The Laser model: structure and properties

Alain Jean-Marie, Fabrice Philippe

Advanced Markov Modeling
11 February 2015

Plan

Progress

The Model

Constituants:

- $2n$ energy levels, forming two "bands"; at most one electron occupies an energy level;
- two special "lasing levels" L et ℓ in the middle of both bands
- a cavity: supports photons.

Transitions (changes of state):

- electrons may change level in each band (thermalization)
- one electron may emit one photon, and passes simultaneously from L to ℓ
- one electron may absorb a photon, and passes simultaneously from ℓ à L
- photons may exit the cavity
- "pumping" makes an electron pass from the lowest to the highest energy level.

Choice of the state space

What is a state of the system? At least two choices:

electron-centric : each electron has a state, its energy level $\in [0..2n-1]$. The number of photons is *a priori* unbounded.

$$\Longrightarrow \qquad \mathcal{E} \subset [0..2n-1]^N \times \mathbb{N}.$$

But the exclution constraint has to be enforced.

level-centric : each energy level contains at most one electron.

$$\Longrightarrow \qquad \mathcal{E} \subset \{0,1\}^{2n} \times \mathbb{N}.$$

Since the total number of electrons is fixed:

$$\Longrightarrow \qquad \mathcal{E} \subset \left\{ \sigma \in \{0,1\}^{2n} \mid \sum_{i=0}^{2n-1} \sigma_i = N \right\} \times \mathbb{N}.$$

Decomposition

The time scales of thermalization and other events (emission, absorption, light, pumping) are very different.
The process may be seen as:

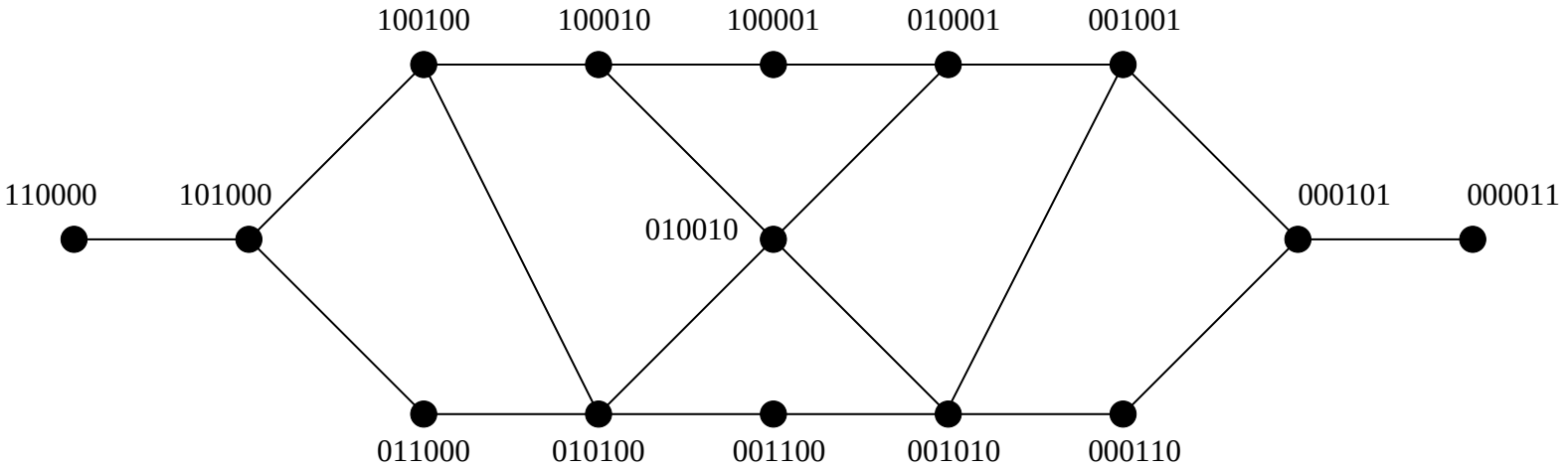
- 2 independent processes in each of the bands
- coupled by *rare* events

$\Longrightarrow$ focus on each band

Progress

Generators

Example of a transition diagram: $n = 6$ levels, $N = 2$ particles.



All transitions represented are two-way. Unwritten transition rates: rate p for transitions to the left (lower energy), pq to the right (higher energy).

Recursive structure

Order states lexicographically: matrices can be represented as

$$M_{n,m} = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$M_{n-1,m-1}$}} & \mathbf{0}_{p\times q} & \mathbf{0}_{p\times r} \\ & & \lambda & \\ \hline & \mu & & \\ \mathbf{0}_{q\times p} & \ddots & \multicolumn{2}{c}{\multirow{2}{*}{$M_{n-1,m}$}} \\ & \mu & & \\ \hline \mathbf{0}_{r\times p} & \mathbf{0}_{r\times q} & \multicolumn{2}{c}{} \end{array}\right).$$

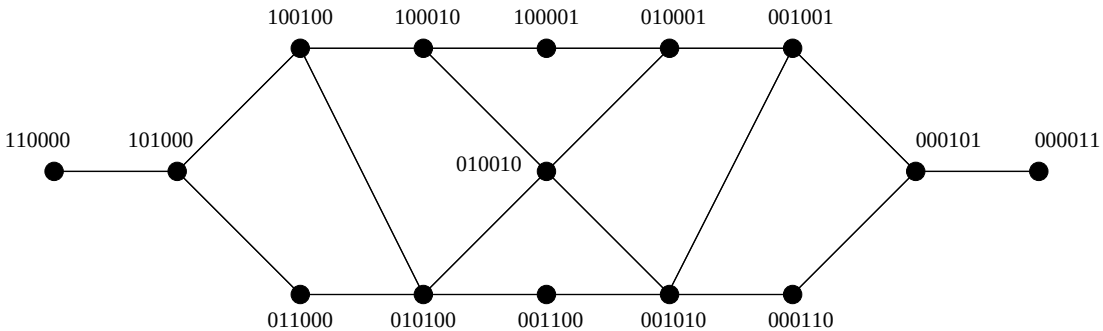
Generators (ctd)

The base matrix is the $n \times n$ generator of the *birth and death* process with n states: (for the laser, $\lambda = p$ and $\mu = pq$)

$$M_{n,1} = \begin{pmatrix} - & \lambda & & & \\ \mu & - & \lambda & & \\ & \ddots & \ddots & \ddots & \\ & & \mu & - & \lambda \\ & & & \mu & - \end{pmatrix}.$$

Generators (ctd)

But if we order by energy level, we get a "quasi-birth-death" process



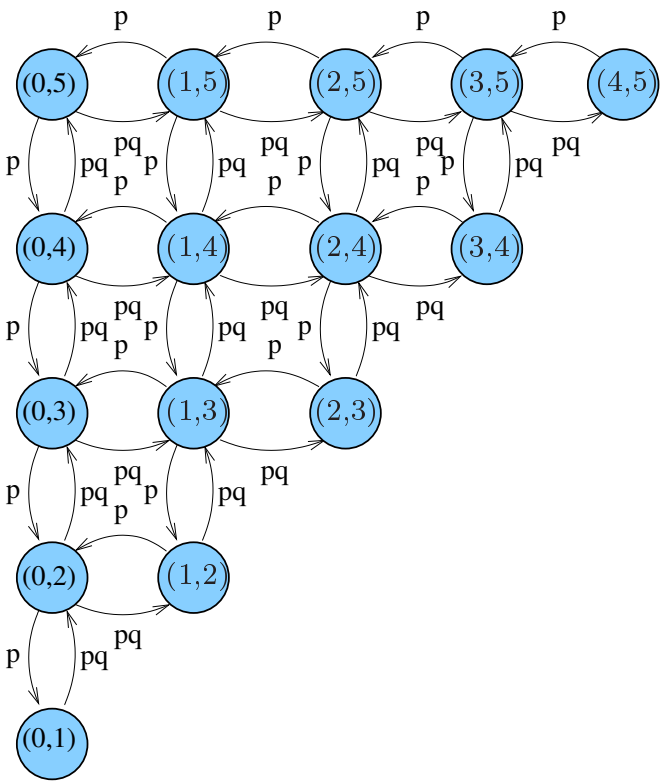
The matrix is then: $M_{6,2} =$

$$
\left(\begin{array}{c|c|cc|cc|ccc|cc|cc|cc|cc}
- & \lambda & & & & & & & & & & & & & & \\
\hline
\mu & - & \lambda & \lambda & & & & & & & & & & & & \\
\hline
 & \mu & - & 0 & \lambda & & & & & & & & & & & \\
 & \mu & 0 & - & \lambda & & & & & & & & & & & \\
\hline
 & & \mu & 0 & - & 0 & \lambda & \lambda & 0 & & & & & & & \\
 & & \mu & \mu & 0 & - & 0 & \lambda & \lambda & & & & & & & \\
\hline
 & & & & \mu & 0 & - & 0 & 0 & \lambda & 0 & & & & & \\
 & & & & \mu & \mu & 0 & - & 0 & \lambda & \lambda & & & & & \\
 & & & & 0 & \mu & 0 & 0 & - & 0 & \lambda & & & & & \\
\hline
 & & & & & & \mu & \mu & 0 & - & 0 & \lambda & 0 & & & \\
 & & & & & & 0 & \mu & \mu & 0 & - & \lambda & \lambda & & & \\
\hline
 & & & & & & & & & \mu & \mu & - & 0 & \lambda & & \\
 & & & & & & & & & 0 & \mu & 0 & - & \lambda & & \\
\hline
 & & & & & & & & & & & \mu & \mu & - & \lambda \\
\hline
 & & & & & & & & & & & & & \mu & -
\end{array}\right).
$$

Generators (end)

And what happens if we adopt the electron-centric model?

$$\mathcal{E} \;=\; \left\{ \mathbf{X} \in [0..2n-1]^N \mid X_1 < X_2 < \ldots < X_N \right\}$$



For instance, still with $n = 6$, $P = 2$, a regular, 2-dimensional grid-like structure

Progress

1. The Model
2. Generators for one band
3. Distributions
   - Stationary distribution
   - Stationary distribution
   - Sampling
4. Speed of convergence

Stationary distribution

The stationary distribution can be computed in *closed form* as:

$$\pi_{n,m}(\boldsymbol{\sigma}) \;=\; \frac{1}{G(n,m)} \prod_{i=0}^{n-1} q^{i\sigma_i} \;=\; \frac{1}{G(n,m)}\, q^{\sum_{i=0}^{n-1} i\sigma_i} \;=\; \frac{q^{U\boldsymbol{\sigma}}}{G(n,m)}\,.$$

The *partition function* $G(n,m)$ is given by:

$$\begin{aligned} G(n,m) \;&=\; \sum_{\boldsymbol{\sigma}\in\mathcal{S}_{n,m}} \prod_{i=0}^{n-1} q^{i\sigma_i} \\ &=\; \frac{q^{m-1}-q^n}{1-q^m}\,\frac{q^{m-2}-q^n}{1-q^{m-1}}\,\cdots\,\frac{1-q^n}{1-q}\,. \end{aligned}$$

Proofs:

- direct check of balance equations
- reversibility and truncation

Reversibility

Definition (Reversible Markov Chains)

A continuous-time Markov chain is reversible if $\{X(t)\}$ have $\{X(-t)\}$ the same generator, that is, if for all pair of states $(i,j) \in \mathcal{E} \times \mathcal{E}$,

$$\pi(i)\ q_{i,j} \quad = \quad \pi(j)\ q_{j,i}\ .$$

This equation is the **local balance** condition. It implies that when transition $j \to k$ exists, transition $k \to j$ must exist as well.
It is sufficient that it be satisfied for *some* family of numbers that need not be, *a priori* a probability distribution. If it does, then this family turns out to be indeed a stationary distribution for $\mathbf{Q}$, possibly up to a normalization constant.

Reversibility (ctd)

The class of Reversible Markov chains has property very useful for computing distributions: it is stable with respect to *truncation*.

Definition (Truncated Markov Chain)

Consider a CTMC over a countable state space $\mathcal{E}$ and generator $\mathbf{Q}$. Let $\mathcal{A} \subset \mathcal{E}$. The truncated Markov chain over the sub-space $\mathcal{A}$ is the CTMC with generator $\mathbf{Q}_{\mathcal{A}}$ defined as:

$$(q_{\mathcal{A}})_{jk} = \begin{cases} q_{jk} & \text{if } j,k \in \mathcal{A} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

In words, the transitions of the process are the same as the original process as long as the state is in $\mathcal{A}$. But transitions outside $\mathcal{A}$ have rate 0. If the process starts inside $\mathcal{A}$, it stays there forever, and it can be considered as a CTMC on the space $\mathcal{A}$.

Reversibility (end)

Theorem (Stationary distribution of truncated reversible chains)

Let $\mathcal{A} \subset \mathcal{E}$. Consider some reversible CTMC, with stationary distribution $\boldsymbol{\pi}$. Then the truncated Markov chain over $\mathcal{A}$ is also reversible, and its stationary distribution is given by: for $i \in \mathcal{A}$

$$\pi'(i) \;=\; \frac{\pi(i)}{\sum_{j \in \mathcal{A}} \pi(j)} \;.$$

Calculating occupancies

Open question: is it possible to calculate marginal probabilities (occupancies). For each energy level i,

$$\mathbb{P}\{N_i = 1\} \;=\; ?$$

The direct approach is:

$$\begin{aligned}\mathbb{P}\{N_i = 1\} &= \sum_{\boldsymbol{\sigma} \in \mathcal{S}_{n,m} \;|\; \sigma_i = 1} \pi_{n,m}(\boldsymbol{\sigma}) \\ &= \frac{1}{G(n,m)} \sum_{\boldsymbol{\sigma} \in \mathcal{S}_{n,m} \;|\; \sigma_i = 1} q^{\sum_{j=0}^{n-1} j\sigma_j} \;.\end{aligned}$$

Too complex $\Longrightarrow$ need for some simpler computation, *e.g.* with a recurrence.

Calculating occupancies (end)

Two recurrences

$$\mathbb{P}\{N_i = 1, N+1 \text{ particles}\} \;=\; q^i \frac{1-q^{N+1}}{q^N - q^n}(1-\mathbb{P}\{N_i = 1, N \text{ particles}\}).$$

$$\begin{aligned}
&\mathbb{P}\{N_i = 1\} \\
&\quad = \quad \frac{\rho^i}{G(n,N)} \sum_{k=0}^{i} \rho^{(i+1)(N-1-k)} G(i,k) G(n-i-1, N-k-1) \\
&\mathbb{P}\{N_i = 0\} \\
&\quad = \quad \frac{1}{G(n,N)} \sum_{k=0}^{i} \rho^{(i+1)(N-k)} G(i,k) G(n-i-1, N-k) \;.
\end{aligned}$$

Sampling from the stationary distribution

It is possible to *directly* sample from this stationary distribution, although the state space is very large.
This is because the distribution has a *recursive* strucure.

Sampling, Forward approach

The "simplex" $\mathcal{S}_{n,m}$ can be decomposed as:

$$\mathcal{S}_{n,m} \;=\; \{1\} \times \mathcal{S}_{n-1,m-1} \;\cup\; \{0\} \times \mathcal{S}_{n-1,m}\;, \qquad 0 < m < n.$$

Then we get the recursion:

$$\begin{aligned} G(n,m) &= q^{m-1} G(n-1,m-1) \;+\; q^m G(n-1,m)\;. \\ \pi_{n,m}(\sigma_0 = 1) &= q^{m-1}\,\frac{G(n-1,m-1)}{G(n,m)} \\ \pi_{n,m}(\sigma_0 = 0) &= q^{m}\,\frac{G(n-1,m)}{G(n,m)} \end{aligned}$$

+ boundary values.

Forward algorithm

Sampling from the stationary distribution, forward method

Data: Two integers n and m, $0 \leq n \leq m$
Result: A vector in $\mathcal{S}_{n,m}$, sampled from the distribution $\pi_{n,m}$
begin
    $k \leftarrow m$
    for i **from** 0 **to** $n-1$ **do**
        if $k = 0$ **then**
            $\sigma[i] \leftarrow 0$
        else if $k = n - i$ **then**
            $\sigma[i] \leftarrow 1$
        else
            $u \leftarrow \texttt{Uniform}([0,1])$
            if $u < q^k \dfrac{G(n-1-i,k)}{G(n-i,k)}$ **then**
                $\sigma[i] \leftarrow 0$
            else
                $\sigma[i] \leftarrow 1$
                $k \leftarrow k-1$
    return $(\sigma[0], \ldots, \sigma[n-1])$

Sampling, Backwards approach

Symetrically, $\mathcal{S}_{n,m}$ is decomposed as

$$\mathcal{S}_{n,m} \;=\; \mathcal{S}_{n-1,m-1} \times \{1\} \;\cup\; \mathcal{S}_{n-1,m} \times \{0\}\;, \qquad 0 < m < n.$$

Then we get:

$$\begin{aligned} G(n,m) &= q^{n-1} G(n-1,m-1) \;+\; G(n-1,m) \\ \pi_{n,m}(\sigma_{n-1}=1) &= q^{n-1}\,\frac{G(n-1,m-1)}{G(n,m)} \\ \pi_{n,m}(\sigma_0=0) &= \frac{G(n-1,m)}{G(n,m)}\;. \end{aligned}$$

+ boundary values.

Backwards algorithm

Sampling from the stationary distribution, backwards method

Data: Two integers n and m, $0 \leq n \leq m$
Result: A vector in $\mathcal{S}_{n,m}$, sampled from the distribution $\pi_{n,m}$
begin
  $k \leftarrow m$
  for i **from** 1 **to** n **do**
    if $k = 0$ **then**
      $\sigma[n-i] \leftarrow 0$
    else if $k = n - i$ **then**
      $\sigma[n-i] \leftarrow 1$
    else
      $u \leftarrow \texttt{Uniform}([0,1])$
      if $u < \dfrac{G(n-i,k)}{G(n-i+1,k)}$ **then**
        $\sigma[n-i] \leftarrow 0$
      else
        $\sigma[n-i] \leftarrow 1$
        $k \leftarrow k-1$
  return $(\sigma[0], \ldots, \sigma[n-1])$

Progress

1. The Model
2. Generators for one band
3. Distributions
   - Stationary distribution
   - Stationary distribution
   - Sampling
4. Speed of convergence

Speed of convergence

The question is to quantify the distance between $\pi_t = \pi_0 e^{\mathbf{Q}t}$ and the stationary distribution π.
The *spectral decomposition* principle says that if $\mathbf{Q} = \mathbf{R}\mathbf{D}\mathbf{R}^{-1}$, then

$$
\begin{aligned}
e^{\mathbf{Q}t} &= \mathbf{R}e^{\mathbf{D}t}\mathbf{R}^{-1} = \mathbf{R}\begin{pmatrix} 1 & & \mathbf{0} \\ & e^{\lambda_2 t} & \\ & & \ddots \\ \mathbf{0} & & e^{\lambda_n t} \end{pmatrix}\mathbf{R}^{-1} \\
&= \mathbf{1}\pi + \sum_{j=2}^{n} e^{\lambda_j t}\mathbf{A}_j \ .
\end{aligned}
$$

Locating the eigenvalues (spectrum) λ_j gives information on the speed of convergence.

Spectra

The basis matrix is:

$$M_{n,1} = \begin{pmatrix} -\lambda & \lambda & & & \\ \mu & -(\lambda+\mu) & \lambda & & \\ & \ddots & \ddots & \ddots & \\ & & \mu & -(\lambda+\mu) & \lambda \\ & & & \mu & -\mu \end{pmatrix} .$$

Its n eigenvalues are 0 and:

$$\omega_k = -(\lambda+\mu) + 2\sqrt{\lambda\mu}\cos\frac{k\pi}{n} , \qquad k = 1..n-1 .$$

Speed of convergence: given by the largest, non-zero one. It is less than

$$-(\lambda+\mu) + 2\sqrt{\lambda\mu} = -(\sqrt{\lambda}-\sqrt{\mu})^2 .$$