ANISOTROPIC GOAL-ORIENTED MESH OPTIMISATION

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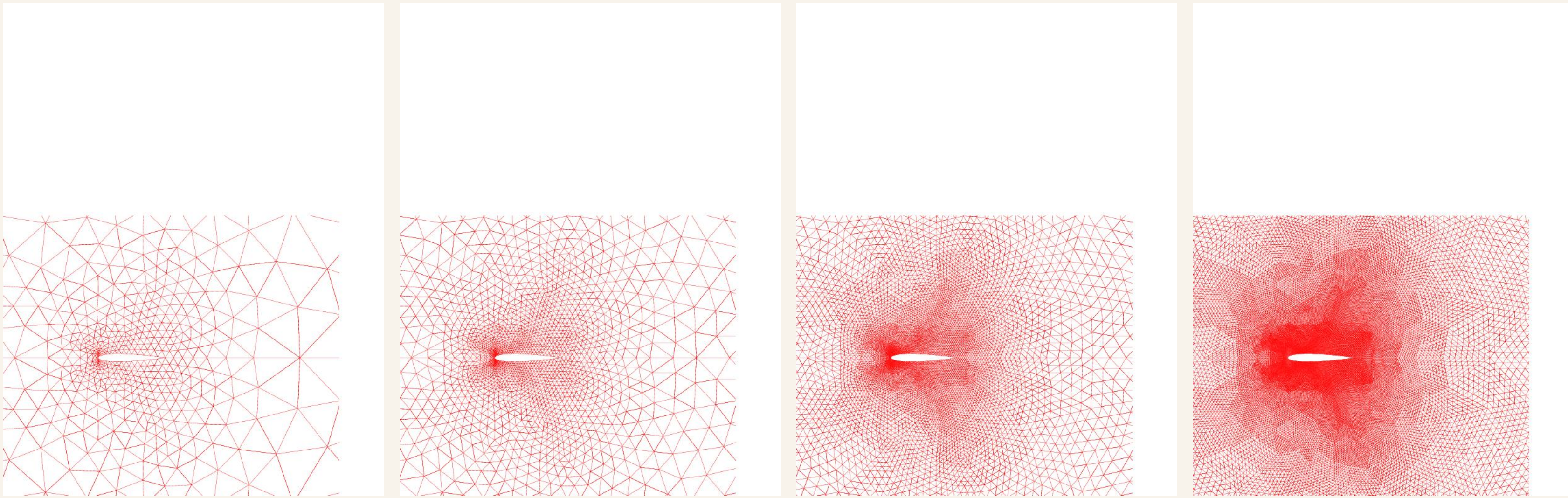
An example

Numerical convergence, uniform embedded refinement

Navier-Stokes, Mach number is 1.2, Reynolds is 100.



	mesh 1	mesh 2	mesh 3	mesh 4
# of nodes	800	3114	12284	48792
L^2 Numerical convergence order			**0.94**	**1.14**

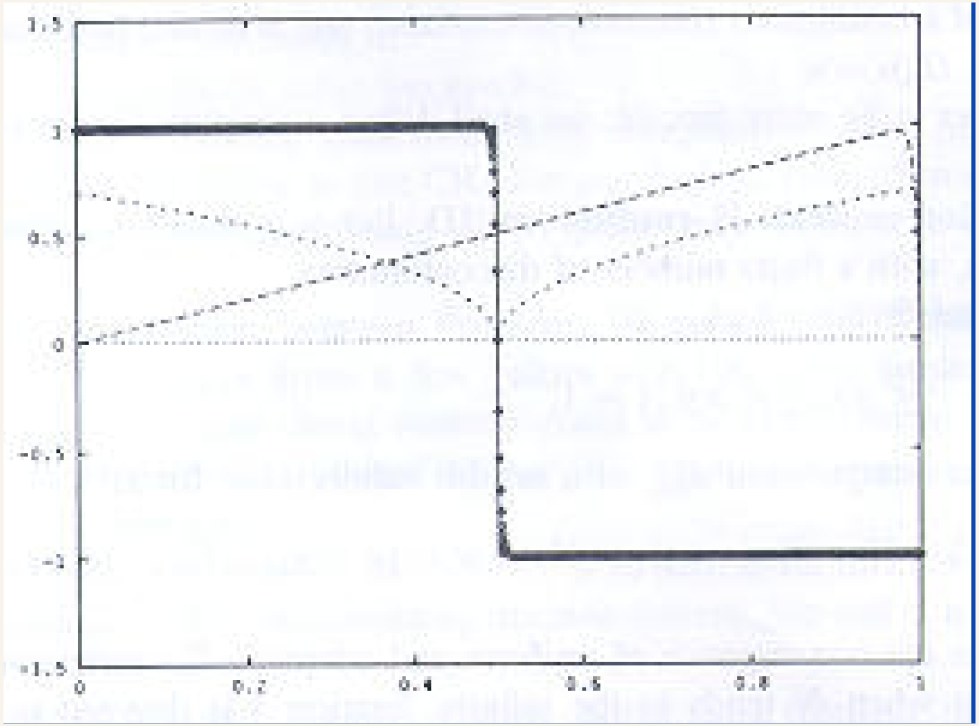
Overview:

1. Anisotropic adapted meshes,
2. Hessian-based unsteady applications,
3. Goal-oriented applications.

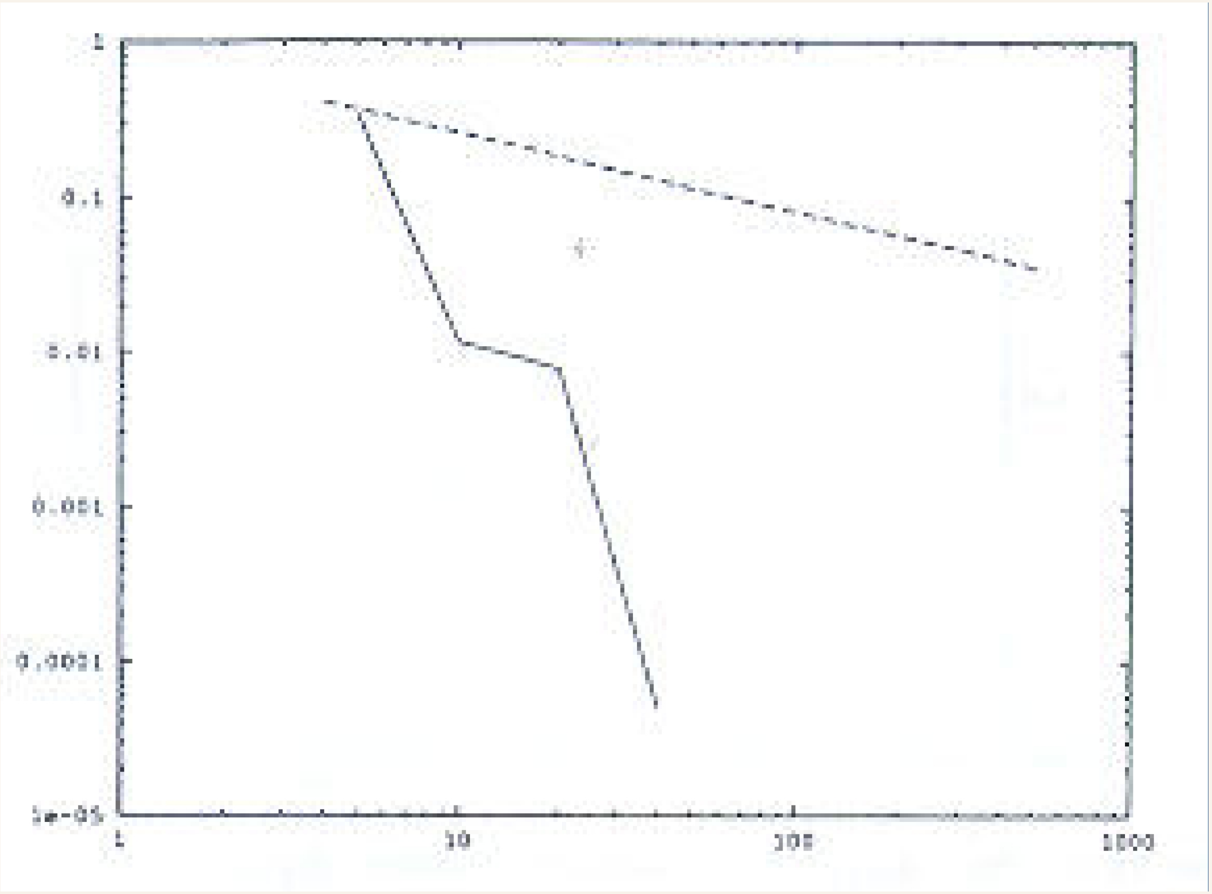
1. Mesh adaptation by P1 interpolation adaptation

1.1. Motivation: what are the conditions for asymptotic convergence?

Two 1D examples : smooth arctangent, discontinuous Heavyside.

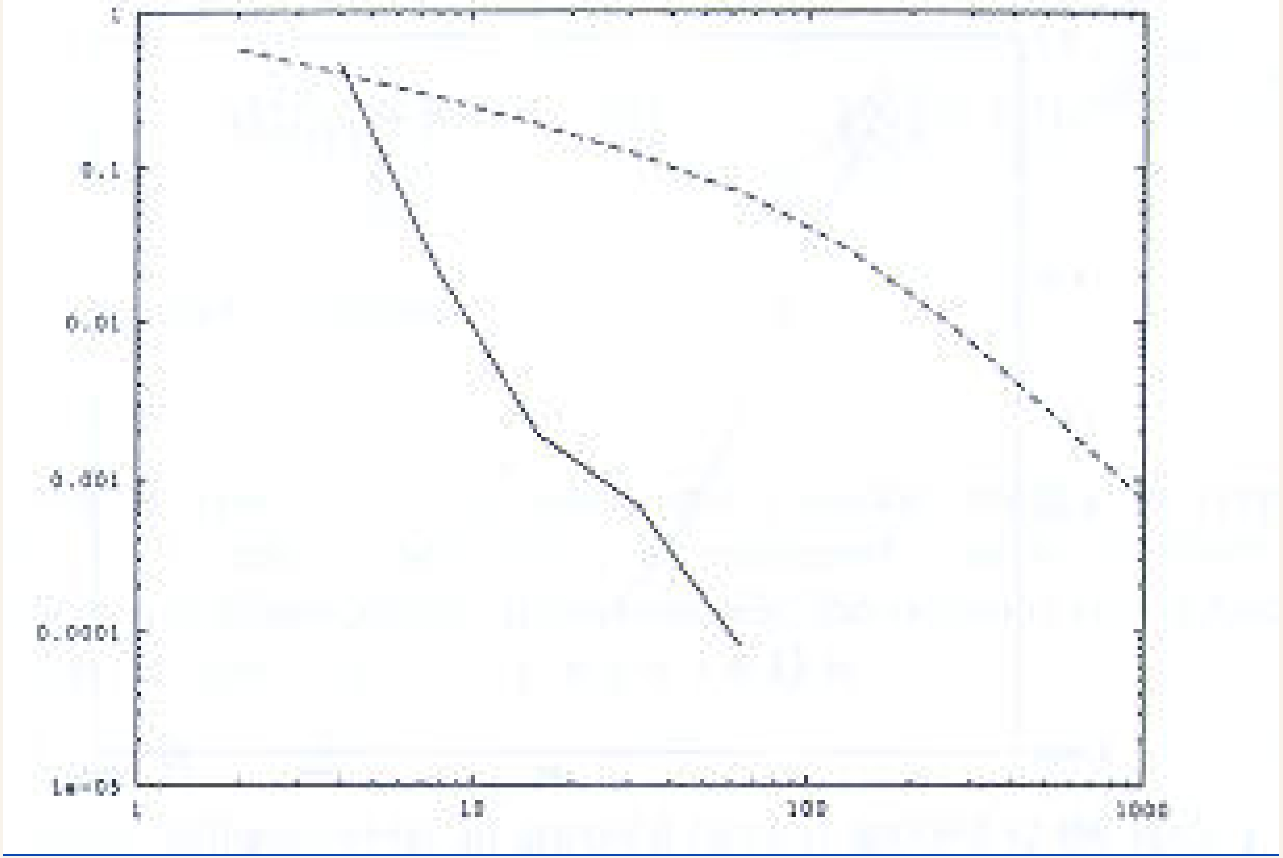


L^p ($p \neq \infty$) convergence to the continuous: Heavyside



Abscissae : number of nodes (from 8 to 512) ; ordinates : interpolation error, dashes : uniform refinement(slope$\approx 1/2$), line : adaptive refinement.

Convergence to the continuous: Arctangent



Uniform refinement: late capturing

Adaptative refinement : early capturing

Early capturing/late capturing

Uniform refinement needs $O(N_S)$ nodes, where N_s is the inverse of the size of the smallest detail (1D).
A good adaptative refinement needs $O(N_d)$ nodes, where N_d is (1D) the number of details (for example: the function is monotone on N_d intervals).

In general, $N_d \quad << \quad N_S$.

We look for a mesh adaption method which is higher order accurate on discontinuities, and, therefore enjoys early capturing of smooth details.

1.2. Minimizing P1-Interpolation error in L^2

(Castro-Diaz *et al.*, Habashi *et al.*, Lipnikov-Vasilevski-Agouzal, Huang, Long Chen, Alauzet *et al.*)
(Loseille, PhD, Paris VI, 2008)

$$||u - \pi_{\mathcal{M}} u||^2 \quad = \quad \int \left(|\frac{\partial^2 u}{\partial \xi^2}|.m_\xi^2 + |\frac{\partial^2 u}{\partial \eta^2}|.m_\eta^2 \right)^2 \; dxdy$$

where ξ and η are directions of diagonalization of the Hessian of u.

$$\min_{\mathcal{M}} \; \mathcal{E}_{\mathcal{M}}$$
$$\text{under the constraint} \quad N_{\mathcal{M}} = N.$$

This can be solved analytically.

Optimal Metric for interpolation

$$\mathcal{M}_{opt} = \frac{C}{N}\,\mathcal{R}^{-1}\begin{pmatrix} \left|\frac{\partial^2 u}{\partial \eta^2}\right|^{-5/6}\left|\frac{\partial^2 u}{\partial \xi^2}\right|^{1/6} & 0 \\ 0 & \left|\frac{\partial^2 u}{\partial \xi^2}\right|^{-5/6}\left|\frac{\partial^2 u}{\partial \eta^2}\right|^{1/6} \end{pmatrix}\mathcal{R}\ . \qquad (1)$$

with: $C \ = \ \int \left(\left|\frac{\partial^2 u}{\partial \xi^2}\right| . \left|\frac{\partial^2 u}{\partial \eta^2}\right|\right)^{\frac{2}{6}}\ dxdy\ .$

- Since the theoretical optimal error can be expressed in terms of the number of nodes,

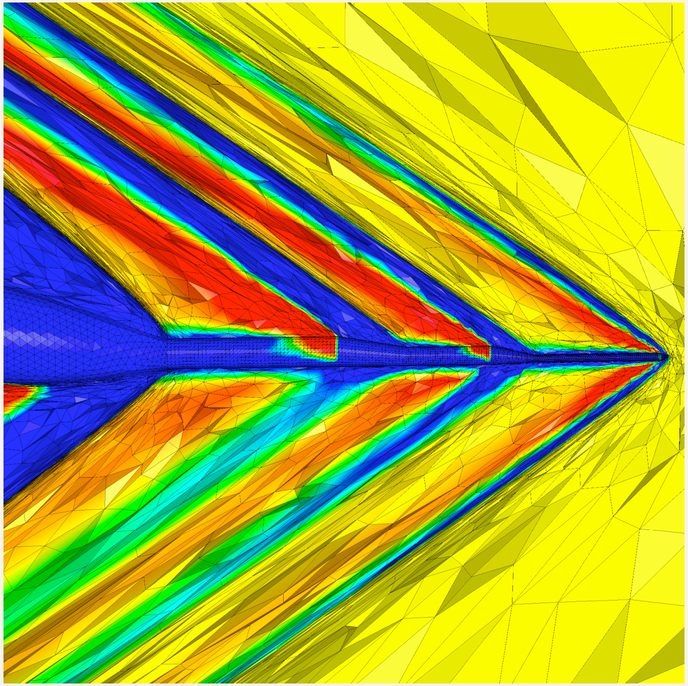
$$\mathcal{E}_{\mathcal{M}_{opt}} = fcn_1(N) \Leftrightarrow N_\varepsilon = fcn_2(\varepsilon)$$

then, in practice, *either* **the number of nodes** *or* **the optimal** L^2 **error can be specified.**

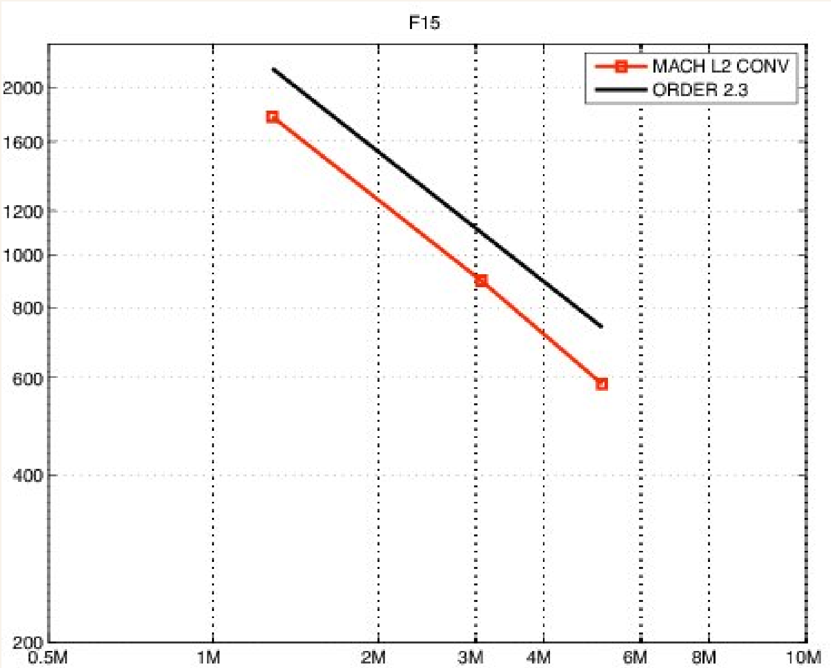
- It can be shown this interpolation adaptation is higher-order and it has been observed that it enjoys early capturing.

For a **PDE**, a sensor field, e.g. the Mach field can be chosen. Then a **Fixed Point** between interpolation-adaptation, PDE solution transfer and PDE recomputing needs be applied.

Example: sonic boom prediction (spike NASA flight measurement)



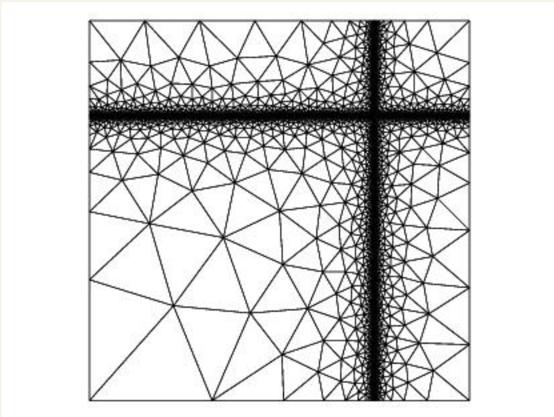
Mesh and pressure field partial view

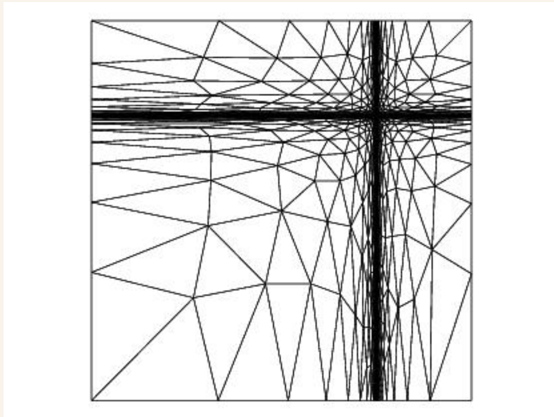


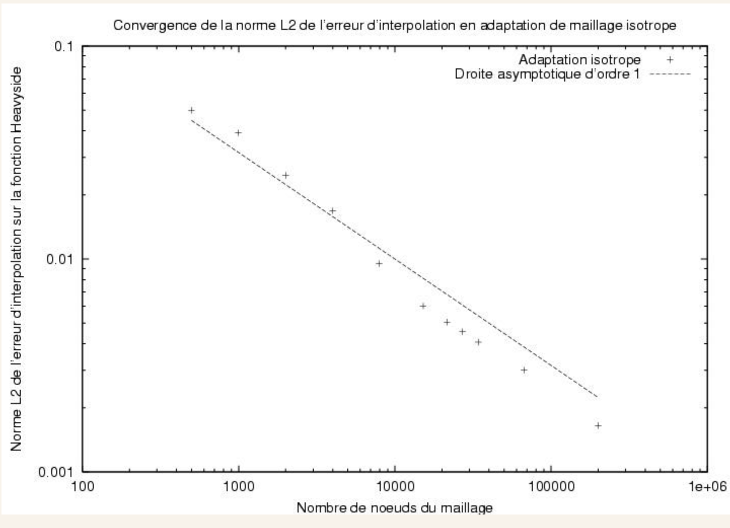
L^2 convergence from 1.1Mnodes to 5Mnodes.

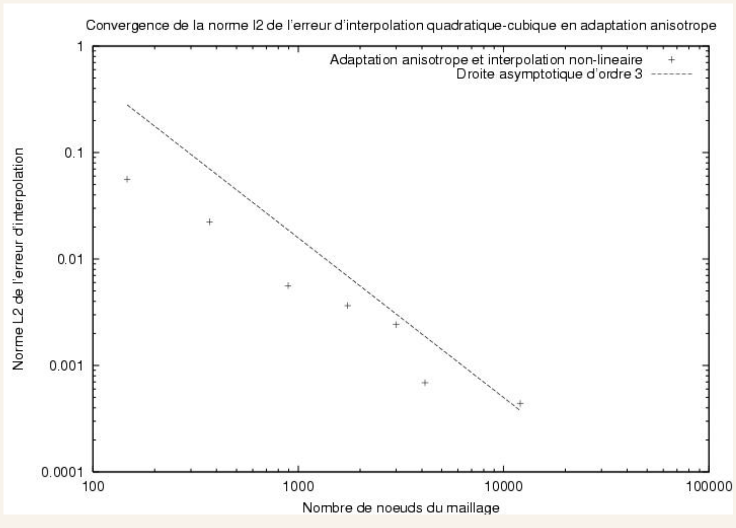
1.3. Conditions for multidim'al higher-order convergence

L^p mesh convergence on two 2D Heavyside functions, $p \neq \infty$









1.3. Conditions for multidim'al higher order convergence

Barrier lemma: best L^p convergence of P_1 interpolation for an isotropic mesh adaptation method in dimension d on discontinuity lying on a surface of dimension $d-1$ satisfies $\alpha \le d/(dp-p)$.
Coudière-Dervieux-Leservoisier-Palmerio, 2001

L^2 Conv. order	Unif. Ref.	Adap. Isotropic	Adap. Anisotropic
2D barrier theory	$\le \frac{1}{2}$	≤ 1	≤ 2
2D Optimal L^p Metric Theory		1	2
Optimal L^2 Metric Num. exp. interp. Heavyside 2D		1	2
3D barrier theory	$\le \frac{1}{3}$	$\le \frac{3}{4}$	≤ 2
Optimal L^2 Metric Num. exp. Euler Spike 3D		not comp'd	2+

2. Hessian-based unsteady applications

$L^\infty(0,T;L^2_x)$ Transient Fixed point Mesh Adaptation:

$$[0,T] \;=\; [0=t_0,t_1]\cup,\ldots\cup[t_i,t_{i+1}]\cup,\ldots\cup[t_{n-1},t_n].$$

Step0: Choose error level ε,

Step1: On $[t_i,t_{i+1}]$, time-discretise: $t_i^0=t_i, t_i^1,\ldots t_i^{n-1}, t_i^n=t_{i+1}$,

Step2: Advance in time the discrete PDE,

Step3: Get Hessians and corresponding L^2 optimal metrics for ε:

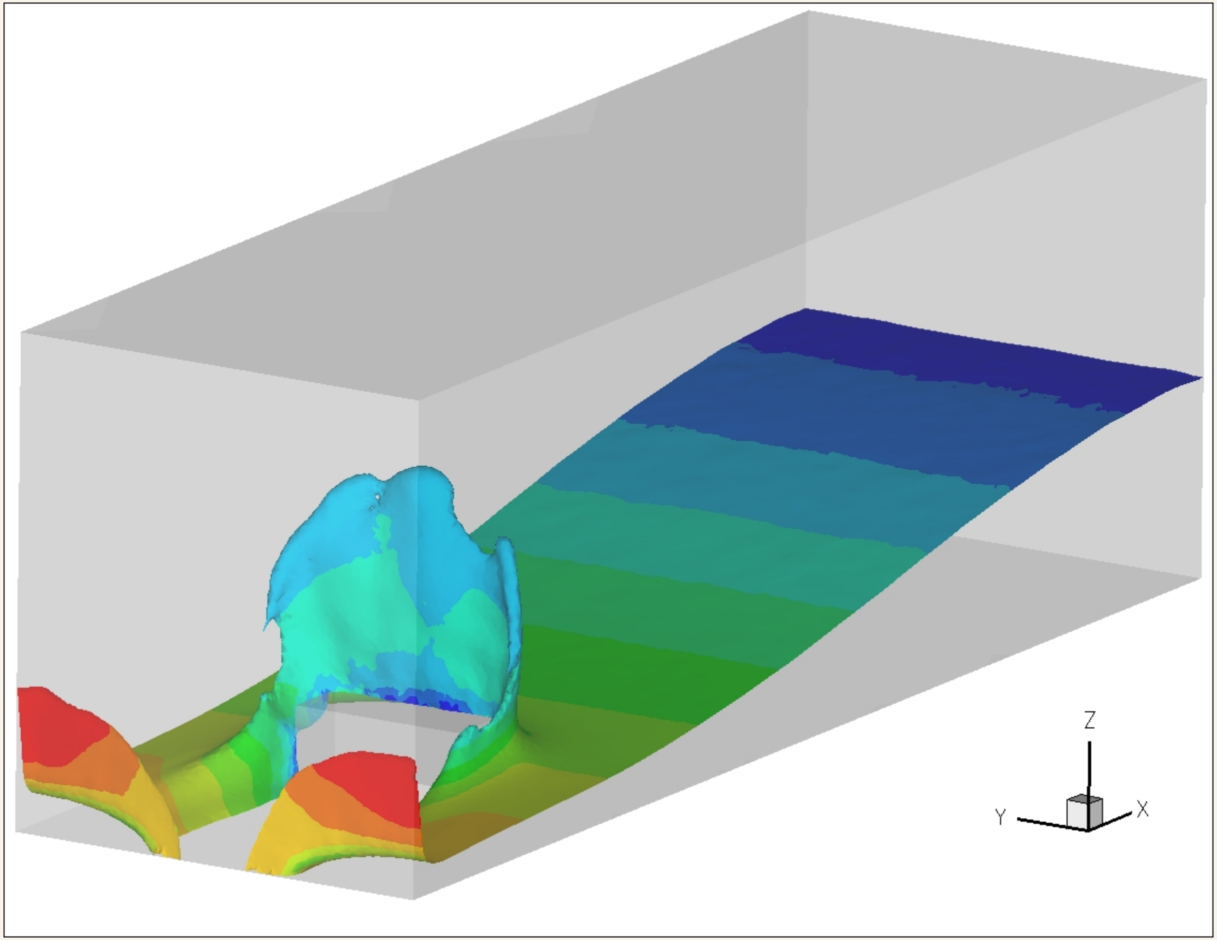
$$H_i^0, H_i^1,\ldots H_i^{n-1}, H_i^n \quad\Rightarrow\quad \mathcal{M}_i^0, \mathcal{M}_i^1,\ldots \mathcal{M}_i^{n-1}, \mathcal{M}_i^n$$

Step4: Use $\mathcal{M}_i=\mathcal{M}_i^0\cap\mathcal{M}_i^1\cap,\ldots\mathcal{M}_i^{n-1}\cap\mathcal{M}_i^n$ for remeshing.

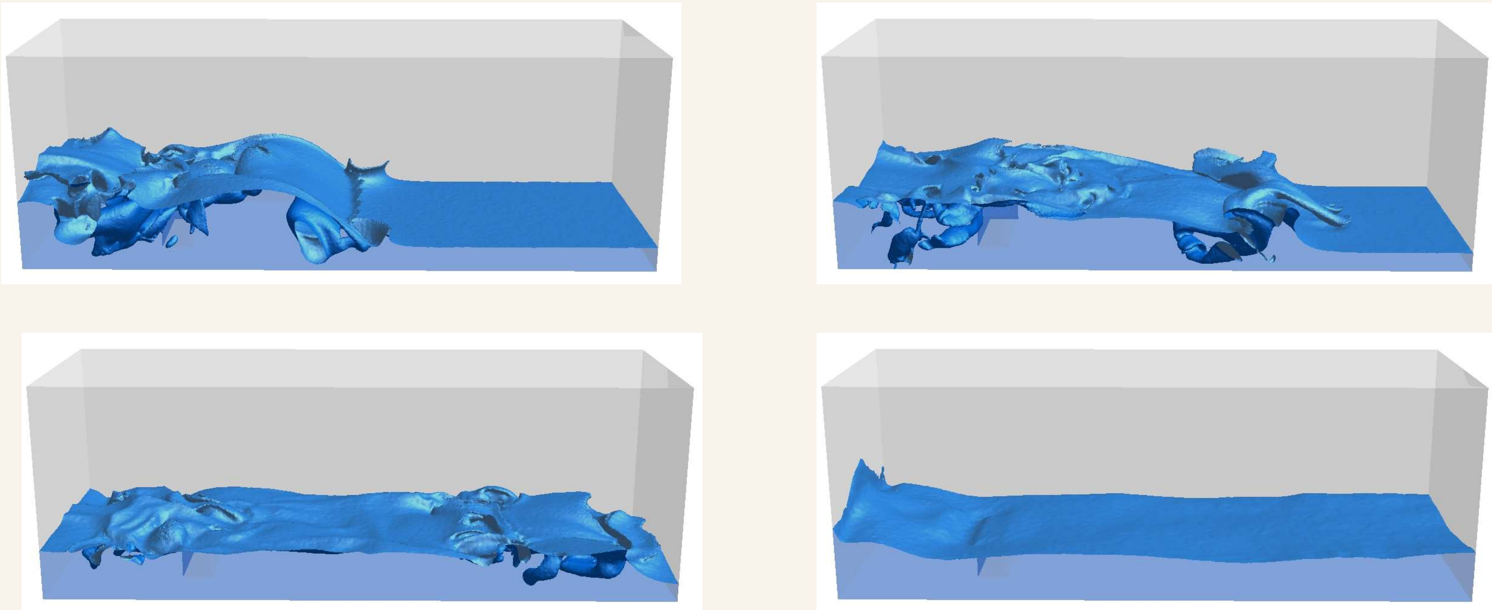
Step5: Go to **Step1** until convergence.

2. An example of 3D unsteady mesh adaptative flow calculation

MARIN test case: geometry, interface, colors from velocity

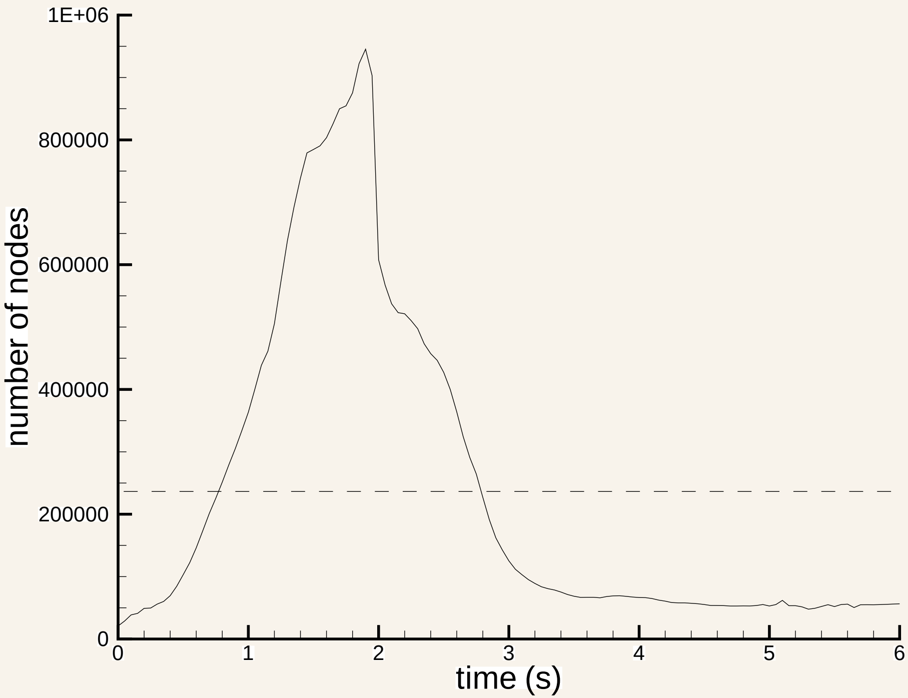


2. An example of 3D unsteady mesh adaptative flow calculation



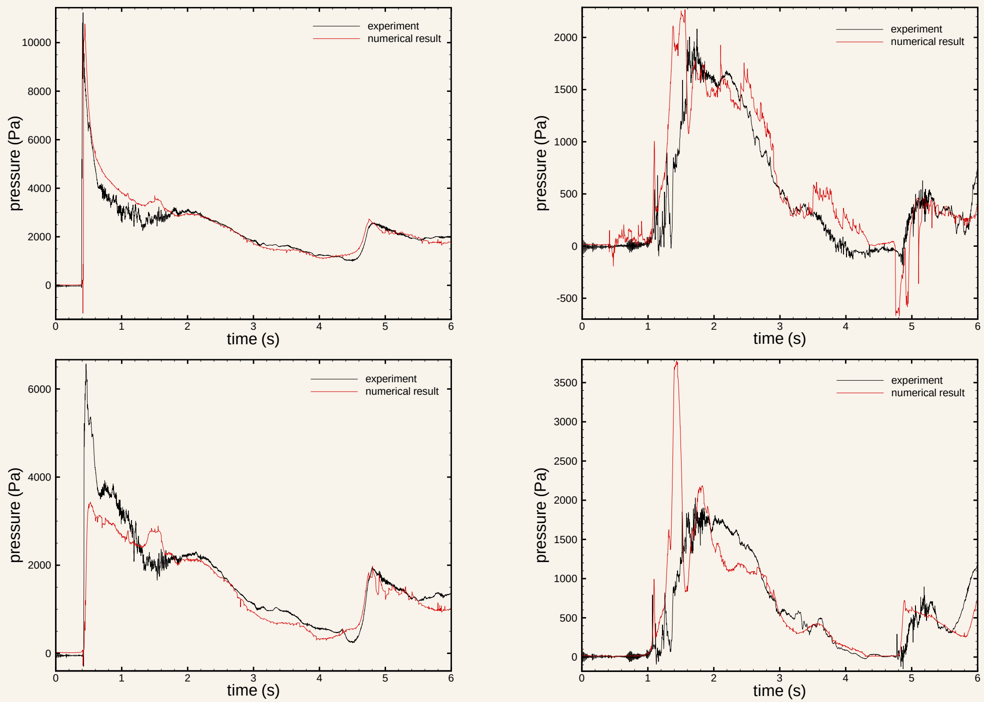
Wave sloshing in a basin with cubic obstacle at different times.

2. 3D unsteady mesh adaptative flow, cont'd



Wave sloshing in a basin: number of mesh nodes as a function of physical time.

2. 3D unsteady mesh adaptative flow, cont'd



Wave sloshing in a basin: comparision computation/measurement for pressure at various spot

3. PDE-approximation-based adaptation

Abstract representation of the Partial Differential Equation:

$$\Psi(W) = 0 \ .$$

Discretisation of the PDE:

$$\Psi_h(\mathbf{W}_h) = \mathbf{0} \ \in \ R^N$$
$$\mathbf{W}_h \ \in \ R^N \ , \ \mathbf{W}_h = [(\mathbf{W}_h)_i] \ .$$

Operators between continuous and discrete:

$$R_h : \ R^N \ \rightarrow \ V \ \subset \ L^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega}) \ \ \mathbf{v}_h \ \mapsto \ R_h \mathbf{v}_h$$
$$T_h : V \ \rightarrow \ R^N \qquad\qquad v \ \mapsto \ T_h v$$

$$W_h(x, y, z) = (R_h \mathbf{W}_h)(x, y, z) \quad ; \quad W - W_h \approx ?$$

3. PDE-approximation-based adaptation

A posteriori estimate:

$$\Psi(W) - \Psi(W_h) = -\Psi(W_h) \Rightarrow W - W_h \approx -[\frac{\partial \Psi}{\partial W}]^{-1}\Psi(W_h),$$

in practice:

$$\delta W_h = -R_h[\frac{\partial \Psi_h}{\partial W_h}]^{-1}T_h\Psi(R_h W_h),$$

quadratures formulas can be used for $\Psi(R_h W_h)$.

A priori estimate:

$$\Psi_h(W) - \Psi_h(W_h) = \Psi_h(W) \Rightarrow W - W_h \approx [\frac{\partial \Psi_h}{\partial W_h}]^{-1}\Psi_h(W),$$

in practice:

$$\delta W_h = R_h[\frac{\partial \Psi_h}{\partial W_h}]^{-1}(\Psi_h - T_h\Psi)(W_{(h)}).$$

3. PDE-approximation-based adaptation

Goal-oriented error:

$$j(W) = (g, W)_{L^2(\Omega)} \ s.t. \ \Psi(W) = 0$$

$$(\frac{\partial \Psi}{\partial W})^* p = g$$

$$j_h = (g, R_h \mathbf{W}_h)_{L^2(\Omega)} \ s.t. \ \Psi_h(\mathbf{W}_h) = 0$$

$$\mathbf{g}_h = T_h g$$

$$(\frac{\partial \Psi_h}{\partial \mathbf{W}_h})^* \mathbf{p}_h = \ \mathbf{g}_h \Leftrightarrow \mathbf{p}_h = [\frac{\partial \Psi_h}{\partial W_h}]^{-*} T_h \ \mathbf{g}_h$$

$$p_h = R_h \mathbf{p}_h \ .$$

A fundamental assumption of the present analysis is that this discrete adjoint is a good enough approximation of continuous adjoint.

3. PDE-approximation-based adaptation

Adjoint-based goal-oriented a posteriori analysis (Giles-Pierce)

$$\delta_1 j = -\left(p_h\ ,\ T_h\Psi(W_h)\right)_{V\times V'}$$

Adjoint-based goal-oriented a priori analysis

$$\delta_2 j = -\left(p_h\ ,\ (\Psi_h - T_h\Psi)(W_{(h)})\right)_{V\times V'}$$

Both formulas assume mesh convergence.

Let us apply the second formula...

3. *A priori* numerical-based adaptation

Steady Euler equations:

$$W \in V = \left[H^1(\Omega)\right]^5, \ \forall \phi \in V,$$
$$\left(\Psi(W)\,,\,\phi\right) = \int_\Omega \phi\, \nabla.\mathcal{F}(W)\ \mathrm{d}\Omega - \int_\Gamma \phi\, \hat{\mathcal{F}}(W).\mathbf{n}\ \mathrm{d}\Gamma = 0$$

where $\Gamma = \partial\Omega$ and $\hat{\mathcal{F}}$ is B.C. fluxes.

Mixed-Element-Volume appoximation:

$$\forall \phi_h \in V_h, \quad \int_{\Omega_h} \phi_h \nabla.\Pi_h \mathcal{F}(W_h)\ \mathrm{d}\Omega_h - \int_{\Gamma_h} \phi_h \Pi_h \hat{\mathcal{F}}(W_h).\mathbf{n}\ \mathrm{d}\Gamma_h = - \int_{\Omega_h} \phi_h\, D_h(W_h) \mathrm{d}\Omega_h,$$

where Π_h is the usual elementwise linear interpolation and where D_h holds for a numerical dissipation term.

3. *A priori* numerical-based adaptation

A priori **adjoint-based** error estimate:

$$(g, W - W_h) \approx ((\Psi_h - \Psi)(W), P), \text{ with } \left[\frac{\partial \Psi}{\partial W}\right]^* P = g,$$

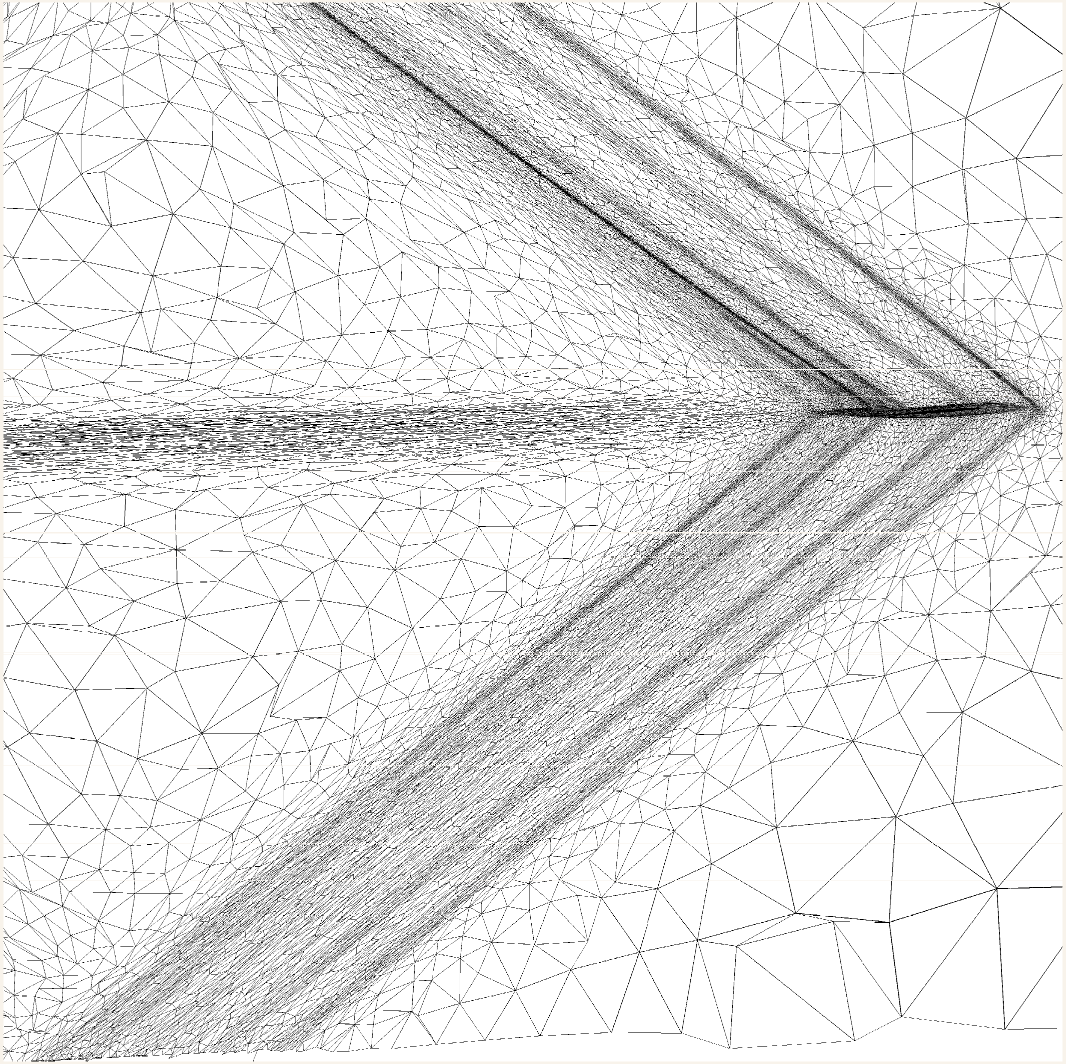
the optimal mesh is obtain after some transformations by solving:

$$\begin{aligned} \text{Find } \mathcal{M}_{opt} = \text{Argmin}_{\mathcal{M}} \int_\Omega |\nabla P||\mathcal{F}(W) - \pi_{\mathcal{M}} \mathcal{F}(W)| d\Omega \\ + \int_\Gamma |P||(\bar{\mathcal{F}}(W) - \pi_{\mathcal{M}} \bar{\mathcal{F}}(W)).\mathbf{n}|\, d\Gamma \end{aligned} \qquad (2)$$

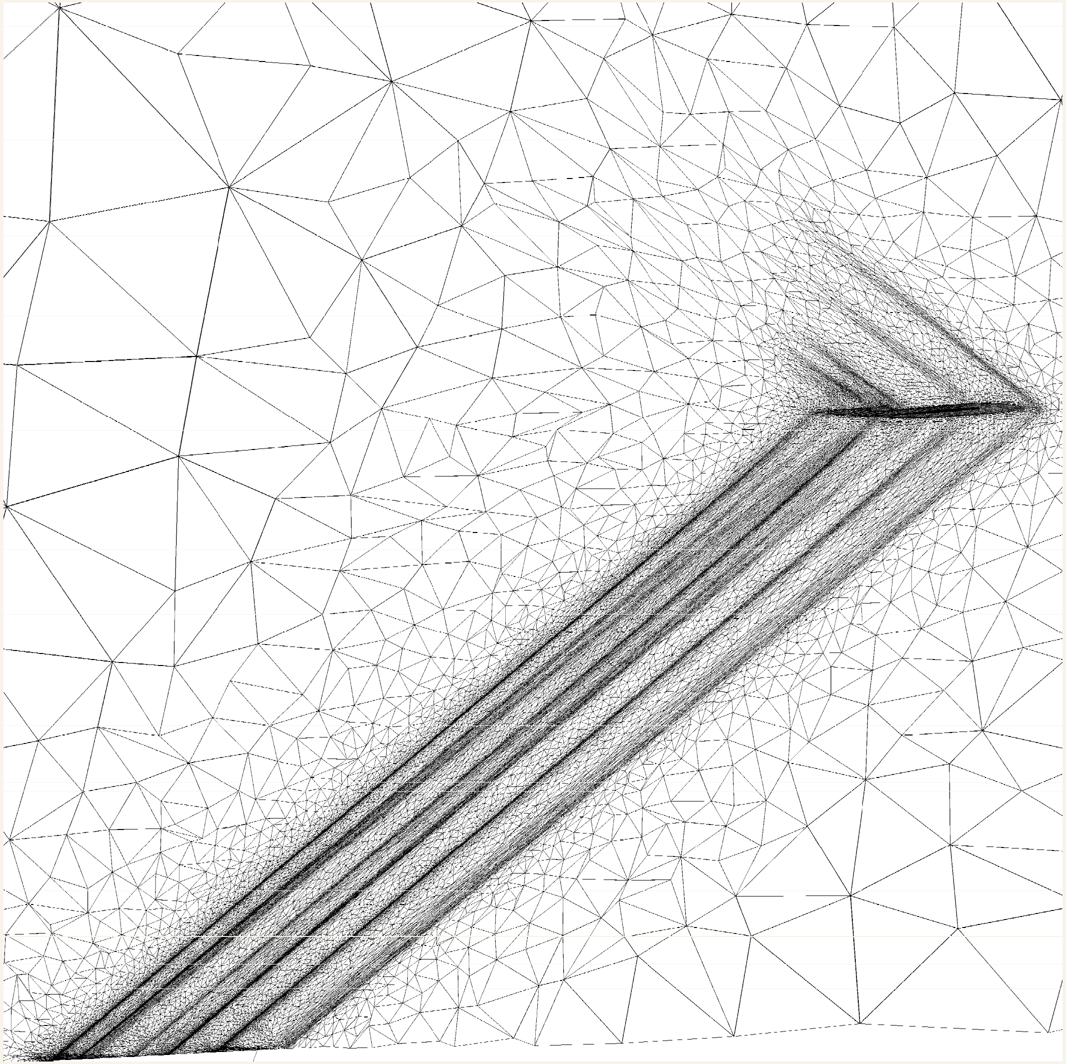
under the constraint $\mathcal{C}(\mathcal{M}) = N$.

Solved analytically as interpolation case.

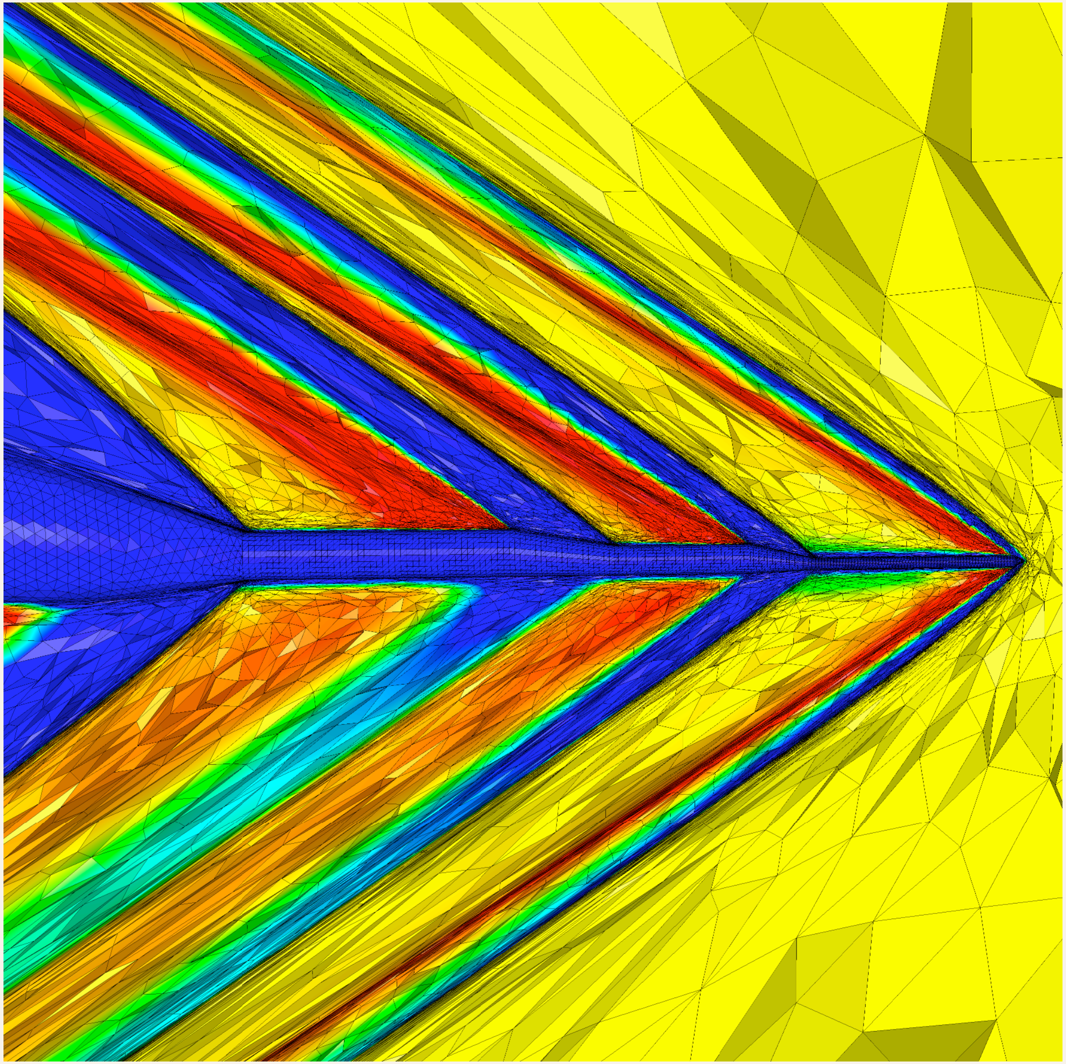
Remark: The adjoint-based formulation is compulsory.



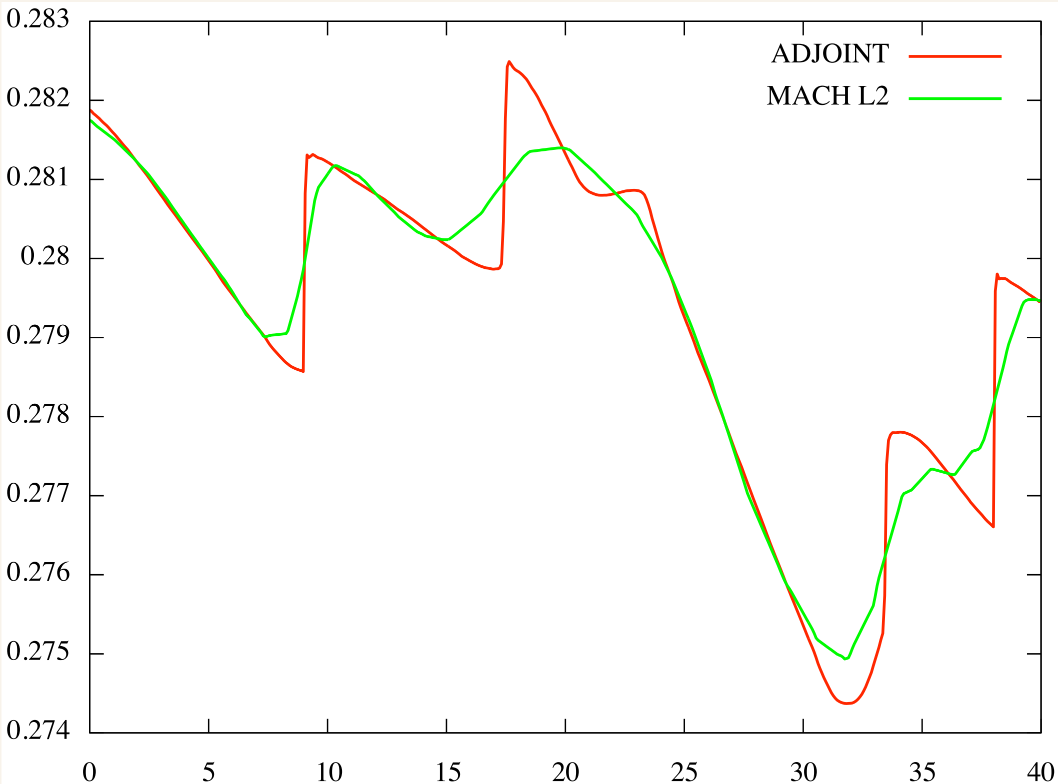
Application to sonic boom : Hessian-based (Mach L2)



Goal-oriented + Hessian-based (“foot print” funct.)



Pressure



Pressure track at ground

Concluding remarks

- Metric-based Anisotropic adaptation is an important tool for mesh convergence and then for approximation error control and certification.
- This kind of study rises much more questions than it solves: other models, other approximations (h-p?), among others.
- A main component is the scientific and technical effort in mesh generation and control, in particular by P.L.George and co-workers.
- For adjoint developement, an important help is given by Automatic Differentiation, in particular with TAPENADE, developed by Hascoet-Pascual.

Current investigations:

- Second-order PDE models.
- Mesh adaptation, correction and accuracy control for large instationary state systems.