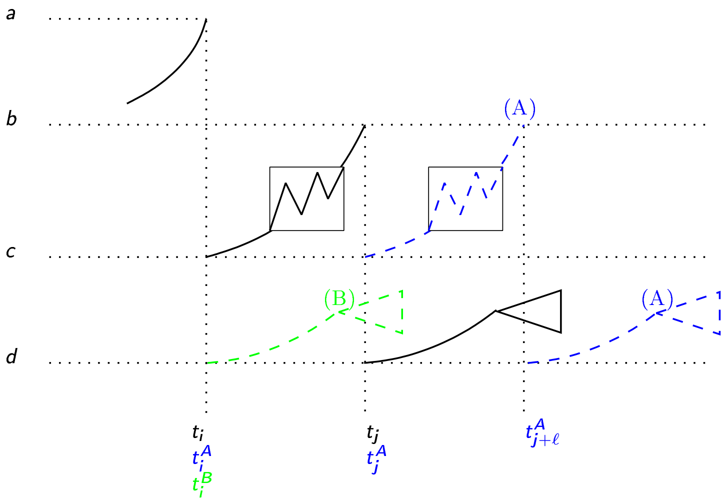
$$= \begin{cases} G(\underline{x}^*, \bar{x}^*, x_0) & \textit{if} \quad x_0 < \bar{x}^* \\ e^{-r\tau(x_0, x^*(x_0))}\left[\pi(x^*(x_0), x^*(x_0) - y^*(x_0)) \right. & \\ \left. \quad + G(\underline{x}^*, \bar{x}^*, y^*(x_0))\right] & \textit{if} \quad x_{ns} \geq x_0 \geq \bar{x}^*. \end{cases}$$

Optimal trajectories



Submodularity (1)

For every solution to problem (P) which is not cyclical, there exists a cyclical solution with the same value.



Submodularity (2)

The assumption

$$\pi(a, a-c) + \pi(b, b-d) \;\leq\; \pi(a, a-d) + \pi(b, b-c)$$

is, with $g(x, y) = \pi(x, x-y)$,

Submodularity

For all $a \geq b \geq c \geq d$

$$g(a, c) + g(b, d) \;\leq\; g(a, d) + g(b, c)$$

A particular case, since $g(x, x) \equiv 0$:

Triangular constraint

$$g(a, c) + g(c, d) \;\leq\; g(a, d)$$

Existence of solutions to Problem (P)

Existence of solutions to (P) $\equiv$ location of solutions to (AP).

Theorem

Under the submodularity assumption, if the solution to (AP) is:

- in the interior $\underline{x} < \bar{x}$, there exists a solution to problem (P), and the solution can be chosen as cyclical.
- on the boundary $\underline{x} = \bar{x}$, there is no solution to (P), but sequences of ε-solutions corresponding to harvests $[\underline{x}, \underline{x} + \varepsilon]$.

Degenerate or not?

Back to the submodularity assumption.

$$\pi(a, a-c) + \pi(b, b-d) \;\leq\; \pi(a, a-d) + \pi(b, b-c)$$

If equality holds: there exists some integrable function $\gamma(\cdot)$:

$$\pi(\bar{x}, \bar{x} - \underline{x}) \;=\; \int_{\underline{x}}^{\bar{x}} \gamma(x)\, \mathrm{d}x$$

Result

Assume that the function $G_d(\cdot)$ is of class C^1, and is $\nearrow$, then $\searrow$, with an unique maximum at x_m.
If the function π satisfies the submodularity assumption

- in the strict sense, then all solutions to Problem (AP) are non-degenerate.
- with equality, then the solution of Problem (AP) is unique and given by $\underline{x} = \bar{x} = x_m$.

Exhausting the resource?

We note "ELB" (Exhaustion locally better). According to the form of the growth function $F(x)$ as $x \to 0$, we have:

$\underline{x} = 0$ optimal?

i) If $F(x) \sim \alpha x^{\beta}$ with $\alpha > 0$ and $\beta > 1$, then the ELB property holds.

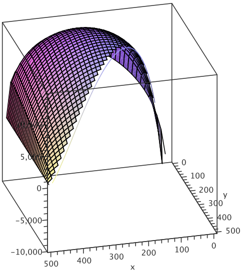
ii) If $F(x) = \alpha x + O(x^2)$, and if $a = r/\alpha$, then:

- ii.1) if $a > 1$, then ELB holds.
- ii.2) if $a < 1$, then ELB does not hold.
- ii.3) if $a = 1$… technical necessary condition for ELB involving $F()$ and $\pi()$.

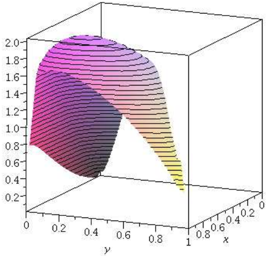
iii) If $F(x) \sim \alpha x^{\beta}$ with $\alpha > 0$ and $0 \leq \beta < 1$, then ELB does not hold.

Illustration

The function $G(\underline{x}^*, \bar{x}^*)$ turns out to be concave in most situations.



$\pi(x, l) = (a - b/x)l$



$\pi(x, l) = (a + bx)l$

Progress

Link with continuous control problem

If the solution of (AP) is on the boundary ($x^* = \underline{x} = \bar{x}$), there does not exist a solution to (P).

However, there exists a sequence of cyclical impulse controls with $\bar{x} - \underline{x} = \epsilon$ (ϵ-optimal solutions of (P)) approaching the value $G_d(x^*) := G(x^*, x^*, x_0), \forall x_0$.

We have:

$$G_d(x) \quad = \quad \frac{\partial \pi}{\partial l}(x, 0)\ \frac{F(x)}{r}\ e^{-r\tau(x_0,x)}$$

Maximizing $G_d(x)$:

$$0 \;=\; \frac{\pi_{Ix}}{\pi_I}(x,0) \;+\; \frac{F'(x)}{F(x)} \;-\; \frac{r}{F(x)}\;.$$

This value is the value of the solution to the continuous singular control problem

$$\max_h \int_0^\infty e^{-rt}\, \frac{\partial \pi}{\partial I}(x,0)\; h\; dt, \quad \dot{x} = F(x) - h.$$

The turnpike x^* solves $G_d'(x) = 0$.

Continuous to impulse

In the other direction, the singular control model "tends" to a mixed impulse/continuous control.
The turnpike:

- If $x_0 < x^*$, the optimal control is $h(t) = 0$ as long as $x(t) < x^*$.
- If $x_0 > x^*$, the optimal control is $h(t) = h_{max}$ as long as $x(t) > x^*$.
- If $x(t) = x^*$, the optimal control keeps $x(t)$ constant.

becomes, when $h_{max} \to \infty$:

- If $x_0 < x^*$, the optimal control is $h(t) = 0$ as long as $x(t) < x^*$.
- If $x_0 > x^*$, harvest down to x^*.
- If $x(t) = x^*$, the optimal control keeps $x(t)$ constant.

Progress

Conclusion

- Difficult to find conditions on F, π which ensure the existence of a solution to (P).
- Find instead conditions on F, π (cost) for the (AP) to have a solution with $\underline{x} < \bar{x}$... or with $\underline{x} = \bar{x}$.
- Find a framework to handle at the same time continuous and impulse control.
- Go to higher dimension

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