Analysis of stochastic min-max systems: results and conjectures

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ANALYSIS OF STOCHASTIC MIN-MAX SYSTEMS: RESULTS AND CONJECTURES

Alain Jean-Marie*and Geert Jan Olsder†

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Abstract

Systems in which the operations min, max and addition appear simultaneously are called min-max systems. Such systems, which are extensions of timed discrete event systems (which on their turn are based on the max-plus algebra, i.e. on the operations max and addition only), have been studied for some years now [4], [3], and [2]. In these references only deterministic systems were studied. In the current paper some stochastic extensions will be considered. It will be shown that extensions of eigenvalues, Lyapunov coefficients exist for these stochastic systems. Some conjectures will be given and they are supported by characteristic examples.

1 Introduction

We will study systems of the form

$$x(k+1) = A \otimes x(k) \vee B \otimes y(k) \ , \tag{1}$$

$$y(k+1) = C \odot x(k) \wedge D \odot y(k) \ . \tag{2}$$

The juxtaposition of the vectors $x = (x_1, \ldots, x_n)'$ and $y = (y_1 \ldots, y_m)'$, where $'$ is the transpose, is the state of the system. The quantities A, B, C and D are matrices of order $n \times n$, $n \times m$, $m \times n$ and $m \times m$ respectively. One writes $A = \{a_{ij}\}$, $B = \{b_{ij}\}$, $C = \{c_{ij}\}$ and $D = \{d_{ij}\}$. We assume on the one hand that the coordinates of x and the entries of A and B are elements of $R \cup \varepsilon$ and on the other hand that the coordinates of y and the entries of C and D are elements of $R \cup \top$. The symbols ε and $\top$ denote $-\infty$ and $+\infty$ respectively. The operator $\otimes$ is the multiplication in the max-plus algebra; $\vee$ is the addition in the same algebra; the operators $\odot$ and $\wedge$ are respectively the multiplication and addition in the min-plus algebra. One has the convention that $\top \otimes \varepsilon = \varepsilon$ and $\top \odot \varepsilon = \top$. The conventional notation of (1) and (2) is:

$$\begin{aligned} x_i(k+1) &= \max(a_{i1} + x_1(k), \ldots, a_{in} + x_n(k), \\ &\qquad b_{i1} + y_1(k), \ldots, b_{im} + y_m(k)), \ \ i = 1, \ldots, n, \\ y_i(k+1) &= \min(c_{i1} + x_1(k), \ldots, c_{in} + x_n(k), \\ &\qquad d_{i1} + y_1(k), \ldots, d_{im} + y_m(k)), \ \ i = 1, \ldots, m. \end{aligned}$$

If an initial condition $(x(0), y(0))$ is given, one can calculate the evolution of the equations (1) and (2) uniquely, i.e. one can calculate $(x(k), y(k))$ for $k = 1, 2, \ldots$.

Before formulating extensions to stochastic systems and problems which will be studied in this paper, we will first briefly give some known properties of the deterministic system (1), (2). System (1), (2) can be thought of to consist of two independent subsystems

$$x(k+1) = A \otimes x(k), \ \ y(k+1) = D \odot y(k), \tag{3}$$

which are connected by the matrices B and C. Properties of these subsystems (3) are well known; see [1].

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Definition .1 *If the matrix A is irreducible, there exists an eigenvalue (denoted by $\lambda_{\max}$) which is unique and an eigenvector $v \neq \varepsilon$ (not necessarily unique) such that $A \otimes v = \lambda_{\max} \otimes v$.*

In the same vein, the matrix D has a unique eigenvalue $\lambda_{\min}$.

Definition .2 *The precedence graph $\mathcal{G}(A)$ of a square $n \times n$ matrix A is a digraph consisting of n nodes (one for each column of the matrix A). This graph has an arc between j and i iff $a_{ij} \neq \varepsilon$. The value a_{ij} is attached to this arc.*

The precedence graph $\mathcal{G}(D)$ is likewise defined.

We will write the system (1), (2) more compactly as $(x(k+1), y(k+1)) = \mathcal{M}((x(k), y(k)))$. The graph $\mathcal{G}(\mathcal{M})$ is made up of the two graphs $\mathcal{G}(A)$ and $\mathcal{G}(D)$, together with arcs between $\mathcal{G}(D)$ and $\mathcal{G}(A)$ (representing the elements $b_{ij} \neq \varepsilon$ of B and the arcs between $\mathcal{G}(A)$ and $\mathcal{G}(D)$ (representing the elements $c_{ij} \neq \top$ of C).

Definition .3 *A scalar λ, $\varepsilon \leq \lambda \leq \top$, is called an eigenvalue of the mapping $\mathcal{M}$ if there exists a vector (x, y), where at least x or y has a finite component (i.e. it belongs to R), such that*

$$(\lambda \otimes x, \lambda \odot y) = \mathcal{M}((x, y)). \tag{4}$$

The vector (x, y) is called an eigenvector of $\mathcal{M}$.

Theorem .4 *Suppose A and D are irreducible and $B \neq \varepsilon$, $C \neq \top$. The mapping $\mathcal{M}$ has a unique eigenvalue λ and a corresponding eigenvector iff $\lambda_{\max} \leq \lambda_{\min}$. One then has $\lambda_{\max} \leq \lambda \leq \lambda_{\min}$.*

The problem studied in this paper deals with stochastic extensions of Theorem .4. It will be assumed that (some of) the elements of the matrices A, B, C and D may be stochastic. Since the uncertainty can depend on k, we will write $A(k)$, $a_{ij}(k)$, $B(k)$, etc. Throughout this paper, unless stated differently, it will be assumed that all stochastic variables involved are i.i.d. with respect to k. The distribution function of $a_{ij}(k)$ will be denoted by $\bar{F}_{a_{ij}}$, i.e. $\bar{F}_{a_{ij}(k)}(\alpha) = P(a_{ij}(k) \leq \alpha)$. Because of the i.i.d. assumption, this distribution does not depend on k. The functions $\bar{F}_{b_{ij}}$. $\bar{F}_{c_{ij}}$ and $\bar{F}_{d_{ij}}$ are likewise defined. Mainly for notational convenience it will be assumed that the variables $a_{ij}(k)$, $b_{ij}(k)$, $c_{ij}(k)$ and $d_{ij}(k)$ are independent, both with respect to k as well as i and j.

Stochastic extensions of the kind just described but then for the simpler systems in (3) have been known for some time: the reader is referred to [1] for details. Consider $x(k+1) = A \otimes x(k)$ and assume that the stochastic variables in A have bounded support (i.e. if a_{ij} is bounded for some realization, it will be bounded for all realizations) and that the precedence graph of A, well defined because of the bounded support assumption, is strongly connected. Under these assumptions the Lyapunov coefficient of A, to be denoted by $\lambda_{\max}$, is well defined and unique. Similar assumptions for $y(k+1) = D \odot y(k)$ lead to a unique Lyapunov coefficient $\lambda_{\min}$ for the latter system. These Lyapunov coefficients can be interpreted as 'eigenvalues' of the respective stochastic systems and they are equal to the eigenvalues defined in Definition .1 and directly thereafter if the systems happen to be deterministic.

2 General Analysis

2.1 The direct approach

Define

$$F_k(\mu; \nu) \stackrel{\text{def}}{=} P(x(k) \leq \mu; y(k) \leq \nu), \tag{5}$$

which is short-hand notation for

$$F_k(\mu_1, \ldots, \mu_n; \nu_1, \ldots, \nu_m) \stackrel{\text{def}}{=} P(x_1(k) \leq \mu_1, \ldots, x_n(k) \leq \mu_n; y_1(k) \leq \nu_1, \ldots, y_m(k) \leq \nu_m). \tag{6}$$

Employing the same vectorial notation, we can write

$$F_{k+1}(\alpha; \beta) = \int \cdots \int K_k(\alpha; \beta; s; t) F_k(ds; dt) \tag{7}$$

where

$$K_k(\alpha;\beta;s;t) = P(x(k+1) \le \alpha; y(k+1) \le \beta | x(k) = s; y(k) = t). \tag{8}$$

Here the vectors α, β, s, t, ds, dt, μ and ν are defined as

$$\begin{array}{ll} \alpha = (\alpha_1, \ldots, \alpha_n), & \beta = (\beta_1, \ldots, \beta_m), \\ s = (s_1, \ldots, s_n), & t = (t_1, \ldots, t_m), \\ ds = (ds_1, \ldots, ds_n), & dt = (dt_1, \ldots, dt_m), \\ \mu = (\mu_1, \ldots, \mu_n), & \nu = (\nu_1, \ldots, \nu_m). \end{array}$$

Inequalities and equalities between vectors (of the same size) are understood to be componentwise. Because of the assumptions made on the stochastic behavior of the elements of A, B, C and D, the kernel K can be written in terms of known distribution functions as follows:

$$\begin{aligned} &K_k(\alpha;\beta;s;t) = \\ &\prod_{i=1}^{n} P(\max(a_{i1}+s_1, \ldots, a_{in}+s_n; b_{i1}+t_1, \ldots, b_{im}+t_m) \le \alpha_i) \times \\ &\prod_{j=1}^{m} P(\min(c_{j1}+s_1, \ldots, c_{jn}+s_n; d_{j1}+t_1, \ldots, d_{jm}+t_m) \le \beta_j) = \\ &\prod_{i=1}^{n}[\prod_{l=1}^{n} P(a_{il}(k)+s_l \le \alpha_i) \prod_{l=1}^{m} P(b_{il}(k)+t_l \le \alpha_i)] \times \\ &\prod_{j=1}^{m}[1 - \prod_{l=1}^{n}(1 - P(c_{il}(k)+s_l \le \beta_j)) \prod_{l=1}^{m}(1 - P(d_{il}(k)+t_l \le \beta_j))] = \\ &\prod_{i=1}^{n}[\prod_{l=1}^{n} \bar{F}_{a_{il}}(\alpha_i - s_l) \prod_{l=1}^{m} \bar{F}_{b_{il}}(\alpha_i - t_l)] \\ &\times \prod_{j=1}^{m}[1 - \prod_{l=1}^{n}(1 - \bar{F}_{c_{il}}(\beta_j - s_k)) \prod_{l=1}^{m}(1 - \bar{F}_{d_{il}}(\beta_j - t_l))]. \end{aligned}$$

Since K_k does not depend on k, we will simply write K.

We would like to find a distribution function F and a scalar λ such that

$$F(\alpha;\beta) = \int \cdots \int K(\alpha-\lambda;\beta-\lambda;s;t)F(ds;dt), \tag{9}$$

where the notation $\alpha - \lambda$ refers to the subtraction of each component of α by λ and likewise for $\beta - \lambda$. Posed differently, we would like to find a distribution function F such that, if substituted for F_k and F_{k+1} into (7),

$$F(\alpha+\lambda;\beta+\lambda) \equiv F_{k+1}(\alpha+\lambda;\beta+\lambda) \equiv F_k(\alpha;\beta) \equiv F(\alpha;\beta).$$

The additions $\alpha + \lambda$ and $\beta + \lambda$ are again to be understood as the addition of λ to each component of the vectors α and β respectively. We will now sketch an algorithm which is expected to yield the stationary distribution F and the 'eigenvalue' λ. It is the functional (infinite dimensional) extension of the vectorial (finite dimensional) case as described as Procedure 3, page 459, of [1]. The quantity η in this procedure indicates the accuracy with which λ will be found.

1. Choose an arbitrary distribution function F.
2. Calculate

$$\underline{c} \stackrel{\text{def}}{=} \inf_{\alpha,\beta}\{\lambda | F_{\text{aux}}(\alpha;\beta|\lambda) - F(\alpha;\beta)\}; \tag{10}$$

$$\bar{c} \stackrel{\text{def}}{=} \sup_{\alpha,\beta}\{\lambda | F_{\text{aux}}(\alpha;\beta|\lambda) - F(\alpha;\beta)\}, \tag{11}$$

   where

$$F_{\text{aux}}(\alpha;\beta|\lambda) \stackrel{\text{def}}{=} \int \cdots \int K(\alpha-\lambda;\beta-\lambda;s;t)F(ds;dt).$$

3. If $\bar{c} - \underline{c} < \eta$, then stop ($\lambda$ is the eigenvalue).

4. Construct a new function F in the following way and go back to step 2. The superscript inv will refer to the inverse function. As an example, F^{inv} is a mapping from $[0,1]$ to R^{m+n}. Construct $F^{\text{inv}}_{\text{new}} = \min(F^{\text{inv}}(\alpha;\beta), F^{\text{inv}}_{\text{aux}}(\alpha;\beta,\underline{c}+\eta/2))$; this minimization is componentwise. The inverse function of $F^{\text{inv}}_{\text{new}}$, i.e. F_{new}, is the new function F.

2.2 The approach with the differences

Later on we will also consider the values of $x_i(k)$ and $y_j(k)$ normalized with respect to $x_1(k)$, i.e. we will consider $v_i(k) \stackrel{\text{def}}{=} x_i(k) - x_1(k), \quad i = 2,\ldots,n$, and $w_i(k) \stackrel{\text{def}}{=} y_i(k) - x_1(k), \quad i = 1,\ldots,m$. The equations for v_i and w_j are respectively

$$\begin{aligned} v_i(k+1) &= \max(a_{i1}, a_{i2}+v_2(k),\ldots,a_{in}+v_n(k); b_{i1}+w_1(k),\ldots,b_{im}+w_m(k)) && (12)\\ &\quad - \max(a_{11}, a_{12}+v_2(k),\ldots,a_{1n}+v_n(k); b_{11}+w_1(k),\ldots,b_{1m}+w_m(k)) && (13)\\ w_j(k+1) &= \min(c_{j1}, c_{j2}+v_2(k),\ldots,c_{jn}+v_n(k); d_{j1}+w_1(k),\ldots,d_{jm}+w_m(k)) && (14)\\ &\quad - \max(a_{11}, a_{12}+v_2(k),\ldots,a_{1n}+v_n(k); b_{11}+w_1(k),\ldots,b_{1m}+w_m(k)). && (15) \end{aligned}$$

Define now

$$\hat{F}_k(\hat{\mu};\hat{\nu}) \stackrel{\text{def}}{=} P(v(k) \le \hat{\mu}; w(k) \le \hat{\nu}), \tag{16}$$

which is short-hand notation for

$$\hat{F}_k(\hat{\mu}_2,\ldots,\hat{\mu}_n;\hat{\nu}_1,\ldots,\hat{\nu}_m) \stackrel{\text{def}}{=} P(v_2(k) \le \hat{\mu}_2,\ldots,v_n(k) \le \hat{\mu}_n; w_1(k) \le \hat{\nu}_1,\ldots,w_m(k) \le \hat{\nu}_m). \tag{17}$$

We now write

$$\hat{F}_{k+1}(\hat{\alpha};\hat{\beta}) = \int\cdots\int \hat{K}_k(\hat{\alpha};\hat{\beta};\hat{s};\hat{t})\hat{F}_k(d\hat{s};d\hat{t}) \tag{18}$$

where

$$\hat{K}_k(\hat{\alpha};\hat{\beta};\hat{s};\hat{t}) = P(v(k+1) \le \hat{\alpha}; w(k+1) \le \hat{\beta} | v(k) = \hat{s}; w(k) = \hat{t}). \tag{19}$$

Here the vectors $\hat{\alpha}$, $\hat{\beta}$, $\hat{s}$, $\hat{t}$, $d\hat{s}$, $d\hat{t}$, $\hat{\mu}$ and $\hat{\nu}$ are defined as

$$\begin{array}{ll} \hat{\alpha} = (\hat{\alpha}_2,\ldots,\hat{\alpha}_n), & \hat{\beta} = (\hat{\beta}_1,\ldots,\hat{\beta}_m),\\ \hat{s} = (\hat{s}_2,\ldots,\hat{s}_n), & \hat{t} = (\hat{t}_1,\ldots,\hat{t}_m),\\ d\hat{s} = (d\hat{s}_2,\ldots,d\hat{s}_n), & d\hat{t} = (d\hat{t}_1,\ldots,d\hat{t}_m),\\ \hat{\mu} = (\hat{\mu}_2,\ldots,\hat{\mu}_n), & \hat{\nu} = (\hat{\nu}_1,\ldots,\hat{\nu}_m). \end{array}$$

The kernel $\hat{K}$ can be written as

$$\begin{aligned} &\hat{K}(\hat{\alpha};\hat{\beta};\hat{s};\hat{t}) = \\ &\quad \prod_{i=2}^{n} P[\max(a_{i1}, a_{i2}+\hat{s}_2,\ldots,a_{in}+\hat{s}_n; b_{i1}+\hat{t}_1,\ldots,b_{im}+\hat{t}_m) \le \\ &\quad \hat{\alpha}_i + \max(a_{11}, a_{12}+\hat{s}_2,\ldots,a_{1n}+\hat{s}_n; b_{11}+\hat{t}_1,\ldots,b_{1m}+\hat{t}_m)] \times \\ &\quad \prod_{j=1}^{m} P[\min(c_{j1}, c_{j2}+\hat{s}_2,\ldots,c_{jn}+\hat{s}_n; d_{j1}+\hat{t}_1,\ldots,d_{jm}+\hat{t}_m) \le \\ &\quad \hat{\beta}_j + \max(a_{11}, a_{12}+\hat{s}_2,\ldots,a_{1n}+\hat{s}_n; b_{11}+\hat{t}_1,\ldots,b_{1m}+\hat{t}_m)] = \\ &\quad \int \prod_{i=2}^{n}[P(a_{i1} \le \hat{\alpha}_i + u)\prod_{l=2}^{n} P(a_{il}+\hat{s}_l \le \hat{\alpha}_i + u)\prod_{l=1}^{m} P(b_{il}(k)+\hat{t}_l \le \hat{\alpha}_i + u)] \times \\ &\quad \prod_{j=1}^{m}[(1-(1-P(c_{j1} \le \hat{\beta}_j + u))\prod_{l=2}^{n}(1-P(c_{jl}+\hat{s}_l \le \hat{\beta}_j + u)))(1-\prod_{l=1}^{m}(1-P(d_{jl}+\hat{t}_l \le \hat{\beta}_j + u)))] \times \\ &\quad \frac{d}{du}[P(a_{11} \le u)\prod_{l=2}^{n} P(a_{1l}+\hat{s}_l \le u)\prod_{l=1}^{m} P(b_{1l}+\hat{t}_l \le u)]du. \end{aligned}$$

Once the kernel $\hat{K}(\hat{\alpha};\hat{\beta};\hat{s};\hat{t})$ is known and then substituted into (18), one can calculate $\hat{F}(\hat{\mu};\hat{\nu})$ as the stationary solution (which itself is a distribution function) of (18). Consider next

$$\begin{aligned}
&\lim_{k\to\infty} P(x_1(k+1)-x_1(k)\le\gamma) = \\
&\quad \lim_{k\to\infty} P[\max\{a_{11}(k), a_{12}(k)+v_2(k),\ldots,a_{1n}(k)+v_n(k), b_{11}(k)+w_1(k),\ldots,b_{1m}(k)+w_m(k)\}\le\gamma] = \\
&\quad \lim_{k\to\infty} P[a_{11}(k)\le\gamma, a_{12}(k)\le\gamma-v_2(k),\ldots,b_{1m}(k)\le\gamma-w_m(k)] = \\
&\quad \bar{F}_{a_{11}}(\gamma)\int\cdots\int\prod_{i=2}^{n}(\bar{F}_{a_{1i}}(\gamma-\hat{s}_i)\prod_{j=1}^{n}(\bar{F}_{b_{1j}}(\gamma-\hat{t}_j)\hat{F}(d\hat{s};d\hat{t}).
\end{aligned}$$

Here we tacitly assumed that limit and integral could be interchanged. The expectation of the stochastic variable $(x_1(k+1)-x_1(k))$, when $k\to\infty$, can now readily be calculated and it must be equal to λ of the previous subsection.

3 Conjectures

Throughout this section we will assume that all stochastic variables in A, B, C and D have bounded support, that the precedence graphs of A and D are strongly connected and that $B\ne\varepsilon$, $C\ne\top$. Two conjectures will be formulated; proofs of them are not (yet) known, but their evidence will be illustrated by characteristic examples in the next sections.

Conjecture .5 *If the Lyapunov coefficients $\lambda_{\max}$ and $\lambda_{\min}$ as defined at the end of section 1 satisfy $\lambda_{\max}<\lambda_{\min}$, then the eigenvalue λ of the total system exists and it satisfies $\lambda_{\max}\le\lambda\le\lambda_{\min}$.*

For the next conjecture it is assumed that $\lambda_{\max}<\lambda_{\min}$. Suppose that the stochastic variables in $A(k)$ could be chosen independent of k and moreover in such a way that the critical cycle in the precedence graph of this A (now independent of k) is as slow as possible. The corresponding λ will be indicated by $\lambda^{\mathrm{s}}_{\max}$. Said differently, $\lambda^{\mathrm{s}}_{\max}$ corresponds to the slowest possible behaviour of $x(k+1)=A\otimes x(k)$ if the uncertain (i.e. stochastic) entries of A might be chosen at will. Similarly $\lambda^{\mathrm{f}}_{\min}$ is defined; it is the fastest cycle which could occur with respect to all possible samples of D, given that they must be independent of time.

Conjecture .6 *If $\lambda^{\mathrm{s}}_{\max}<\lambda^{\mathrm{f}}_{\min}$, then the support of $w(k)$ and $v(k)$ is bounded. If on the other hand, $\lambda^{\mathrm{s}}_{\max}>\lambda^{\mathrm{f}}_{\min}$, then this support is not necessarily bounded.*

4 Example 1

Consider the following system with $m=n=1$;

$$x(k+1) = \max(a_{11}+x(k), 3+y(k)), \tag{20}$$
$$y(k+1) = \min(3+x(k), 5+y(k)) \tag{21}$$

where for each k the entry a_{11} has value 0 with probability 0.5 and value 4 also with probability 0.5. For ease of notation, we have omitted the subscripts '1' for the states x_1 and y_1. This same convention will also be valid for other one-dimensional vectors later on in this section. The kernel $\hat{K}_k$, as defined in (19), becomes for this system

$$\hat{K}_k(\hat{\alpha};\hat{\beta};\hat{s};\hat{t}) = P(w(k+1)\le\hat{\beta}|w(k)=\hat{t}) \tag{22}$$

and is independent of $\hat{\alpha}$ and $\hat{s}$. By abuse of notation we will simply write $\hat{K}_k(\hat{\beta};\hat{t})$. Since

$$w(k+1)=y(k+1)-x(k+1)=\min(3,w(k)+5)-\max(a_{11},w(k)+3),$$

we get

$$\hat{K}_k(\hat{\beta};\hat{t}) = P(-\max(a_{11},\hat{t}+3)\le\hat{\beta}-\min(3,\hat{t}+5)) =$$

$$1 - P(a_{11} \le -\hat\beta + \min(3, \hat t + 5))P(\hat t + 3 \le -\hat\beta + \min(3, \hat t + 5))$$

$$= \begin{cases} 1 - P(a_{11} \le -\hat\beta + \hat t + 5), & \text{if } \hat t \le -2 \text{ and } \hat\beta \le 2, \\ 1, & \text{if } \hat t \le -2 \text{ and } \hat\beta > 2, \\ 1 - P(a_{11} \le -\hat\beta + 3), & \text{if } \hat t > -2 \text{ and } \hat t \le -\hat\beta, \\ 1, & \text{if } \hat t > -2 \text{ and } \hat t > -\hat\beta. \end{cases}$$

Again with some abuse of notation, (18) becomes for this example

$$\hat F_{k+1}(\hat\beta) = P(w(k+1) \le \hat\beta) = \int \hat K_k(\hat\beta; \hat t) \frac{d}{dt} \hat F_k d\hat t,$$

which yields

$$\begin{array}{ll} \hat F_{k+1}(\hat\beta) = 1, & \text{if } \hat\beta > 2, \\ \hat F_{k+1}(\hat\beta) = \frac{1}{2}\hat F_k(\hat\beta - 5) - \frac{1}{2}\hat F_k(-\hat\beta) + 1, & \text{if } -1 < \hat\beta \le 2, \\ \hat F_{k+1}(\hat\beta) = \frac{1}{2}\hat F_k(\hat\beta - 5) + \frac{1}{2}\hat F_k(\hat\beta - 1) + 1 - \hat F_k(-\hat\beta), & \text{if } \hat\beta \le -2. \end{array} \tag{23}$$

To obtain the stationary solution $\hat F$ we substitute $\hat F = \hat F_k = \hat F_{k+1}$ which leads to a functional equation. This functional equation has the degenerate solution $\hat F \equiv 1$. Some trial and error leads to the nondegenerate solution (we now write F_{stat} rather than $\hat F$)

$$F_{\text{stat}}(\hat\beta) = \begin{cases} 0, & \text{if } \hat\beta \le -1, \\ 2/3, & \text{if } -1 < \hat\beta \le 1, \\ 1, & \text{if } 1 < \hat\beta. \end{cases}$$

Another possible way of obtaining the stationary solution is to use the algorithm described in Section 2. Though this algorithm seemed to converge, it is not very efficient and for low dimensional examples trial and error often leads faster to the correct solution.

Since $x(k+1) - x(k) = \max(a_{11}, w(k) + 3)$, we can calculate

$$P(x(k+1) - x(k) \le \gamma) = \bar F_{a_{11}}(\gamma) F_{\text{stat}}(\gamma - 3) = \begin{cases} 0, & \text{if } \gamma \le 2, \\ 1/3, & \text{if } 2 < \gamma \le 4, \\ 1, & \text{if } \gamma > 4. \end{cases}$$

This leads to

$$E[x(k+1) - x(k)] = 10/3,$$

in the stationary situation, and it is this value which can be interpreted as the eigenvalue of the system.

There is another way to calculate the eigenvalue, at least for this example. If the initial condition would be $(0,0)'$, an arbitrary choice, then there are two possibilities for $(x(1), y(1))'$, viz. $(3,3)'$ and $(4,3)'$ which we normalize to $(0,0)'$ and $(0,-1)'$ by subtracting the first component $x(k)$ from each of the components. If the initial condition would have been equal to the latter of these normalized states (the former is equal to the initial condition already treated) then there are again two possibilities for the next state, viz, $(4,3)'$ and $(2,3)'$ which are normalized to $(0,-1)'$ and $(0,1)'$ respectively. Only the latter state is 'new'. If $(0,1)'$ would have been the initial condition then the only next normalized state is $(0,-1)'$. Thus we have constructed a Markov chain with a unique recurrent class consisting of two normalized states, viz. $(0,-1)'$ and $(0,1)'$. If $(x(k), y(k))'$ happens to be $(1,0)$ (normalized), then with chance 1/2 we remain in this state at $k+1$, with chance 1/2 we go to $(0,-1)'$. If $(x(k), y(k))'$ happens to be $(0,-1)$ (normalized), then at the next step the state will be $(1,0)'$ (normalized again). In the stationary situation the state is with chance 2/3 in $(1,0)'$ and with chance 1/3 in $(0,-1)'$. From there it is easy to calculate the eigenvalue, i.e. the average of $x(k+1) - x(k)$ (or of $y(k+1) - y(k)$, which happens to be the same) equals 10/3.

4.1 Generalisation of Example 1

The example in this section resembles the previous one. The only difference is that now for each k the entry a_{11} has value 0 with probability 0.5 and value χ also with probability 0.5; χ is a parameter. If it is assumed that $\chi > 1$, then some easy calculations lead to

$$\begin{array}{ll} \hat{F}_{k+1}(\hat{\beta}) = 1, & \text{if } \hat{\beta} > 2, \\ \hat{F}_{k+1}(\hat{\beta}) = \frac{1}{2}\hat{F}_k(\hat{\beta}-5) - \frac{1}{2}\hat{F}_k(-\hat{\beta}) + 1, & \text{if } 3-\chi < \hat{\beta} \leq 2, \\ \hat{F}_{k+1}(\hat{\beta}) = \frac{1}{2}\hat{F}_k(\hat{\beta}-5) + \frac{1}{2}\hat{F}_k(\hat{\beta}-5+\chi) + 1 - \hat{F}_k(-\hat{\beta}), & \text{if } \hat{\beta} \leq 3-\chi. \end{array} \tag{24}$$

This is in complete accordance with (23) if one substitutes $\chi = 4$.

$\chi = 5$

Let us now take $\chi = 5$. Some trial and error leads to the following stationary solution of (24):

$$F_{\text{stat}}(\hat{\beta}) = \begin{cases} 0, & \text{if } \hat{\beta} \leq -2, \\ 2/3, & \text{if } -2 < \hat{\beta} \leq 2, \\ 1, & \text{if } 2 < \hat{\beta}. \end{cases}$$

Some further calculations, simply mimicking the example with $\chi = 4$, yield

$$E[x(k+1) - x(k)] = 11/3,$$

once the trancient behaviour has died out.

It is also possible to obtain the same result by means of the construction of a Markov chain of the normalized states. For this case the normalized states in the stationary situation happen to be $(0,-2)'$ (where the system is with probability 2/3) and $(0,2)'$ (with probability 1/3).

$\chi = 6$

We now substitute $\chi = 6$ into (24) and the stationary solution $\hat{F}$ satisfies

$$\begin{array}{ll} \hat{F}(\hat{\beta}) = 1, & \text{if } \hat{\beta} > 2, \\ \hat{F}(\hat{\beta}) = \frac{1}{2}\hat{F}(\hat{\beta}-5) - \frac{1}{2}\hat{F}(-\hat{\beta}) + 1, & \text{if } -3 < \hat{\beta} \leq 2, \\ \hat{F}(\hat{\beta}) = \frac{1}{2}\hat{F}(\hat{\beta}-5) + \frac{1}{2}\hat{F}(\hat{\beta}+1) + 1 - \hat{F}(-\hat{\beta}), & \text{if } \hat{\beta} \leq -3, \end{array} \tag{25}$$

which is rewritten as

$$\hat{F}(\hat{\beta}) = 1, \quad \text{if } \hat{\beta} > 2, \tag{26}$$

$$\hat{F}(\hat{\beta}) = \frac{1}{2}\hat{F}(\hat{\beta}-5) - \frac{1}{2}\hat{F}(-\hat{\beta}) + 1, \quad \text{if } -2 < \hat{\beta} \leq 2, \tag{27}$$

$$\hat{F}(\hat{\beta}) = \frac{1}{2}\hat{F}(\hat{\beta}-5) + \frac{1}{2}, \quad \text{if } -3 < \hat{\beta} \leq -2, \tag{28}$$

$$\hat{F}(\hat{\beta}) = \frac{1}{2}\hat{F}(\hat{\beta}-5) + \frac{1}{2}\hat{F}(\hat{\beta}+1), \quad \text{if } \hat{\beta} \leq -3. \tag{29}$$

We try a piecewise constant solution of the form $\hat{F}(\hat{\beta}) = \delta_i$ when $i \leq \hat{\beta} < i+1$, where i runs through all integers. One has to determine the δ_i's. For $i = -4, -5, \ldots$ one assumes a solution of the form $\delta_i = cr^{i-4}$, where $0 < r < 1$. If one substitutes al this into (25), an infinite number of linear equations is obtained which have the nontrivial solution

$$\begin{array}{ll} \delta_i = 1, & \text{if } i = 2, 3, \ldots, \\ \delta_1 = \frac{16}{3}(2c - cr^3 + 2), & \\ \delta_0 = \frac{1}{6}(-32cr^2 + 19cr - 26), & \\ \delta_{-1} = \frac{16}{3}(2cr^2 - cr + 2), & \\ \delta_{-2} = \frac{1}{6}(cr^3 - 32c - 26), & \\ \delta_{-3} = \frac{1}{2}(cr^4 + 1), & \\ \delta_i = cr^{i-4}, & \text{if } i = -4, -5, \ldots, \end{array}$$

where r is the smallest real solution of $2r = r^6 + 1$ (this equation has two real solutions: $r = 1$ and the other real solution, the one we need, is close to 1/2). The constant c is determined by $c(4 - 2r^5 - r^4) = 1$. It turns out that the conditions $\delta_i \leq \delta_{i+1}$, $\forall i$, necessary for $\hat{F}$ to be a distribution function, are fullilled. Note that unlike the previous examples, the stationary distribution function does not have a compact support anymore.

The other solution method, by means of the finite Markov chain on the normalized states, does not work here: if one starts with an arbitrary initial condition, the number of normalized states grows to infinity (as $k \to \infty$). This is not surprising since $\hat{F}$ has infinite support.

$\chi > 6$

For sufficiently small values of $\hat{\beta}$ (i.e. for $\hat{\beta} \leq 3 - \chi$), $\hat{F}$ satisfies

$$\hat{F}(\hat{\beta}) = \frac{1}{2}\hat{F}(\hat{\beta} - 5) + \frac{1}{2}\hat{F}(\hat{\beta} - 5 + \chi)$$

which is the equivalent of (29) for $\chi = 6$. For this equation we try (again) a solution of the form $\hat{F}(\hat{\beta}) = cr^{[\hat{\beta}]}$, where $[\hat{\beta}]$ is the integer part of $\hat{\beta}$. This results in $2r^{\chi-5} = r^{\chi} + 1$ (for $\chi = 6$ this coincides with the previous results). This polynomial in r has a real solution smaller than 1 if $\chi \leq 9$. It has the real solution $r = 1$ if $\chi = 10$ and it does not have real solutions for $\chi \geq 11$.

The above results are in accordance with our conjectures. The average value of a_{11}, i.e. $\lambda_{\max}$, equals $\chi/2$ and this value must be smaller than or equal to $\lambda_{\min} = d_{11} = 5$. The fact that $\hat{F}$ did not have finite support for $\chi \geq 6$ can be made plausible as follows. The lower- and upperbound of the support of a_{11} are respectively 0 and χ. The eigenvalue $\lambda_{\min}$ belongs to the interval $[0, \chi]$. For $\chi \leq 4$ this 'overlap' does not exist, i.e. $\lambda_{\min}$ does not belong to the interval $[0, \chi]$. The case $\chi = 5$ is just on the borderline of the two possibilities.

5 Example 2

Consider now the following system with $m = n = 1$;

$$x(k+1) = \max(x(k), b_{11}(k) + y(k)), \tag{30}$$

$$y(k+1) = \min(x(k), 1 + y(k)) \tag{31}$$

where for each k the entry b_{11} has value 0 with probability 0.5 and value χ also with probability 0.5. For ease of notation, we have omitted the subscripts '1' for the states x_1 and y_1 again. Whatever the value of χ, we always have $\lambda_{\max} = 0$ and $\lambda_{\min} = 1$. Therefore it is conjectured that the eigenvalue of the total system always exists and this will be made plausible by means a closer scrutinization of the example for some specific χ values.

Let us take $\chi = 5$. If the initial condition would be $(0,0)'$, then the next state is either $(0,0)'$ again (with probabiblity 1/2) or $(5,0)'$ (also with probability 1/2). As before, we will normalize these states such that $x = 0$ and thus this next state is either $(0,0)'$ or $(0,-5)'$. The 'normalization factors', i.e. the scalars which are subtracted from the original vector such as to obtain the normalized factor, are 0 and 5 respectively in these cases. If one continues the evolution of the system, it follows that from $(0,-5)'$ one can only reach $(0,-4)'$, which is already normalized (normalization factor is 0). From the latter state one can reach $(0,-3)'$ and $(0,-4)'$, with normalization factors 0 and 1 respectively, etc. It turns out that one obtains a finite Markov chain in this way, with the states $(0,0)'$, $(0,-5)'$, $(0,-4)'$, $(0,-3)'$, $(0,-2)'$ and $(0,-1)'$. The transition matrix and the matrix which contains the normalization factors are respectively

$$\begin{pmatrix} .5 & 0 & 0 & 0 & 0 & .5 \\ .5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & .5 & .5 & .5 & .5 \\ 0 & 0 & .5 & 0 & 0 & 0 \\ 0 & 0 & 0 & .5 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & & & & & 0 \\ 5 & & & & & \\ & 0 & 1 & 2 & 3 & 4 \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \end{pmatrix}.$$

The stationary distribution is now easily calculated; it is $1/33(2,1,16,8,4,2)'$ and with the aid of these probabilities and the normalization factors one obtains $\lambda = 31/33$.

In exactly the same way it can be shown that $\lambda = 1/3$ if $\chi = 1$; $\lambda = 3/5$ if $\chi = 2$, $\lambda = 7/9$ if $\chi = 3$, $\lambda = 15/17$ if $\chi = 4$. Together with $\lambda = 31/33$ for $\chi = 5$ it is clear how the system behaves for larger values of χ. A conclusion might be that the upperlimit $\lambda_{\min}$ is only approached asymptotically. This is not true as is shown by (almost) the same example. The difference will be that now $b_{11}(k) = 3$ with probability 1/2 and $b_{11}(k) = 4$ also with probability 1/2. In this case it can be shown, using the standard method, that $\lambda = \lambda_{\min} = 1$.

6 Example 3

Consider now the following system with $m = n = 1$;

$$x(k+1) = \max(x(k), 3 + y(k)), \tag{32}$$

$$y(k+1) = \min(c_{11}(k) + x(k), 6 + y(k)), \tag{33}$$

where for each k the entry c_{11} has value 1 with probability 0.5 and value 3 also with probability 0.5. As before, we have omitted the subscripts '$1'$ for the states x_1 and y_1 again. This example will show a pitfall of stochastic min-max systems, which is related to the chronological ordering of events.

It is shown easily that, as in the previous section, there is a unique recurrence class, which consists of the five (normalized) states $(0,1)'$, $(0,-3)'$, $(0,-1)'$, $(0,3)'$ and $(0,-5)'$. The transition matrix and the matrix which contains the normalization factors are respectively for this example,

$$\begin{pmatrix} 0 & .5 & .5 & 0 & 1 \\ .5 & 0 & 0 & .5 & 0 \\ .5 & 0 & .5 & 0 & 0 \\ 0 & .5 & 0 & 0 & 0 \\ 0 & 0 & 0 & .5 & 0 \end{pmatrix}, \begin{pmatrix} & 0 & 2 & & 0 \\ 4 & & & 6 & \\ 4 & & 2 & & \\ & 0 & & & \\ & & & 5 & \end{pmatrix}.$$

Of course one can calculate the Lyapunov coefficient again (which turns out to be 48/19), but the fact we want to emphasize here is that in the transition from $(0,3)'$ to $(0,-5)'$, which becomes from $(0,3)'$ to $(6,1)'$ in the real, nonnormalized, situation, there is a problem with causality. If $x(0) = 0$, $y(0) = 3$, one gets $x(1) = 6$ and $y(1) = 1$. Hence $y(1)$ occurs earlier than $y(0)$. If one allows 'overtaking' of events, then the analysis given to calculate the Lyapunov coefficient is correct. If one does not allow such overtaking, then restrictions in the model must be built in such as to avoid this. With such an restriction, the value of the Lyapunov coefficient will become larger. For the reader who wishes to go through this example, with overtaking, in detail, we give the distribution function $F_{\text{stat}}(\hat{\beta})$:

$$F_{\text{stat}}(\hat{\beta}) = \begin{cases} 0, & \text{if } \hat{\beta} \leq -5, \\ 1/19, & \text{if } -5 < \hat{\beta} \leq -3, \\ 5/19, & \text{if } -3 < \hat{\beta} \leq -1, \\ 11/19, & \text{if } -1 < \hat{\beta} \leq 1, \\ 17/19, & \text{if } 1 < \hat{\beta} \leq 3, \\ 1, & \text{if } 3 < \hat{\beta}. \end{cases}$$

7 Properties of the Markov Chain

The markovian analysis of the system proceeds in a way parallel to that of [5]. Assume that $m = n = 1$, and let, for $k \geq 0$:

$$w(k) = w_1(k) = x(k) - y(k).$$

It is easily seen from the definition of the stochastic recurrence that, on the one hand:

$$w(k+1) = \max(a_{11}, b_{11} - w(k)) - \min(c_{11}, d_{11} - w(k)) , \tag{34}$$

and on the other hand:

$$x(k+1) - x(k) = \max(a_{11}, b_{11} - w(k)) . \tag{35}$$

From (34), it is clear that the process $\{w(k)\}$ is a homogeneous Markov chain, under the condition that the sequences $\{a_{11}\}$,$\{b_{11}\}$, $\{c_{11}\}$ and $\{d_{11}\}$ be i.i.d. From (35), it is seen that the stationary distribution of the increments $x(k+1) - x(k)$ can be computed from that of $w(k)$ (when it exists).

One should notice here that the existence of an underlying Markov chain depends on the fact that the operator "+" is distributive over "max" and "min". Such a Markov chain will still exist for other systems with this property. An important class of such systems is built with "k-out-of-n" operators, which generalize min and max.

A common belief in the "(max, +) only" case is that when the random elements of A have a discrete finite support (resp. a bounded support), then the state space of $v(k)$ is also finite (resp. bounded). This is *not* true however, as will be shown shortly. This property also does not necessarily hold in our case of min-max systems. However, it is still possible to compute the eigenvalue λ in simple systems, thanks to the regularity of the infinite part of this chain (see the next subsections).

We now give a counter example to the belief expressed above[1]. Consider $x(k+1) = A(k) \otimes x(k)$ with x being two-dimensional and with

$$A(k) = \begin{pmatrix} 0 & a_{12}(k) \\ a_{21}(k) & 0 \end{pmatrix},$$

where the variables $a_{12}(k)$ and $a_{21}(k)$ are i.i.d. and are chosen from the finite set $\{-\phi, -1, 1, \phi\}$, all four values are chosen with positive probabilities. The quantity ϕ is an arbitrary irrational number in the interval $[1, 2]$.

It is claimed now that all numbers of the form $p - q\phi$ and $-p + q\phi$, with $p, q = 1, 2, \ldots$, and provided that $p - q\phi \in (-\phi, +\phi)$, are states of the Markov-chain $v_1(k)$. As a corollary we get that the states of the chain are dense in the interval $(-\phi, +\phi)$.

The proof of the claim just made is as follows. Define $|p| + |q|$ to be the *value* of the number $p - q\phi$ or $-p + q\phi$. Suppose that for some k we have that $v_1(k+1) = p - q\phi \in (-\phi, +\phi)$ (or $v_1(k+1) = -p + q\phi \in (-\phi, +\phi)$). It will be shown now that a $v_1(k)$ exists from which $v_1(k+1)$ is reached with positive probability and such that $v_1(k) \in (-\phi, +\phi)$ and this $v_1(k)$ has a value which is lower than the value of $v_1(k+1)$. If $v_1(k+1) \in (0, +\phi)$, then one of the following two choices, $a_{12}(k) = 1, a_{21}(k) = -\phi$ or $a_{12}(k) = \phi, a_{21}(k) = -\phi$, will lower the value of $v_1(k)$. If, on the other hand, $v_1(k+1) \in (-\phi, 0)$, then one of the following two possibilities, $a_{12}(k) = 1, a_{21}(k) = \phi$ or $a_{12}(k) = \phi, a_{21}(k) = \phi$, will lower the value of $v_1(k)$. Since $\phi \in (1, 2)$, the pairs $(0, q)$ and $(p, 0)$ are not allowed for $p, q \geq 1$, and one ends up, by repeating the above procedure of decreasing the value of k, in one of the states $-\phi$,-1,1,ϕ, all having value 1, or 0. These latter states are easily seen to be reachable from any $v_1(0)$. The conclusion is that $v_1(k+1) = p - q\phi$, with arbitrary p and q subject to $p - q\phi \in (-\phi, +\phi)$, can be reached with positive probability from any $v_1(0)$, which concludes the proof.

7.1 Back to Example 1

The state space of the Markov chain $w(k)$ is *a priori* $\mathbf{Z}$. Its evolution is given by the equation:

$$w(k+1) \;=\; \max(a_{11}, 3 - w(k)) \;-\; \min(3, 5 - w(k)) \;.$$

A subdivision of the cases, taking into account the fact that $0 \leq a_{11} \leq 4$, gives:

$$\begin{array}{lllll} \text{if } w(k) \leq -1 & \text{then} & w(k+1) & = & -w(k), \\ \text{if } 0 \leq w(k) \leq 2 & \text{then} & w(k+1) & = & \max(a-3, -w(k)), \\ \text{if } 3 \leq w(k) & \text{then} & w(k+1) & = & w(k) + a - 5. \end{array}$$

A further investigation reveals that all states $n \in \mathbf{Z}$ are transient, except 1 and -1, which constitutes the unique recurrent class. As mentioned previously, the stationary distribution of $w(k)$ is $(1/3, 2/3)$ on $(-1, 1)$. Using (35), one computes again that in the stationary system, $E[x(k+1) - x(k)] = 10/3$.

[1]This example is due to F.M. Dekking of Delft University of Technology, which we gratefully acknowledge.

7.2 Example 4

Consider now the case where a_{11}, b_{11}, c_{11} and d_{11} have Bernoulli distributions, with parameters a, b, c and d respectively. In other words, $P[a_{11} = 1] = a = 1 - P[a_{11} = 0]$. For ease of notation, we shall use: $\bar{a} = 1 - a, \bar{b} = 1 - b, \bar{c} = 1 - c$ and $\bar{d} = 1 - d$.

It turns out that only the states $\{-1, 0, 1, 2, \ldots\}$ are recurrent. The probability transition matrix restricted to these states is:

$$\mathbf{P} = \begin{pmatrix} 0 & b\bar{c} & bc + \bar{b}\bar{c} & \bar{b}c & 0 & 0 & \ldots \\ \bar{a}b\bar{c}d & \bar{a}\bar{c}(1-bd) & (1-\bar{a}\bar{c})(1-bd) & 0 & 0 & 0 & \ldots \\ & & +(1-\bar{a}\bar{c})bd & & & & \\ 0 & \bar{a}d & ad + \bar{a}\bar{d} & a\bar{d} & 0 & 0 & \ldots \\ 0 & 0 & \bar{a}d & ad + \bar{a}\bar{d} & a\bar{d} & 0 & \\ 0 & 0 & 0 & \bar{a}d & ad + \bar{a}\bar{d} & a\bar{d} & \ddots \\ 0 & 0 & 0 & 0 & \bar{a}d & ad + \bar{a}\bar{d} & \ddots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots \end{pmatrix} . \tag{36}$$

The "tail" of this Markov chain is therefore a birth and death process, and the solution of the equilibrium equations for the stationary distribution π give:

$$\pi(n) = \pi(2) \left(\frac{a\bar{d}}{\bar{a}d} \right)^{n-2} ,$$

as well as the ergodicity condition of the chain: $a(1-d) < d(1-a)$, or:

$$a < d .$$

In particular, $\pi(3) = a\bar{d}\pi(2)/\bar{a}d$, and there remains to solve for π on the states $\{-1, 0, 1, 2\}$, and normalize.

Once this is done, the value of λ is obtained as:

$$\begin{aligned} \lambda &= b\pi(-1) + bd(1 - \bar{a}\bar{c})\pi(0) + d(\sum_{n=1}^{\infty} \pi(n)) \\ &= b\pi(-1) + bd(1 - \bar{a}\bar{c})\pi(0) + d(\pi(1) + \frac{\bar{a}d}{d-a}\pi(2)) . \end{aligned} \tag{37}$$

7.3 Example 5

Consider now the case $a_{11} = 0$, $a_{22} = 1$ and $a_{12} = 0$. Assume in addition that $a_{21} \geq 0$. Our stochastic recurrence takes now the form:

$$x(k+1) = \max(x(k), y(k) + a_{21}), \tag{38}$$

$$y(k+1) = \min(x(k), y(k) + 1) , \tag{39}$$

from which one obtains the following evolution for $w(k)$:

$$\begin{array}{lll} \text{if } w(k) < 0 & \text{then} & w(k+1) = a_{21} - w(k) \\ \text{if } w(k) = 0 & \text{then} & w(k+1) = a_{21} \\ \text{if } 1 > w(k) > 0 & \text{then} & w(k+1) = \max(a_{21} - w(k), 0) \\ \text{if } w(k) \geq 1 & \text{then} & w(k+1) = \max(a_{21}, w(k)) - 1 . \end{array}$$

It follows that if a_{21} is integer, then all $x \in \mathbf{R} \setminus \mathbf{N}$ are transient. Now, observe the following.

The eigenvalue of the system can be computed as:

$$\begin{array}{ll} y(k+1) - y(k) = 0, & \text{if} \quad z(k) = 0, \\ y(k+1) - y(k) = 1, & \text{if} \quad z(k) \geq 1 . \end{array}$$

Consequently, if the Markov chain $w(k)$ is stationary,

$$\lambda \;=\; E[y(k+1)-y(k)] \;=\; P[w(k)>0] \;=\; 1-\pi(0)\ . \tag{40}$$

If for instance $a_{21} \geq 2$ almost surely, then $w(k) > 0$ a.s., and $\lambda = 1$.

Assume now that a_{21} takes the two values 0 and N with probabilities p and $1-p$, respectively. The state space of the Markov chain is $\{0, 1, \ldots, N-1, N\}$, with probability transition matrix:

$$\mathbf{P} \;=\; \begin{pmatrix} 1-a & 0 & \ldots & 0 & 0 & a \\ 1-a & 0 & \ldots & 0 & a & 0 \\ 0 & 1-a & 0 & \ldots & a & 0 \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & & & 1-a & a & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix} . \tag{41}$$

The stationary distribution can be explicitly computed as:

$$\begin{aligned} \pi(0) &= \frac{(1-p)^{N-1}}{1+p(1-p)^{N-1}}; \\ \pi(i) &= \frac{(1-p)^{N-1-i}}{1+p(1-p)^{N-1}}, \quad 1 \leq i \leq N-1; \\ \pi(N) &= \frac{p(1-p)^{N-1}}{1+p(1-p)^{N-1}}\ . \end{aligned}$$

The eigenvalue is therefore:

$$\lambda \;=\; \frac{1-(1-a)^N}{1+a(1-a)^{N-1}}\ .$$

7.4 Example 6

Consider now the "symmetric" case of the previous example, that is: $a_{11} = 0$, $a_{22} = 1$, $a_{21} = 0$ and $a_{12} \geq 0$. Assume that a_{12} takes integer values, and that $w(0)$ is also integer. Then:

$$\begin{array}{lll} \text{if } w(k) \leq -1 & \text{then} \quad w(k+1) &= -\min(a_{12}+w(k), 1) \\ \text{if } w(k) = 0 & \text{then} \quad w(k+1) &= -\min(a_{21}, 1) \\ \text{if } w(k) \geq 1 & \text{then} \quad w(k+1) &= w(k)-1\ . \end{array}$$

The chain has only one recurrent class $\{-1, 0, 1\}$. Its probability transition matrix on this set is:

$$\mathbf{P} \;=\; \begin{pmatrix} 1-p-q & q & p \\ 1-p & p & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{42}$$

with $p = P[a_{12} = 0]$ and $q = P[a_{21} = 1]$. The eigenvalue of the system turns out to be:

$$\lambda \;=\; \pi(-1) \;=\; \frac{1-p}{1+p+q-p^2}\ .$$

Note that only the value of the distribution of a_{12} at the points 0 and 1 is relevant to the value of λ.

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