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1. Anisotropic mesh adaptation
2. Anisotropic goal-oriented mesh adaptation
3. Extension to unsteady
4. Applications to blast waves
5. Applications to linear acoustics
Riemannian metric space: \( (\mathcal{M}(\mathbf{x}))_{\mathbf{x} \in \Omega} \)

- **Distance:**

  \[
  \text{Distance}(a, b) = \ell_{\mathcal{M}}(ab) = \int_0^1 \sqrt{t \mathbf{ab} \cdot \mathcal{M}(\mathbf{a} + t \mathbf{ab}) \cdot \mathbf{ab}} \, dt
  \]

- **Complexity \( C \):**

  \[
  C(\mathcal{M}) = \int_{\Omega} d(\mathbf{x}) \, d\mathbf{x} = \int_{\Omega} \sqrt{\det(\mathcal{M}(\mathbf{x}))} \, d\mathbf{x}.
  \]

- **Matrix writing:**

  \[
  \mathcal{M}(\mathbf{x}) = d^{2/3}(\mathbf{x}) \mathcal{R}(\mathbf{x}) \begin{pmatrix} r_1^{-2/3}(\mathbf{x}) & r_2^{-2/3}(\mathbf{x}) & r_3^{-2/3}(\mathbf{x}) \end{pmatrix}^t \mathcal{R}(\mathbf{x}).
  \]
1. Anisotropic mesh adaptation: unit mesh

- **Main idea:** change the **distance evaluation** in the mesh generator
  [Vallet, 1992], [Casto-Diaz et Al., 1997], [Hecht et Mohammadi, 1997]

- **Fundamental concept:** **Unit mesh**

Adapting a mesh

\[
\text{Work in adequate Riemannian metric space}
\]

Generating a uniform mesh w.r. to \( M(x) \)

\[
\mathcal{H} \text{ unit mesh } \iff \forall e, \ell_M(e) \approx 1 \text{ and } \forall K, |K|_M \approx \begin{cases} 
\sqrt{3}/4 & \text{in 2D} \\
\sqrt{2}/12 & \text{in 3D}
\end{cases}
\]

Mesh-adaptive computation of acoustics
1. Anisotropic mesh adaptation: continuous interpolation error

For any $K$ which is unit for $\mathcal{M}$ and for all $u$ quadratic positive form ($u(x) = \frac{1}{2} t^t x H x$):

$$\|u - \Pi_h u\|_{L^1(K)} = \frac{\sqrt{2}}{240} \det(\mathcal{M}^{-\frac{1}{2}}) \cdot \text{trace}(\mathcal{M}^{-\frac{1}{2}} H \mathcal{M}^{-\frac{1}{2}})$$

Continuous interpolation error:

$$\forall x \in \Omega, \quad |u - \pi_{\mathcal{M}} u|(x) = \frac{1}{10} \text{trace}(\mathcal{M}(x)^{-\frac{1}{2}} |H(x)| \mathcal{M}(x)^{-\frac{1}{2}})$$

equivalent because:

$$\frac{1}{10} \text{trace}(\mathcal{M}(x)^{-\frac{1}{2}} |H(x)| \mathcal{M}(x)^{-\frac{1}{2}}) = 2 \frac{\|u - \Pi_h u\|_{L^1(K)}}{|K|}$$

for any $K$ which is unit with respect to $\mathcal{M}(x)$. 
We proposed a **continuous mesh framework** to solve this problem.

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<th><strong>Discrete</strong></th>
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<td>Element $K$</td>
<td>Metric tensor $\mathcal{M}(x_K)$</td>
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<td>Mesh $\mathcal{H}$ of $\Omega_h$</td>
<td>Riemannian metric space $\mathcal{M} = (\mathcal{M}(x))_{x \in \Omega}$</td>
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<tr>
<td>Number of vertices $N_v$</td>
<td>Complexity $C(\mathcal{M}) = \int_\Omega \sqrt{\det(\mathcal{M}(x))} , dx$</td>
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<td>Linear interpolate $\Pi_h u$</td>
<td>Continuous linear interpolate $\pi_{\mathcal{M}} u$</td>
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We call multi-scale adaptation the minimisation of the $L^p$-norm, with $p < \infty$, of the continuous interpolation:

Find $M_{opt} = (M_{opt}(x))_{x \in \Omega}$ of complexity $N$ such that

$$E_{M_{opt}}(u) = \min_M \| u - \pi_M u \|_{M,L^p(\Omega)}$$

$$= \min_M \left( \int_{\Omega} |u(x) - \pi_M u(x)|^p \, dx \right)^{\frac{1}{p}}$$

A well-posed problem solved by a calculus of variations.
# 1. Anisotropic mesh adaptation: multiscale adaptation

## Optimal metric

\[ M_{L^p} = D_{L^p} \left( \det |H_u| \right)^{-1 \frac{2p+3}{p}} \mathcal{R}_u^{-1} |\Lambda| \mathcal{R}_u \]

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1. **Global normalization:** to reach the constraint complexity \( N \)

\[ D_{L^p} = N^{\frac{2}{3}} \left( \int_\Omega (\det |H_u|)^{\frac{p}{2p+3}} \right)^{-\frac{2}{3}} \quad \text{and} \quad D_{L^\infty} = N^{\frac{2}{3}} \left( \int_\Omega (\det |H_u|)^{\frac{1}{2}} \right)^{-\frac{2}{3}} \]

2. **Local normalization:** sensitivity to small solution variations, depends on \( L^p \) norm chosen

3. **Optimal anisotropy directions** based on Hessian eigenvectors

4. **Diagonal matrix of anisotropy strengths**, defined from the absolute values of Hessian eigenvalues
1. Anisotropic mesh adaptation: multiscale adaptation

**Fixed point algorithm**

- Compute flow
- Compute metric field
- Build new mesh
- Interpolate old data on new mesh
Background and properties:

[Castro Diaz et al., 1997], [Habashi et al., 2000], [Frey and Alauzet, 2005], …

- Genericity, does not depend on the PDE and on the numerical scheme
- Anisotropy easily deduced
- The multiscale (i.e. $L^p$) version provides an optimal mesh without neglecting weaker details.
1. Anisotropic mesh adaptation: application

An example: supersonic steady flow around an aircraft.
Objectif

Deriving the best mesh to observe a given functional $j(w) = (g, w)$ depending of the solution $w$ of a PDE and enough regular to be observed through its Jacobian $g$.

How?

Control of the approximation error on the output functional: $j(w) - j(w_h)$.

Exemples

- vorticity in wake $j(w) = \int_\gamma \| \nabla \wedge (u - u_\infty) \|_2^2 d\gamma$
- drag, lift: use to quantify the performance of a design, etc...
Background:
[Becker-Rannacher], [Giles-Pierce], [Venditti-Darmofal, 2002], [Rogé-Martin, 2008], . . .

- Explicit use of the PDE
- Strong dependency on the numerical scheme
- Anisotropy hard to prescribe

Given a functional $j(w)$
- We only know $w_h$
- How to control $j(w) - j(w_h)$
2. Goal-oriented mesh adaptation: formal derivation

Continuous and discrete equations

\[(\Psi(w), \phi) = 0 \quad \text{and} \quad (\Psi_h(w_h), \phi_h) = 0\]

Continuous and discrete adjoint equations

\[
\left(\frac{\partial \Psi}{\partial w}(w)\phi, w^*\right) = (g, \phi) \quad \text{and} \quad \left(\frac{\partial \Psi_h}{\partial w}(w_h)\phi_h, w_h^*\right) = (g, \phi_h)
\]

Adjoint estimation

- Dual formula [Giles et Süli, 2002]

\[
j(w) - j(w_h) \approx (g, w - w_h) = -(w^*, \Psi(w_h)) = (w_h^*, \Psi_h(w))
\]

A posteriori \quad A priori
2. Goal-oriented mesh adaptation: formal derivation

**A priori error estimation** [A. Loseille and A. Dervieux and F. Alauzet, Fully anisotropic goal-oriented mesh adaptation for 3D steady Euler equations, JCP, 2010]

\[ j(w) - j(w_h) = (g, w - w_h) = \underbrace{(g, w - \Pi_h w)}_{\text{Approximation error}} + \underbrace{(g, \Pi_h w - w_h)}_{\text{Interpolation error}} \]

\[ = \left( (\Psi_h - \Psi)(w), w_{h}^{*}\right) + R_3 \]

- Search for continuous model \( E(\mathcal{M}) \) to evaluate \((\Psi_h - \Psi)(w)\).
- Find \( \mathcal{M} \) that minimises \((E(\mathcal{M}), w_{h}^{*})\).
2. Goal-oriented mesh adaptation: application

Application to sonic boom:

- Adjoint functional:
  \[ j(W) = \int_\gamma \left( \frac{p - p_\infty}{p_\infty} \right)^2 \, d\gamma \]

- Adaptation variable: Mach number
2. Goal-oriented mesh adaptation: application
Even close to the aircraft (2 lengths), the adjoint-based adaptation strongly supersedes the multiscale method.
Problematics:

- Evolution of physical phenomena in time.
- One may need a good prediction of solution evolution into the whole computational domain. In this case, the unsteady multiscale method need be applied. We refer to Alauzet et al. JCP (2007).
- A target observation can be specified: the goal oriented version is needed.
- We neglect time-discretisation errors in the present study.
2. Extension to unsteady flows (Euler model)

\[
(\Psi(W), \Phi) = \int_Q \Phi \partial_t W \, dQ + \int_Q \Phi \nabla \mathcal{F}(W) \, dQ - \int_\Sigma \Phi \hat{\mathcal{F}}(W) \, d\Sigma
\]

\[
(\Psi_h(W), \Phi_h) = \int_Q \Phi_h \Pi_h \partial_t W \, dQ + \int_Q \Phi_h \nabla \Pi_h \mathcal{F}(W) \, dQ - \int_\Sigma \Phi_h \Pi_h \hat{\mathcal{F}}(W) \, d\Sigma
\]

with \( Q = \Omega \times (0, T], \Sigma = \partial \Omega \times (0, T]. \)

Let:

\[
j(w) = (g, w)_Q
\]

\[
j(w) - j(w_h) \approx \int_Q W^* (\partial_t W_h - \partial_t W + \nabla \mathcal{F}_h(W) - \nabla \mathcal{F}(W)) \, dQ + \text{BT}
\]

\[
= \int_Q W^* (\partial_t W_h - \partial_t W) \, dQ + \int_Q \nabla W^* (\mathcal{F}(W) - \mathcal{F}_h(W)) \, dQ + \text{BT}
\]

\[
= \int_Q W^* (\Pi_h \partial_t W - \partial_t W) \, dQ + \int_Q \nabla W^* (\mathcal{F}(W) - \Pi_h \mathcal{F}(W)) \, dQ + \text{BT}
\]

Boundary integrals ("\( \text{BT} \)"") are transformed in a similar manner.
Solve this problem in the continuous framework

Find $\mathbf{M}_{opt} = (\mathcal{M}_{opt}(x))_{x \in Q}$ of complexity $N$ such that

$$E(\mathcal{M}_{opt}) = \min_{\mathcal{M}} \left( \int_Q W^* \left( \pi_{\mathcal{M}} W_t - W_t \right) dQ + \int_Q \nabla \cdot W^* (\mathcal{F}(W) - \pi_{\mathcal{M}} \mathcal{F}(W)) dQ + BT \right)$$

A calculus of variations gives

$$\mathcal{M}_{opt} = \mathcal{M}_{opt}^{L1} \left( \sum_{i=1}^{5} \left( |W^*_h(W_i)| |H(W_{i,t})| + \sum_{j=1}^{3} |\nabla_{x_j} W^*_h(W_i)| |H(\mathcal{F}_{x_j}(W_i))| \right) \right)$$
2. Extension to unsteady flows: discrete case

Discrete State System and functional:

\[ \Psi_{h}^{n+1}(W^{n}, W^{n+1}, \phi^{n}) = 0 \iff W = W_{sol} \]

\[ j = J(W_{sol}) \]

Discrete Adjoint State System writes:

\[ W^{*,N} = \left( \frac{\partial \Psi^{N}_{h}}{\partial W^{N}} \right)^{-T} \left( \frac{\partial J}{\partial W^{N}} \right)^{T} \]

\[ W^{*,n} = \left( \frac{\partial \Psi^{n}_{h}}{\partial W^{n}} \right)^{-T} \left[ \left( \frac{\partial J}{\partial W^{n}} \right)^{T} - \left( \frac{\partial \Psi^{n+1}_{h}}{\partial W^{n}} \right)^{T} W^{*,n+1} \right] \forall n = N - 1, 0 \]

\[ \implies \text{Adjoint State is computed backwards in time.} \]
2. Extension to unsteady flows (Euler model)

Adjoint is advanced forward in time:

- Computing $W^{*,n}$ from the adjoint state $W^{*,n+1}$ needs the knowledge of states $W^n, W^{n+1}$.
- Higher-Order scheme with intermediate storage (like explicit Runge-Kutta schemes) demands even more storage/recompute effort.

Our approach:

- Storage of the solution on checkpoints $\implies$ forward/backward computation only between two checkpoints.
- Interpolate Adjoint states between two adaptation sub-intervals.
Optimal Metric computation needs:

- Adjoint state: $W^*$ (computed backwards in time)
- Adjoint state gradient: $\nabla W^*$
- Hessian of the Euler fluxes: $H(\mathcal{F}(W))$
- Hessian of time derivative: $H(W_t)$

Continuous states $\Leftarrow$ approximated by the discrete ones
Gradients and Hessians $\Leftarrow$ derivative recovery ($L^2$ projection)
2. Extension to unsteady flows (Euler model)

Fixed-point loop $j$

\[ t_0, t_i, \Delta t_i, t_{i+1}, T = t_{n_{\text{adap}}} \]

Solution state and adjoint state sampling

\[ \tilde{\Psi}(W) = 0 \]
\[ \tilde{\Psi}^*(W, W^*) = 0 \]

\[ |H_{i,j,k}^{GO}| = \bigcap_{k=1}^{n_k} |H_{i,j,k}^{GO}| \]

Mesh-adaptive computation of acoustics
Blast-like initialisation inside a circle of radius $r_0 = 0.15$ around $x_0 = (1.2, 0.0)$, given by: $\rho = 10.0$, $v = (0, 0)$ and $e = 25.0$.

The cost function $j$ was the impulse over the target surface $S$ in Figure below:

$$j(W) = \frac{1}{2} \int_S (p - p_\infty)^2 ds.$$
4. APPLICATION TO A BLAST WAVE

State evolution

Mesh-adaptive computation of acoustics
4. APPLICATION TO A BLAST WAVE

Mesh-adaptive computation of acoustics
Figure: Evolution of the mesh in time.

Mesh-adaptive computation of acoustics.
Nonlinear “blast” wave.
Second Example, results
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Mesh-adaptive computation of acoustics
Second Example, results
For acoustic analysis, the use of anisotropic meshes seems less mandatory.

Uniform meshes allow a higher accuracy with lower cost per node, but need good absorbing boundaries.

For a particular family of problems, noise emission and noise observation ("micro") are localised in a small portion of the domain and much resolution can be useless.

In that case, the goal-oriented formulation helps focalising the mesh effort on the propagation from source to micro.
Application to acoustics

Comparison between Goal-Oriented approach and multiscale $L^p$: 

![Diagram showing comparison between Goal-Oriented approach and multiscale $L^p$.](attachment:image)
Application to acoustics

Comparison between Goal-Oriented approach and multiscale $L^P$: 
Application to acoustics

Comparison between Goal-Oriented approach and multiscale $L^p$: 

![Diagram showing comparison between Goal-Oriented approach and multiscale $L^p$.]
Application to acoustics

Comparison between Goal-Oriented approach and multiscale $L^p$: 

Mesh-adaptive computation of acoustics
Application to acoustics

Comparison between Goal-Oriented approach and multiscale $L^p$: 

![Graphical representation of mesh-adaptive computation of acoustics]
Comparison between Goal-Oriented approach and multiscale $L^p$: 
Application to acoustics

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Comparison between Goal-Oriented approach and multiscale $L^p$:

Mesh-adaptive computation of acoustics
Application to acoustics

Comparison between Goal-Oriented approach and multiscale $L^P$:
Application to acoustics

Comparison between Goal-Oriented approach and multiscale $L^p$:

![Graphical representation of mesh-adaptive computation of acoustics]

Mesh-adaptive computation of acoustics
Comparison between Goal-Oriented approach and multiscale $L^p$: 

- Mesh-adaptive computation of acoustics
Application to acoustics

Comparison between Goal-Oriented approach and multiscale $L^P$:
Application to acoustics

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Application to acoustics

Comparison between Goal-Oriented approach and multiscale $L^p$: 

- **Mesh-adaptive computation of acoustics**
Application to acoustics

Comparison between Goal-Oriented approach and multiscale $L^p$: 
Application to acoustics

Comparison between Goal-Oriented approach and multiscale $L^p$:
Application to acoustics

Mesh-adaptive computation of acoustics
Application to 2D acoustics(2)
Application to 2D acoustics (2)

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Mesh-adaptive computation of acoustics
Application to 2D acoustics

Mesh-adaptive computation of acoustics
Application to 2D acoustics(2)
Application to 2D acoustics (2)
Conclusion:

- New mesh adaptation algorithm which prescribes the spatial mesh of an unsteady simulation as the optimum of a goal-oriented error analysis;
- Extension to unsteadiness is applied in an implicit mesh-solution coupling which needs a non-linear iteration, the fixed point;
- The new algorithm is applied to a blast wave test case and a noise propagation test case and shows on these calculations the favourable behavior expected from an adjoint-based method (automatic selection of the mesh necessary for the target output)
Perpectives:

- Accurate integration of time errors in the mesh adaptation process with a more general formulation of the mesh optimisation problem (work in progress)
- Higher order adjoint schemes and 3D unsteady test-cases (work in progress)
- Application to turbulent aeroacoustics (3D Navier-Stokes equations)