Finite Element for Flow Problems (Papers)
FEF05 IACM Special Interest Conference supported by ECCOMAS, April 4-6, 2005, Swansea, Wales, UK

Unified Level Set for Bi-Fluid Flow Simulation

Anne-Cécile Lesage[1,2*], Olivier Allain[2], Alain Dervieux[2,1]

[1] LEMMA, 938-A, avenue de la république, 06550 La Roquette-sur-Siagne, France
[2] INRIA, BP 93, 2004 route des Lucioles, 06902 Sophia-Antipolis, France
contact: alain.dervieux@inria.fr

ABSTRACT

Interface motion simulation has been studied from the early start of computer simulation. This subject sets a difficult approximation problem: - Lagrangian interface methods are facing the complexity in managing the collision and separation of interface components. - Eulerian interface methods are facing the difficulty in keeping accuracy in advecting a discontinuous density field.
The Level Set principle introduced in the end of 80's by Osher and Sethian proposes to advect a relatively smooth field ϕ the zero contour of which is the interface to represent. Since the function to advect is smooth, this method opens the door to high accuracy. It combines well to a finite-element discretisation since a central feature of FEM is the derivation from degrees of freedom, and by interpolation, of the interface location. In order to maintain function ϕ sufficiently smooth, a re-distancing update has to be applied. It consists in replacing ϕ by a signed distance to function ϕ's zero contour. This is generally done by converging a pseudo-unsteady Hamilton-Jacobi equation. We observe that the overall process, made of a physical time step involving a subcycling, is close to the popular dual-time stepping algorithm applied in CFD for advancing an evolution implicitly in time, when using an explicit pseudo-unsteady solver.
The Hamilton-Jacobi re-distancing model is difficult to discretize such that the interface approximation is not deteriorated. An possible explanation lies in some *incompatibility* between the advection and the distance property when the velocity is not uniform. What can be the consequence? Remember that, in a similar way to the initial advective Lax Scheme error behavior, a spatial error introduced at each time step can result in an inconsistant (in time) numerical method. As a compromise, it is proposed in the literature to limit the impact of this deterioration by demanding better conservation of mass during the re-distancing step.
Third, the Level Set method extends to interface motions in presence of interface tension. A typical approach is to use a volumic approximation of the surface tension force, à la Brackbill. This volumic field presents variations normal to interface that may deteriorate its integration along the interface. The accuracy of the surface tension term can be improved by a Hamilton-Jacobi extension from the interface values, with the precautions mentioned above. Surface tension couples in many cases with a contact-angle model which looks like a boundary condition for the ϕ field. Is this condition compatible with the advection of ϕ? With its re-distancing?
To summarize, the Level Set method uses today a blend of physical time-advancing and of pseudo-unsteady artificial Hamilton-Jacobi systems resolution at each time step. It is not always easy to make this orchestra play in harmony. This paper proposes some remarks in the direction of improving the synergy between these ingredients.

1 Main features of the Level Set Method

Let us consider the numerical solution of the motion of two incompressible fluids:

$$\partial_t(\rho u) + div(\rho u \otimes u) - \Delta u + \nabla p = f$$

$$\partial_t \rho + div(\rho u) = 0 \qquad \rho = \rho_l \ or \ \rho_g$$

$$div\ u = 0$$

Let H the step function such that $H(x) = 1$ if $x > 0$ and $H(x) = 0$ else. The Level Set method introduced by Osher et Sethian ([3]) relies on a double model involving the characteristic function χ of one phase

$$\frac{\partial \chi}{\partial t} + V{\cdot}\nabla\, \chi = 0 \quad (\chi = 0\ or\ 1)\ .$$

combined with a smoother function ϕ such that $\chi = H(\phi)$:

$$\frac{\partial \phi}{\partial t} + V{\cdot}\nabla\, \phi = 0\ ;\ \ \chi = H(\phi) \tag{1}$$

The formal accuracy of the advection of a step function as χ is severely limited while the accuracy of the Level Set method can be stated as follows:

Proposition. 1
Let Φ be a $L^2(Q)$ function where $Q = \Omega\times]0,T[$ is the flow integration domain in space and time and we assume that:

$$(f_\Phi(\eta) \equiv mes_Q(-\eta \leq \Phi \leq \eta)) \rightarrow \theta(\eta) \text{ when } \eta \rightarrow 0. \tag{2}$$

If Φ is the level set function which zero value contour represents the interface, the contour $\Gamma : (\Phi = 0)$ has a zero thickness.

Let $(\Phi_h)_h$ is a sequence of $L^2(Q)$ with:

$$\Phi_h \rightarrow \Phi \text{ in } L^2(Q) \text{ strongly}\ . \tag{3}$$

then for all real number $p \geq 1$:

$$(\chi_h \equiv H \circ \Phi_h) \rightarrow (\chi \equiv H \circ \Phi) \text{ strongly in } L^p(Q) \tag{4}$$

where H is defined by $H(x) = 0$ if $x < 0$ and 1 in the other case.
A more accurate estimate is stated now:
Proposition. 2
If it is assumed that

$$\begin{cases} f_\Phi(\eta) \leq K_1\eta \\ \parallel \Phi_h - \Phi \parallel_{L^2(Q)} \leq K_2 h^k \end{cases} \tag{5}$$

with h, η sufficiently small, k the convergence order on Φ and K_1,K_2 independant of h, η then for all real number $p \geq 1$, there is a constant $C(p)$ independant of h such as

$$\parallel H \circ \Phi_h - H \circ \Phi \parallel_{L^p(Q)} \leq C(p) h^{\frac{2k}{3p}} \tag{6}$$

Let us assume that the advection velocity $V(x,t)$ and initial interface are sufficiently smooth. The signed distance ϕ_0 to initial interface is a smooth function in a neighborhood of the interface. Let us take it as an initial condition for ϕ , then, during some time interval ϕ will stay a smooth function. Then the above level set analysis can apply to a level set initialized by a signed distance during some time which do not depend on time and space discretisation. We examine now in which conditions the replacement of ϕ by a signed distance can be applied a every time step.
If a small error is introduced at the occasion of this replacement, even of order $(\Delta x)^k$, the impact on overall accuracy can be as large than $\frac{(\Delta x)^k}{\Delta t}$, which results in inconsistency!

Starting from problems for which only the normal velocity to boundary is known, Sethian has remarked that the inconsistency between signed distance and advected solution can be reduced by choosing an adequate extension:

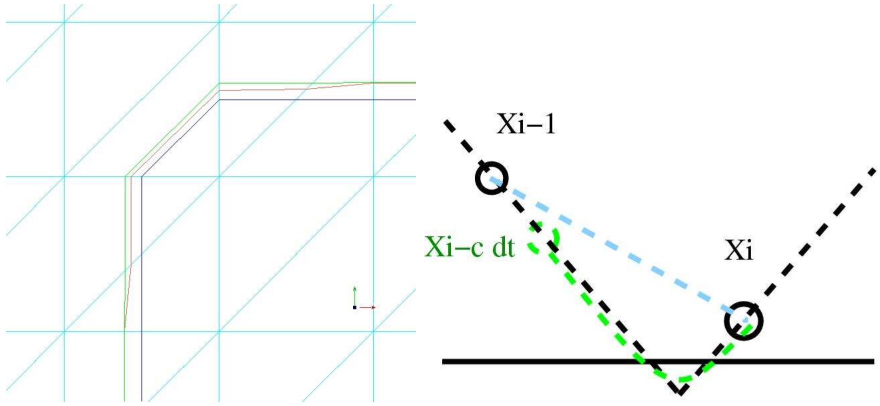

Figure 1: Left: Motion of an interface with a corner, from bottom-right to top-left: initial condition, FEM inaccurate interpolation, exact reconstruction. Right: 1D Method of characteristics for a subgrid component: black dashes show the exact reconstruction between X_{i-1} and X_i

Proposition. 3
Assuming that:

$$\bar{V}(x,y,t) \;=\; \lambda(x,y,t)\nabla\phi + W \quad , \quad W.\nabla\phi \;=\; 0 \; ,$$
$$\nabla\lambda \; . \; \nabla\phi \;=\; 0 \; ,$$

then:

$$\phi_t \;+\; \bar{V}.\nabla\phi = 0 \quad \textbf{and} \quad ||\nabla\phi|| \;=\; 1 \; .\Box$$

The physics gives us λ at the interface and it is necessary to extend it from this data. This can be done by solving untill convergence a pseudo-unsteady Hamilton-Jacobi system:

$$\bar{V}_\tau \;+\; \nabla\phi\nabla\bar{V} = 0 \;\; ; \;\; \bar{V}(x,y) = V(x,y) \;\; \text{if} \;\; \phi(x,y) = 0 \; .$$

Again some smoothness for $\bar{V}$ in a neighborhood of the (presumed) smooth interface can be expected and higher order accuracy can be kept.

2 Accurate advection of a non-smooth function

Up to now we have not considered cases for which the coupled of signed distance ϕ and its advective velocity cannot be locally smooth enough for a higher order convergence. This can happen in various ways: angles in the interface itself, coalescence between several components of the interface, typically.
In this section, we do not examine the issue of a lack of smoothness of the velocity field, but we focus on the lack of smoothness of ϕ. Assuming a smooth velocity, it is possible to apply a characteristic method to advect ϕ. The backward characteristic method writes:

$$\frac{d}{dt}X_i(t) \;=\; V(x,t^n) \; ,$$
$$X_i(t^{n+1}) = (x_i, y_i) \; ,$$
$$\phi_i^{n+1} \;=\; (\Pi\phi^n)(X_i(t^n)) \; .$$

The “interpolation” operator Π can be improved in such a way that the signed distance is reconstructed with as much accuracy as possible. To do this, we can use the fact that the gradient of the signed distance is almost everywhere of unit norm. From the values at vertices of a triangulation, an exact reconstruction inside each element and an exact advection is possible at the neighboring of an angular corner. We refer to see Fig. 1(left): the error that appears near the corner is completely avoided in the improved formulation. We can consider in the same set of singular events

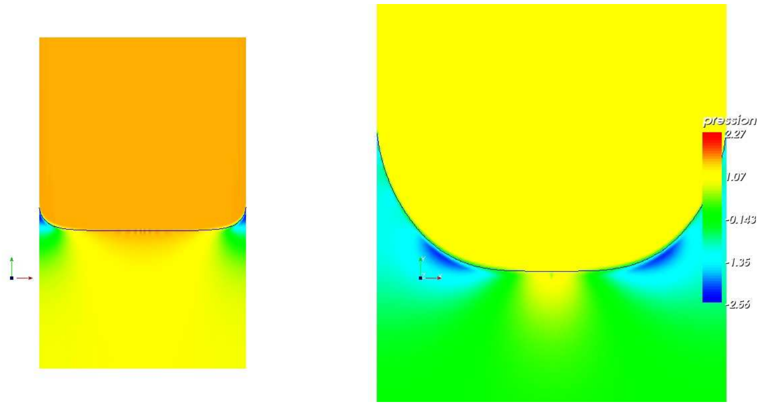

Figure 2: Reorientation sequence with a contact angle of 5.5 degrees

a subgrid case where a small strip of one phase has width too small to allow a capturing of its motion in a given mesh with a P_1 interpolation. This is displayed in a 1D case in Fig. 1(right). The P_1 interpolation between vertices X_{i-1} and X_i cannot see the small subgrid interval in which ϕ is negative. Conversely, using the unity gradient property, this can be reconstructed and exactly advected with the characteristic method. It is interesting to observe that the resulting integrated time derivative of ϕ

$$\int \phi_t dt \;=\; -\left(\Delta t^* \phi_x^{downwind} + (\Delta t - \Delta t^*)\phi_x^{far-upwind}\right)$$

shows a downwind part as in the VOF method.
In the case of curve interfaces such as in Fig. 2, the accuracy of the advective step has to be further improved by building from the node values of ϕ^n a higher-order order interpolation. Second order accuracy is easily obtained by using nodal estimates of the gradient of ϕ^n.
The combination between smooth and non-smooth reconstruction methods can be built on the basis of singularity detection as in TVD techniques.
Boundary treatment is applied as in regular characteristics methods.

3 Concluding remarks

We have reviewed several parts in a Level Set algorithm in which the knowledge of the distance property of the ϕ function can be exploited in order to improve the accuracy and stability of the advection step.
Both non-smooth events and smooth ones can be managed in a same algorithm.
Currently we are studying the coherence between the representation of ϕ in high order re-distancing and advection.

References

[1] C.W. Hirt and B.D. Nichols, Volume of Fluid (VOF) methods for the dynamics of free boundaries, J. Comput. Phys. 30, 201-255, 1981.

[2] J. Brackbill, D.B. Kothe, C. Zemach, A continuum method for modeling surface tension, J. Comput. Phys. 100, 335354, 1992.

[3] S. Osher and J.A Sethian, Front propagating with curvature dependant speed: Algorithms based on Hamilto-Jacobi formulations, J. Comput. Phys. 79;12, 1988.