Advanced Markov Modeling 2015

Lecture 6: Illustrations and Examples / 1
Monte Carlo simulation of lasers
(A work with J. Arnaud and L.Chusseau, IES)

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11 Feb. 2015

Outline

Outline

A birth-death process

Planck: Energy exchanges between matter (particles) and light (wave, frequence ω): only by integer multiples of energy quantum $\delta = \hbar\omega$

Emission (absorption) of a photon corresponds to a particle going down (up) between two energy levels ε, $\varepsilon + \delta$.

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Emission (absorption) of a photon corresponds to a particle going down (up) between two energy levels ε, $\varepsilon + \delta$.

Einstein: Time evolution of the photon number $m(t)$ is a birth-death process,
jump probabilities in interval $[t, t + dt]$:

$$\pi(m \to m-1) \propto n_{\delta\uparrow} m,$$
$$\pi(m \to m+1) \propto n_{\delta\downarrow}(m + 1).$$

At time t: $n_{\delta\uparrow}(t)$ is the number of particles that may jump from a level ε to level $\varepsilon + \delta$.

Thermal bath

Canonical ensemble: System in contact with a (large) heat bath, temperature T, define $q = e^{-k_B/T} \in (0,1)$.
Energy exchanges only (not particles).

Boltzmann: The probability for the system to have energy U at equilibrium is $\propto q^U$.

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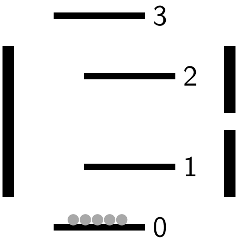
Which simple Markov chains have this kind of stationnary distributions?
Example: one particle, equidistant energy levels $\varepsilon_n = n\varepsilon$.
(geometric distribution, constant-rate birth-death)

$$\pi(\varepsilon_n \to \varepsilon_{n-1}) \propto p,$$
$$\pi(\varepsilon_n \to \varepsilon_{n+1}) \propto pq^{\varepsilon}.$$

Outline

How it works

Atomic laser: 4-level atoms, mirrors



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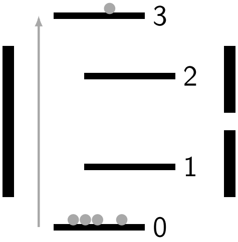

Pumping:

population inversion

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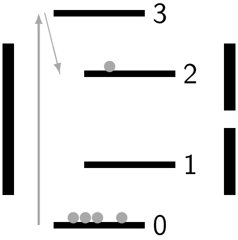

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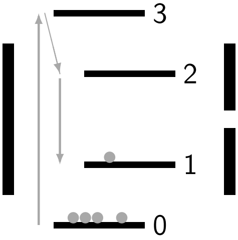

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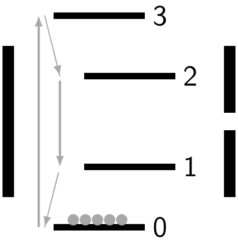

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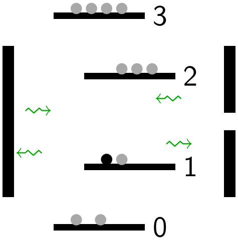

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Lasing levels (1,2):

How it works

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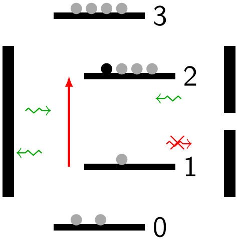

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Lasing levels (1,2):
Stimulated absorption

How it works

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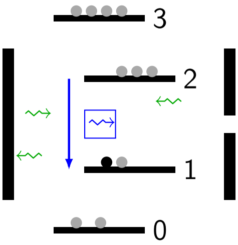

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Lasing levels (1,2):
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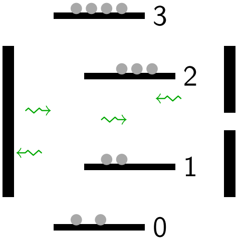

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Lasing levels (1,2):
Stimulated absorption / emission

$$\begin{cases} \pi(m \to m-1) & \propto n_1 m \\ \pi(m \to m+1) & \propto n_2(m+1) \end{cases}$$

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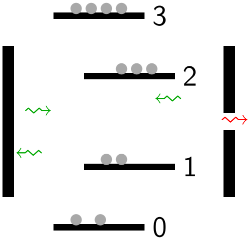

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Coherent emission (absorption by sink)

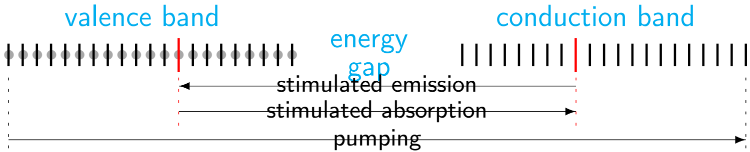
Semi-conductor lasers

Electrons, two bands of energy levels, 0-1 electron by level

(lasing levels)

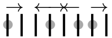
Semi-conductor lasers

Electrons, two bands of energy levels, 0-1 electron by level



(lasing levels)

Other/further moves:

upward and downward thermalization 

(Auger effect, coherent pumping, spontaneous emission, ...)

States and events

n electrons, n energy levels in each band

A state: a repartition of the n electrons among the $2n$ levels, and the number m of photons in the optical cavity.

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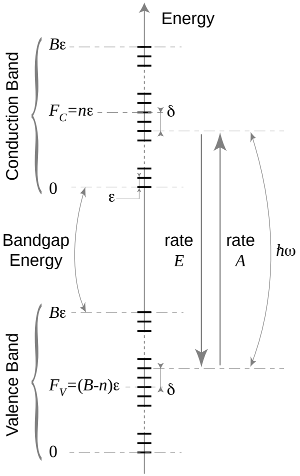
- optronic: stimulated emission, stimulated absorption → interaction bands / cavity
- photonic: coherent emission → cavity

Processes

Homogeneous Poisson processes

pumping (quiet), thermalization



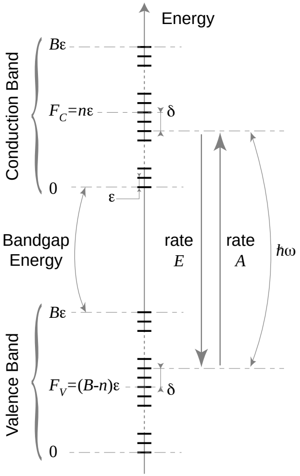
Processes

Homogeneous Poisson processes

pumping (quiet), thermalization

Cox processes

stimulated emission/absorption, laser emission



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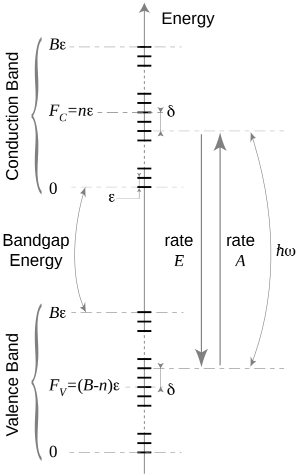

Processes

Homogeneous Poisson processes
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Cox processes
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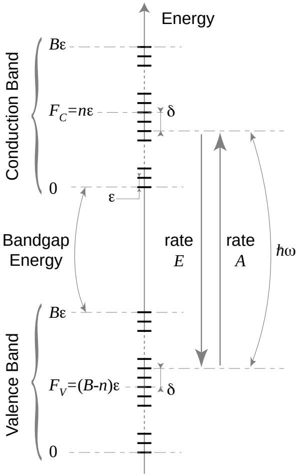
- laser emission

(+ periodic events if regular pumping)

Superposition of similar processes
- upward thermalization,
- downward thermalization



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Events	Rates	Variables and parameters
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downwards	$pN_\downarrow$	$N_\downarrow$ electrons may move a level down.
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Pumping	J	(Poissonian pump)

Outline

Laser Noise

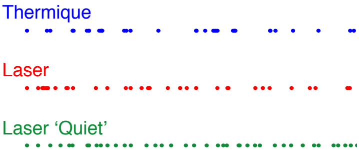
Thermal light : $\pi(m) \propto q^{m\delta}$

Generating function $\phi(z) = \frac{1-q^\delta}{1-zq^\delta}$.

Mean, variance: $\langle m \rangle = \frac{q^\delta}{1-q^\delta}$, $V_m = \frac{q^\delta}{(1-q^\delta)^2} = \langle m \rangle + \langle m \rangle^2$.

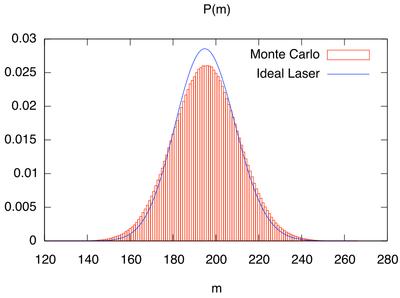
Laser light: poissonian

Is it possibly sub-poissonian? Under which conditions?



Cavity

Statistics of the number of photons in the cavity (stationnary)
Fano factor $\mathcal{F}$: variance/mean (1 for Poisson variables)

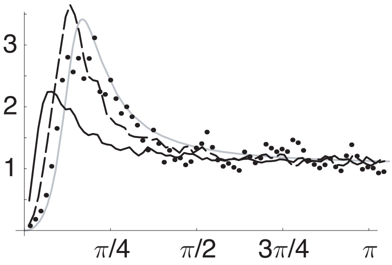


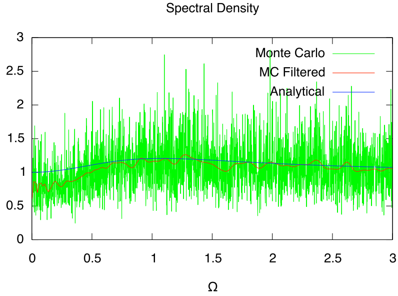
Laser noise

Intensity of the laser: stationnary process Q, let $\Delta Q = Q - \langle Q \rangle$.

Spectral density $\mathcal{S}(\Omega) = \frac{1}{\langle Q \rangle} \mathcal{S}_{\Delta Q}(\Omega)$, $\quad \mathcal{F} = \int \mathcal{S}(\Omega) d\Omega$.

Frequences Ω for which $\mathcal{S}(\Omega) < 1$?

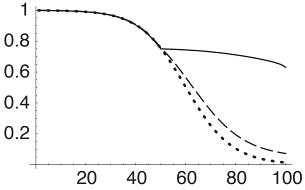


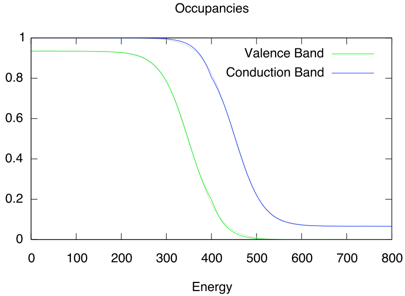


Careers

Levels occupancies

Spectral hole burning at lasing levels





Outline

Algorithm

Simulation of 1 trajectory of the CTMC (MC?)

Known as dynamic MC, kinetic MC, Doob-Gillespie, etc... (MC?)

Method

At time t_k:

- simulate the waiting time τ before next event
- choose the event according to rates
- $t_{k+1} = t_k + \tau$

Implementation

Initialization $(t = 0)$

Any state, e.g., all electrons in VB, no photon in the cavity.

Loop (while $t < T$)

- random number $r = \mathcal{U}(0,1)$ for the waiting time:

$$\tau = \frac{-\ln r}{\Lambda}, \quad \Lambda = \sum_{i \geq 1} \lambda_i,$$

- random number $r' = \mathcal{U}(0,1)$ for the next event:

$$\text{index} = \min\{i : \sum_{j=1}^{i} \lambda_j \geq r'\Lambda\},$$

- update rates and state,
- perform other statistical computations:

  occupancies, histogram, spectral density. $(t = t + \tau)$

Typical values

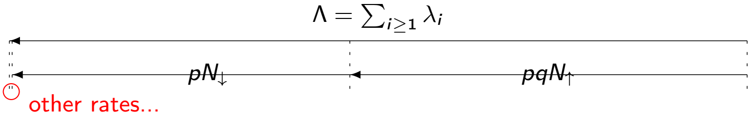
800 energy levels in each band, 800 electrons

Simulation duration $T = 100$–10^5

Rates:

Pumping	J	$J = 100$–500
Absorption (sink)	m/τ	$\tau = 2$
Emission	$m+1$ or 0	$<m>=\tau J$
Absorption	m or 0	
Thermalization	$pN_\downarrow$	$p = 50000$
	$pqN_\uparrow$	$q = 0.9$

0.5 10^{12} events for 6 10^6 useful points



Outline

Pseudorandom generators

Rather large choice. How to choose?

- **period?** number of events to be generated? (e.g., 0.5 10^{12} events)
- **speed?** is it critical?
- **biases?** which randomness is desirable?

See GSL, Dieharder, ...

Stationnarity

- How do we know?
  In many cases, it must be checked *a posteriori*...
- Ergodicity assumptions (time averages / ensemble averages)
- Predictible averages?
  Population (balance) equations at equilibrium?
- Which one to choose? (most frequent? fastest to compute?)

Spectral density approximation

Direct computation? Fourier transform of autocorrelation?
Not incremental.
Estimation by periodogram (duration T, K points $\Omega_k = \frac{2k\pi}{T}$).

$$S(\Omega) \approx S_T(\Omega) = \frac{1}{T} \left| \sum_{n=1}^{N} e^{-i\Omega t_n} \right|^2$$

Correct approximation?

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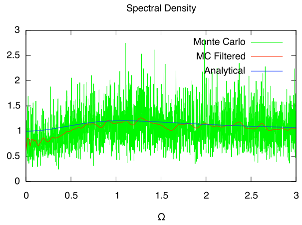
Correct approximation?

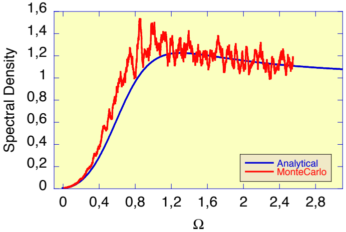
In average, and for large T!

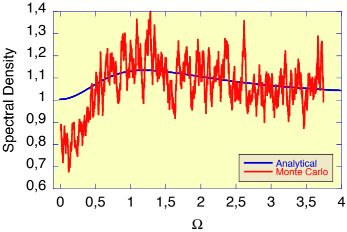
Variance is independent of T...



Spectral density approximation

Bartlett: average N parts of a unique simulation of duration NT.
Here for $N = 10$:

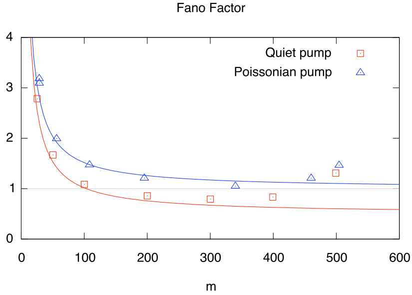




A little better...
For one curve, more than a week...

Searching for optimal parameters

Efficient thermalization ($p = 50000$), 800 levels in each band
Variable pumping rate:



For each point, nearly a week...

Outline

Technical track : Distributing simulations

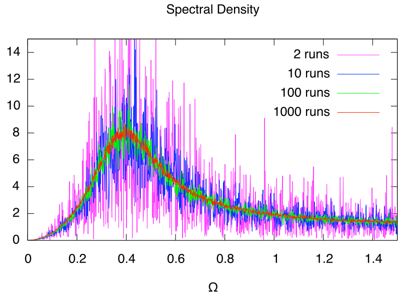
Condor server
Computers of the UFR rooms (300-600), evening and week end

Method

- initialisation (stationnary state)
- 1 initial occupancies computation
- k small mixing runs, *average* duration $T/100$
- k simulation runs, duration T
- 1 final occupancies computation

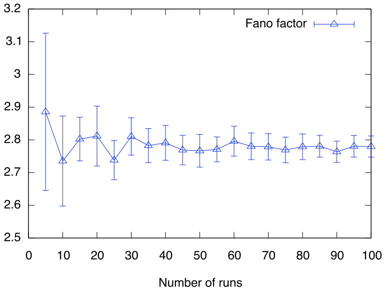
Improvements

Variance of periodogram



Improvements

Fano factor vs runs



error bars: from 0.3 (10 runs) to 0.03 (1000 runs)

Theoretical track : Getting rid of thermalization?

Rare events: pumping, photon emission/absorption, photon escape.
$\rightarrow$ their rates depend on the occupation of few levels only

Between two rare events, only thermalization occurs — so,
the number of electrons in each band is left unchanged

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Consider thermalization processes in a band of B levels, with N electrons inside.

- Can we compute a random state after K steps? ($K \approx 10^5$)
- Better: Can we compute the occupancy of a given level?

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- Can we compute a random state after K steps? ($K \approx 10^5$)
- Better: Can we compute the occupancy of a given level?
- Even better: May we consider that $K = \infty$?
  $\rightarrow$ simple N-recursive formula for the occupancy of a given level!