HIGH-ORDER ADAPTIVE METHOD APPLIED TO HIGH-SPEED FLOWS

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Introduction

Mesh adaptation: Initially understood as devoted to improving a mesh, mesh adaptation is progressively specifying its problematic:

- A. *Problem setting*: what properties define the mesh we look for?

- B. *Problem solving*: how can we specify and build the mesh?

- C. *Analysis of solution*: what are the bonus properties of the adapted mesh?

Introduction

(i)- Hessian-based criteria easily produce optimal specification for mesh density and local stretching. But they rely on local interpolation error and link with PDE's is difficult.

Castro-Diaz *et al.* IJNMF 1997, Habashi *et al.* IJNMF 2000, Huang *et al.* JCP 2005, Courty *et al.* ANM 2006, Alauzet *et al.* IMR 2006

(ii)- PDE-based estimates are closer to PDE goals but produce stretching criteria less easy to exploit.

Venditti-Darmofal *et al.* JCP 2002, Formagggia *et al.* ANM 2004

We explain first a context where both combine well.

Introduction

- τ_h is a 3D tetrahedrization.
- $V_h = \{\psi \in \mathcal{C}^0$, linear by element$\}$
- Let Π_h be the P_1 interpolation from vertices.
- Discrete Euler equations:

$$W_h \in (V_h)^5 and \; \forall \; \phi_h \in (V_h)^5 \quad :$$
$$\int_\Omega \mathcal{F}(W_h).\nabla\phi_h d\Omega - \mathcal{D}_h(W_h,\phi_h) - \int_{\partial\Omega},\phi_h\bar{\mathcal{F}}(W_h).nd\partial\Omega = 0 \;(*)$$

Assuming P_1 exactness of (*) we can ($^+$) derive by an *a priori estimate*:

$$(g, W_h - W) \;\approx\; (g, \Pi_h W - W) - \int_\Omega \nabla P_h.(\mathcal{F}(W) - \Pi_h\mathcal{F}(W))d\Omega$$
$$+ \;\int_{\partial\Omega} P_h(\bar{\mathcal{F}}^{out}(W) - \Pi_h\bar{\mathcal{F}}^{out}(W)).nd\partial\Omega.$$

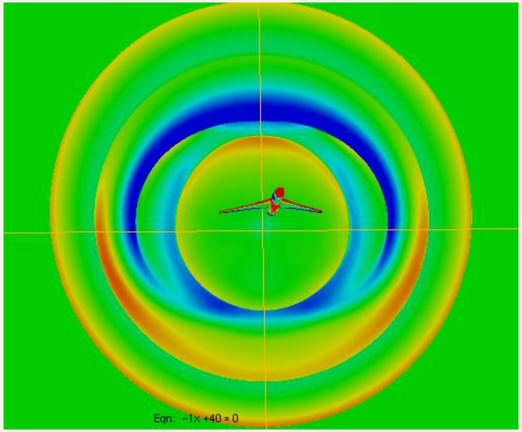
P_h: discrete adjoint with g as RHS.

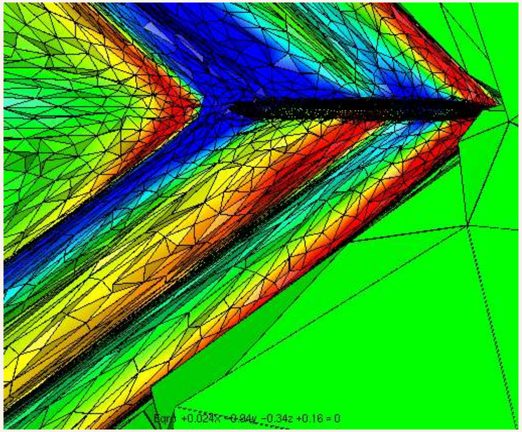
($^+$) Alauzet-Loseille-Dervieux-Mesri, 2007.

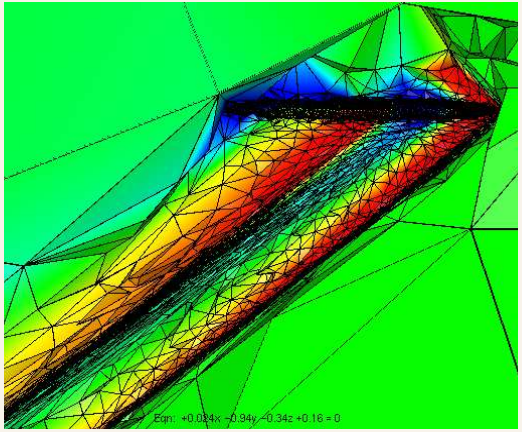
Introduction

$$\begin{aligned} Min \quad & (|g|, |\Pi_h W - W|) - \int_{\Omega} |\nabla P_h| . (|\mathcal{F}(W) - \Pi_h \mathcal{F}(W)|) d\Omega \\ & + \int_{\partial\Omega} |P_h| |(\bar{\mathcal{F}}^{out}(W) - \Pi_h \bar{\mathcal{F}}^{out}(W)).n| d\partial\Omega. \end{aligned}$$

A variational analysis allows exhibit the metric minimizing a weighted L^p norm of interpolation errors.







We explain now this variational analysis.

Introduction

Problem setting: L^p interpolation:

Given a function u in $\Omega \subset R^n$, of bounded derivatives except on a set of smooth curves with bounded length.

Mesh set $(\tau_h)_h$ contains simplexes (2D-triangulations/3D-tetrahedrizations)

Let $\Pi_h u$: continuous, affine by element, $(\Pi_h u)(X_i) = u(X_i)\ \forall\ i$ vertex.

Find the mesh which gives the smallest P_1-*interpolation error* $\Pi_h u - u$ in L^p norm.

Analyse the resulting convergence when mesh size increases.

Three contexts under study:

- **Case C1**: u is smooth, analytically available, or from a PDE.
- **Case C2**: u is assumed to be known analytically but it is singular.
- **Case C3**: Only a PDE solution is known through its numerical approximate u_h given on the current mesh.

PLAN

1) Optimal mesh for smooth function
2) Discontinuous fonction
3) 2D Numerical experiment
4) 3D Flow problems

1. OPTIMAL MESH FOR SMOOTH SOLUTIONS

Problem statement:

Let u a smooth function, with Hessian denoted:

$$\mathcal{H}_u = \mathcal{R}_u^{-1}\text{Diag}(\frac{\partial^2 u}{\partial {\alpha_1}^2}, \frac{\partial^2 u}{\partial {\alpha_2}^2}, \frac{\partial^2 u}{\partial {\alpha_3}^2})\mathcal{R}_u \ .$$

The goal is to obtain the best mesh if a given size for interpolating u.

The mesh is parameterised by a Riemannian metric, a $n \times n$ field defined on the computational domain:

$$\mathcal{M} = \mathcal{R}_{\mathcal{M}}^{-1}\text{Diag}(\lambda_1, \lambda_2, \lambda_3)\mathcal{R}_{\mathcal{M}} \ .$$

To $\mathcal{M}$ corresponds the following Riemannian distance between two arbitrary points of the computational domain:

$$dist_{\mathcal{M}}(X,Y) == \textstyle\int_0^1 \sqrt{\vec{XY}.\mathcal{M}.\vec{XY}(x'\vec{X} + (1-x')\vec{Y})} \ \ dx'.$$

To $\mathcal{M}$ the corresponds a equivalence class of unitary meshes $(\tau_{\mathcal{M}})$, i.e. satisfying:

The distance $dist_{\mathcal{M}}(X_i, X_j)$ *between any ends of edge* ij *of the mesh is equal to one.*

1. OPTIMAL MESH FOR SMOOTH SOLUTIONS

A two-step study can be applied ($^{+}$):

After anisotropy direction optimization the dominent error term reduces to:

$$e_{\mathcal{M}}(a) = \sum_{i=1}^{n} h_i^2 \left| \frac{\partial^2 u}{\partial {\alpha_i}^2} \right| \quad ,$$

Directional mesh size optimization problem writes:

$$\min_{\mathcal{M}} \mathcal{E}(\mathcal{M}) \; = \; \min_{h_i} \left(\int_\Omega \left(\sum_{i=1}^{n} h_i^2(\mathbf{x}) \left| \frac{\partial^2 u}{\partial {\alpha_i}^2}(\mathbf{x}) \right| \right)^p \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{p}} \; ,$$

under the constraint:

$$\mathcal{C}(\mathcal{M}) = \int_\Omega \; \prod_{i=1}^{n} h_i^{-1}(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \int_\Omega d(\mathbf{x}) \, \mathrm{d}\mathbf{x} = N.$$

See in particular:
($^{+}$)Alauzet *et al.*, Leservoisier *et al.*, Long Chen *et al.*, Weizhan Huang.

1. OPTIMAL MESH FOR SMOOTH SOLUTIONS

Lp Optimal metric:

The metric $\mathcal{M}$, a $n \times n$ field on Ω, represents the mesh. After solving the optimality conditions:

$$\mathcal{M}_{\mathbf{L}^p} = D_{\mathbf{L}^p} \left(\det |H_u|\right)^{\frac{-1}{2p+n}} \mathcal{R}_u^{-1} \, |\Lambda_u| \, \mathcal{R}_u$$

with

$$D_{\mathbf{L}^p} = N^{\frac{2}{n}} \left(\int_\Omega \left| \prod_{i=1}^{n} \frac{\partial^2 u}{\partial {\alpha_i}^2} \right|^{\frac{p}{2p+n}} \right)^{-\frac{2}{n}} \quad , \quad \Lambda_u = \mathrm{Diag}\left(\frac{\partial^2 u}{\partial {\alpha_1}^2}, \frac{\partial^2 u}{\partial {\alpha_2}^2}, \frac{\partial^2 u}{\partial {\alpha_3}^2}\right).$$

The positive number $D_{\mathbf{L}^p}$ is a global normalization term to obtain a mesh with N vertices. The field $(\det |H_u|)^{\frac{-1}{2p+n}}$ is a local normalization term accounting for the sensitivity of the $\mathbf{L}^p$ norm. The three last matrices account for stretching control.

1. OPTIMAL MESH FOR SMOOTH SOLUTIONS

Mesh convergence order:

Error at optimum writes:

$$\mathcal{E}(\mathcal{M}_{\mathbf{L}^p}) = nN^{-\frac{2}{n}} \left(\int_\Omega \left| \prod_{i=1}^{n} \frac{\partial^2 u}{\partial \alpha_i^2} \right|^{\frac{p}{2p+n}} \right)^{\frac{2p+n}{pn}} .$$

As u is assumed to be twice continuously differentiable, the error committed with the optimal metric $\mathcal{M}_{\mathbf{L}^p}$ satisfies:

$$\mathcal{E}(\mathcal{M}_{\mathbf{L}^p}) \leq \frac{C(n,p,u)}{N^{2/n}}$$

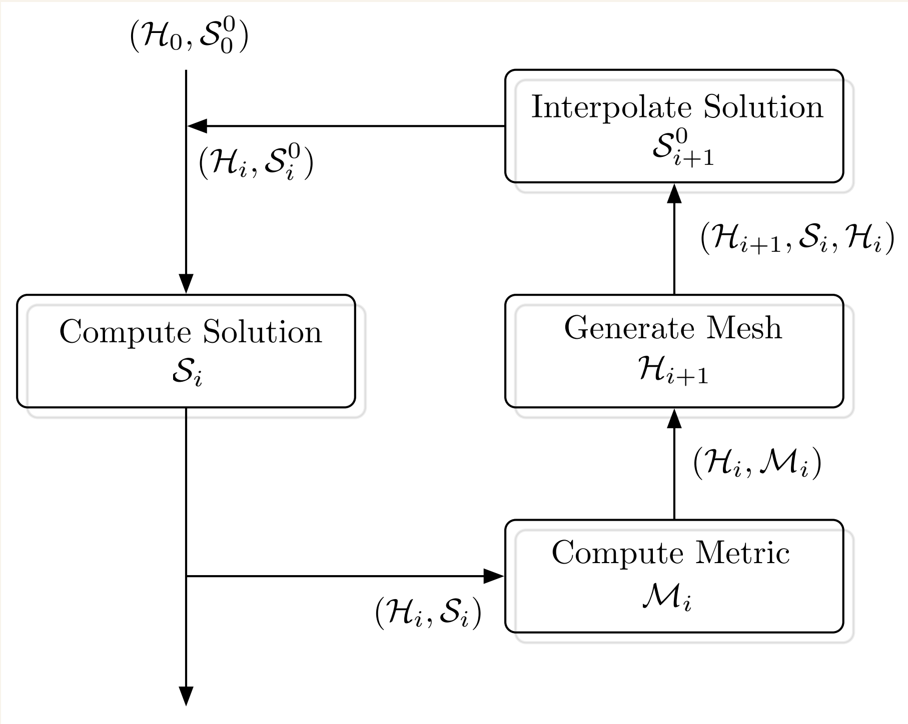
which expresses the second-order convergence.

The fact that $C(n,p,u)$ does not depend on N shows that optimal meshes classes are embedded.

N.B. All above extends to a weighted sum of L^p interpolation error!

1. OPTIMAL MESH FOR SMOOTH SOLUTIONS

Fixed point adaptation algorithm: Pure interpolation or adaptation for a discretised EDP. N is specified:



2. DISCONTINUOUS SOLUTIONS

1D example:

Interpolate optimally:

$$u(x) = u_{smooth}(x) + \sigma H(x - x_0) \ .$$

- Hessian is approximated on the iterated mesh.
- mesh size tends to zero on discontinuity.

Mesh becomes non-valid. The fixed point fails.

Now, an infinitely thin capture of the discontinuity is not necessary for accuracy, since it remains other errors on smooth region.

2. DISCONTINUOUS SOLUTIONS

1D model:

$$\int_0^1 |e_{\mathcal{M}}(x)|^p dx = \int_0^1 \left(m^2(x) g(N)(x)\right)^p dx.$$

where:

$$g(N)(x) = \delta(N)^{-2} |u(x+\delta(N)) - 2u(x) + u(x-\delta(N))| \ .$$

Model validity, smooth, unsmooth regions:

$$g(N)(x) \approx \left| \frac{\partial^2 u}{\partial x^2}(x) \right| \ .$$

$$m(x)^2 g(N)(x) \ \geq \ \left(\frac{m(x)}{\delta(N)}\right)^2 |\sigma| \ + \ R \ \geq \ |\sigma| \ + \ R$$

$$\int_{x_0}^{x_0+m} \left(m^2 g(N)\right)^p = \frac{m\sigma^p}{p+1} \left(\frac{m}{\delta}\right)^{2p} \geq \frac{m\sigma^p}{p+1}.$$

2. DISCONTINUOUS SOLUTIONS

Formal second-order accuracy: *Independantly of the size of $\delta(N)$ supposed sufficiently small, the optimally adapted interpolation converges to the discontinuous function u with "formal" second-order accuracy.*

$$m_{opt}(x) = \frac{1}{N} \left(\int_0^1 g(N)(y)^{\frac{p}{2p+1}}\, dy\right) g(N)(x)^{-\frac{p}{2p+1}}$$

$$\mathcal{E}_{opt} = \left(\int_0^1 (m_{opt}^2 g(N))^p\right)^{\frac{1}{p}} = \frac{2}{N^2} \ \left(\int g(N)^{\frac{p}{2p+1}}\right)^{\frac{2p+1}{p}} \ .$$

$$|g(N)| \le |\sigma| \delta(N)^{-2} \ .$$

$$|g(N)|^{\frac{p}{2p+1}} \le \ |\sigma|^{\frac{p}{2p+1}} \delta(N)^{-\frac{p}{2p+1}} \ \le |\sigma|^{\frac{p}{2p+1}} \delta(N)^{-1}$$

$$\Rightarrow \qquad \mathcal{E}_{opt} \le \frac{K}{N^2} \ .$$

2. DISCONTINUOUS SOLUTIONS

Case of a real mesh: The issue is to adequately choose $g(N)$. We observe that $\int_0^1 g(N)(x)^{\frac{p}{2p+1}}\, dx$ is uniformly bounded. In the discontinuity, $m_{opt}(x_0)$ is of order:

$$m_{opt}(x_0) = K\, N^{-1}\, g(N)^{-\frac{p}{2p+1}} + R = K\, N^{-1}\, \delta(N)^{\frac{2p}{p+1}} + R.$$

Writing $\delta(N)^{\frac{2p}{p+1}} = N^{\alpha}$, and imposing

$$m_{opt,\,2N} = \frac{1}{2^{2p}} m_{opt,\,N}$$

in the discontinuity then $\alpha = 1 - 2p$. Consequently, an upper bound for $\delta(N)$ is given by:

$$\bar{\delta}(N) = K'\, N^{\frac{1-4p^2}{2p}} \tag{1}$$

We get $m_{opt}(x_0) = K\, N^{-2p}$. Dividing N by 2 divides $m_{opt}(x_0)$ by 2^{2p} and:

2. DISCONTINUOUS SOLUTIONS

Real mesh second-order convergence: *Second-order convergence holds on discontinuity for a mesh specified by:*

$$\int_0^1 |e_{\mathcal{M}}(x)|^p dx = \int_0^1 \left(m^2(x) g(N)(x)\right)^p dx.$$

$$g(N)(x) = \delta(N)^{-2} |u(x + \delta(N)) - 2u(x) + u(x - \delta(N))| \ .$$

$$\bar{\delta}(N) = K' \, N^{\frac{1-4p^2}{2p}}$$

2. DISCONTINUOUS SOLUTIONS

Application to Case C3, numerical solution:

Minimal step size:

$$Min\,\Delta x \;=\; const.N^{-\frac{15}{4}}.(*)$$

Our strategy is to choose it for the finer adapted mesh to be applied and then to use (*) to fix the $Min\,\Delta x$ for the coarser ones in order to ensure second-order convergence. The same formula is applied to multi-dimensional cases.

3. 2D NUMERICAL EXPERIMENT

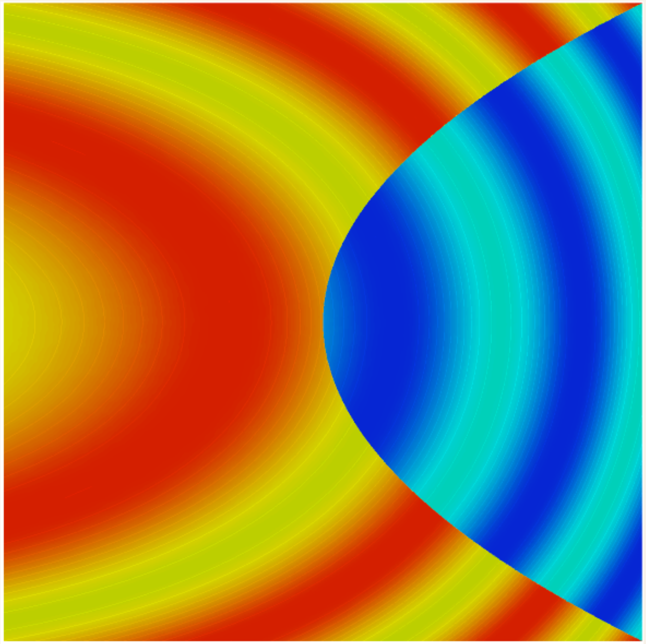
Function to interpolate:

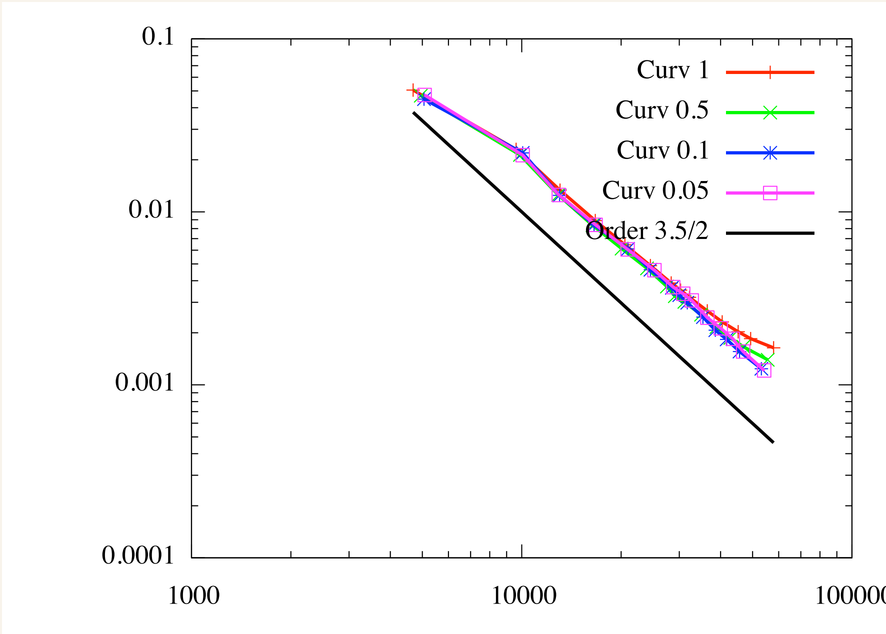
We adapt for the best interpolation of:

$$\begin{array}{ll} sin(2\pi(exp(x)+0.5+y^2)) & \text{if} \quad x \ > \ 0.5y^2 \\ sin(2\pi(exp(x)+0.5+y^2))+5 & \text{else.} \end{array}$$

3. 2D NUMERICAL EXPERIMENT

Results:





Convergence of adaptative P1 interpolation on a complex discontinuous function. Top, function under study, bottom, convergence in L^2 norm as a function of the number of nodes. Each curve correspond to a curvature of the discontinuity.

4. 3D FLOW PROBLEMS

Flow model:

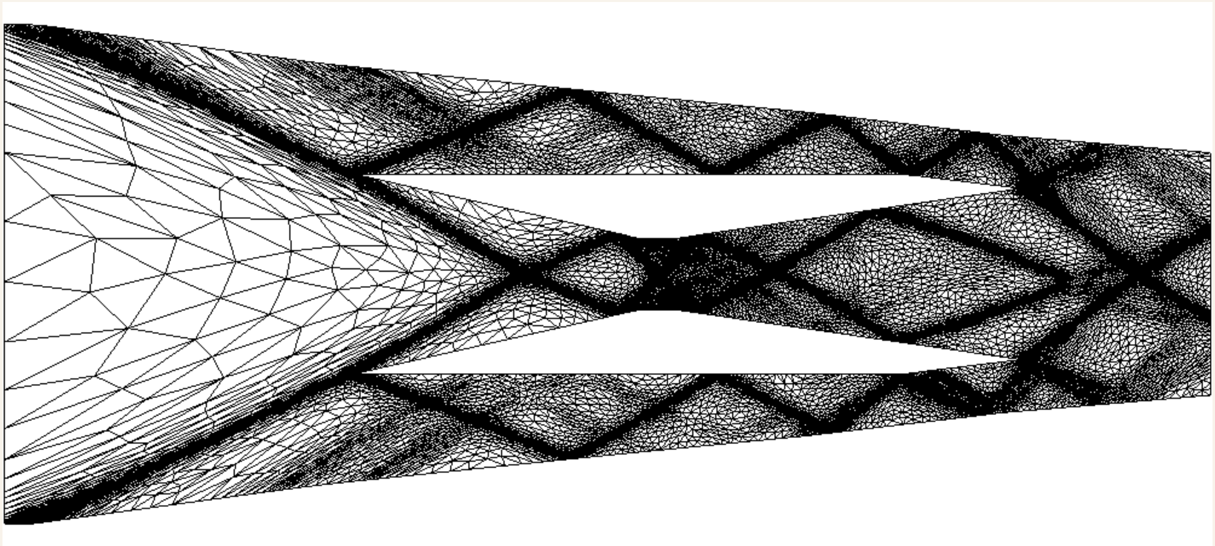
$$\frac{\partial W}{\partial t} + \nabla \cdot F(W) = 0\,,$$

where $W = {}^t(\rho, \rho u, \rho v, \rho w, \rho E)$
Mixed-Element-Volume approximation: values on vertices, P^1 interpolation.

4. 3D FLOW PROBLEMS

Scramjet internal flow(1):

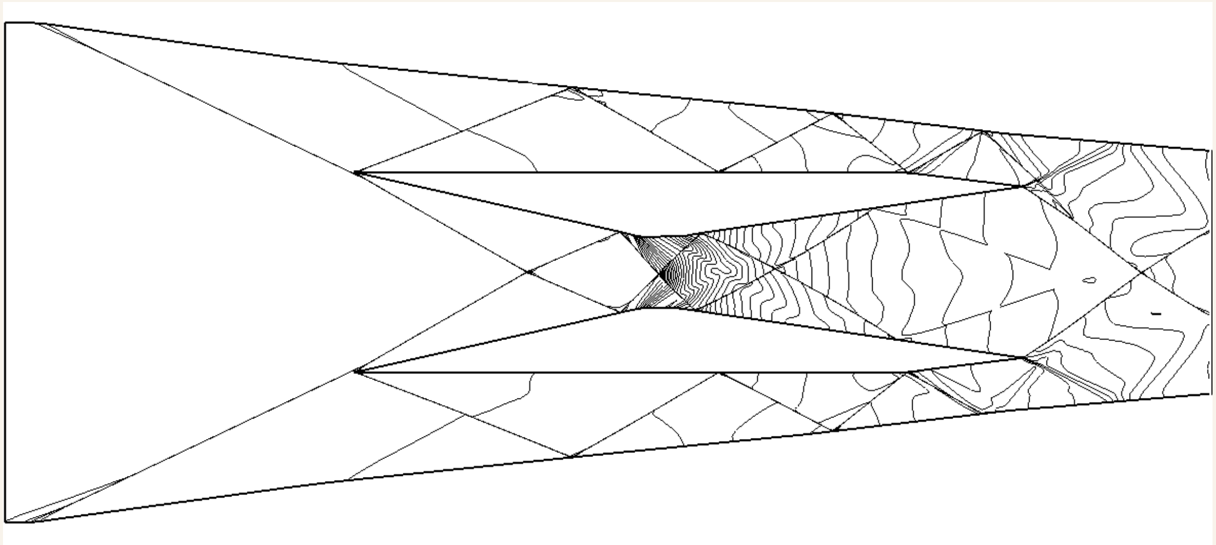
Final anisotropic mesh with $\mathbf{L}^1$ norm based mesh adaptation.



4. 3D FLOW PROBLEMS

Scramjet internal flow(2):

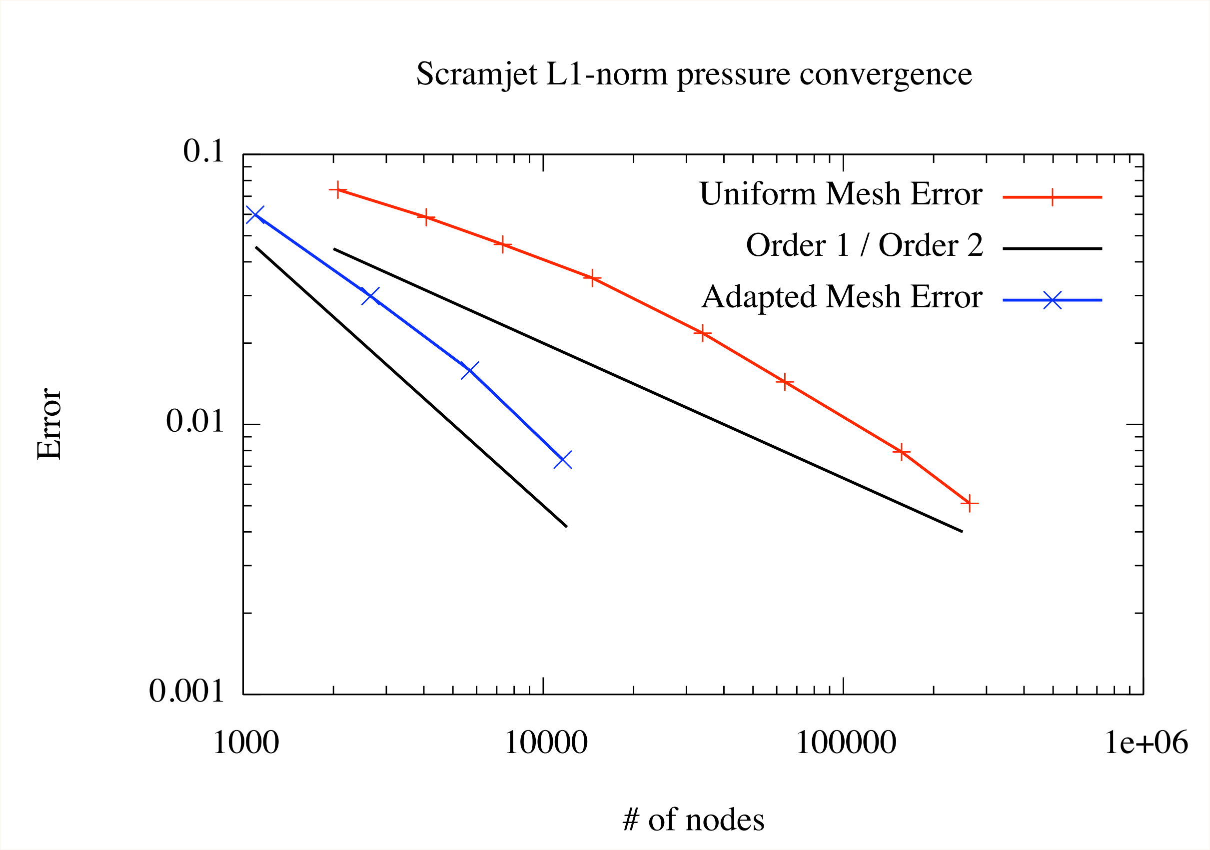
Final density iso-lines.



4. 3D FLOW PROBLEMS

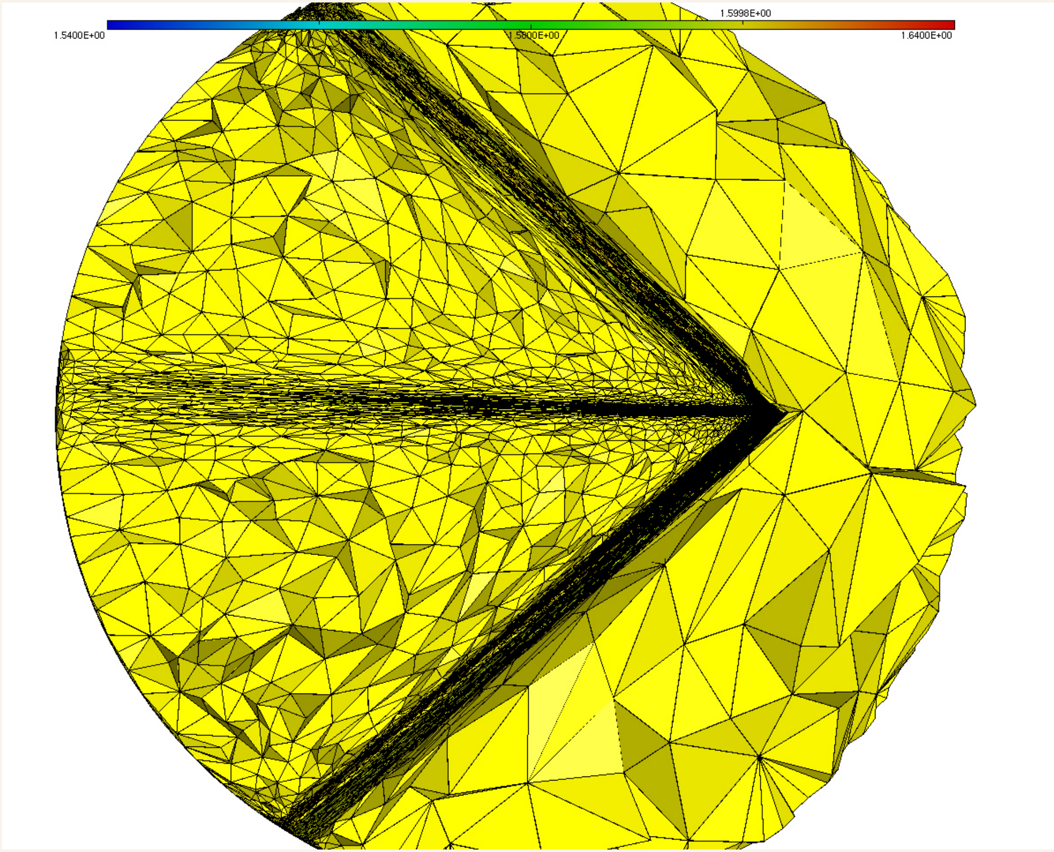
Scramjet internal flow(3):

Order of mesh convergence in $\mathbf{L}^1$ norm for the pressure.



4. 3D FLOW PROBLEMS

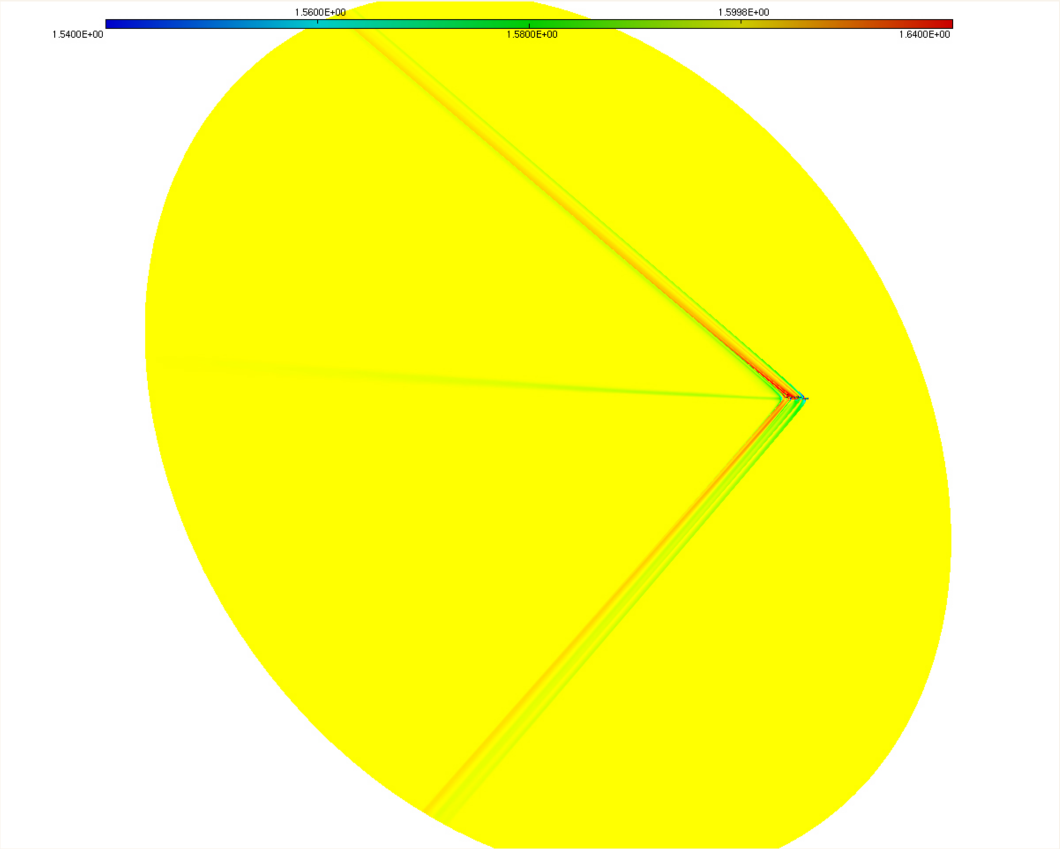
Flow around a supersonic aircraft(1):
Final anisotropic mesh with $\mathbf{L}^2$ norm adaptation.



4. 3D FLOW PROBLEMS

Flow around a supersonic aircraft(2):

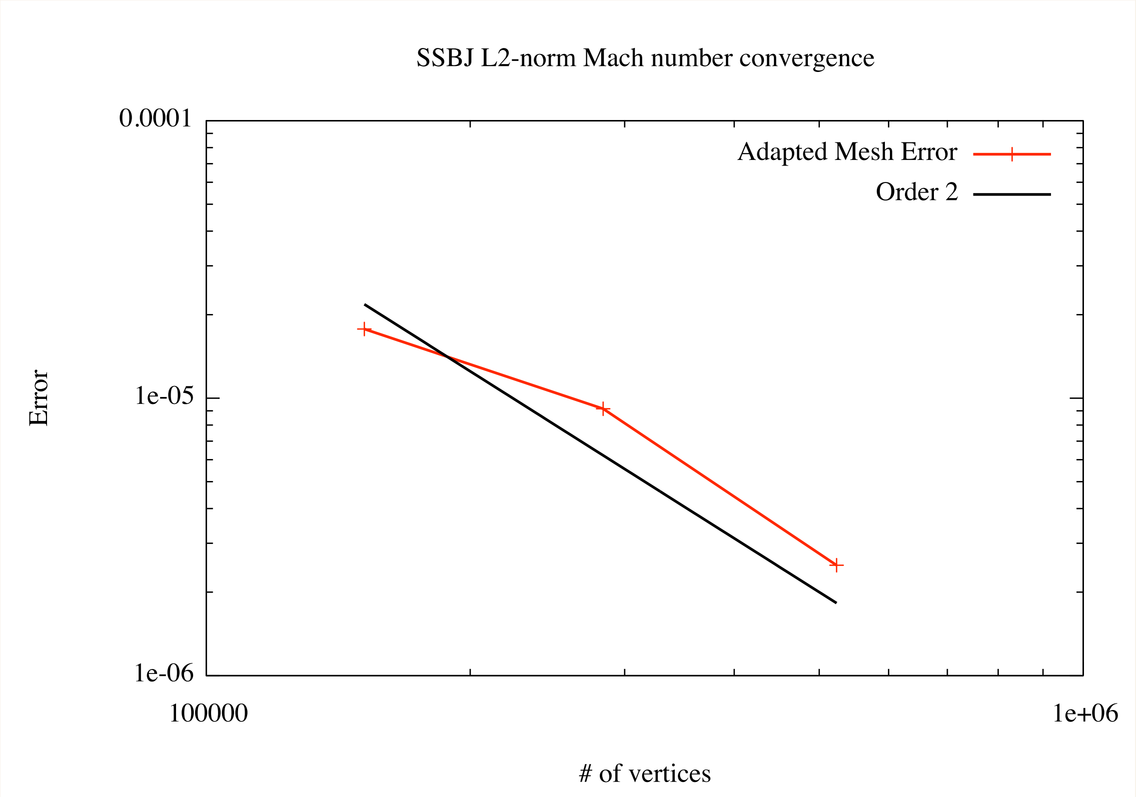
Final Mach number iso-value in the symmetry plane.



4. 3D FLOW PROBLEMS

Flow around a supersonic aircraft(3):

Order of mesh convergence in $\mathbf{L}^2$ norm for the Mach number.



5. CONCLUDING REMARKS

Many ingredients in a simulation tool can provoke a degradation of mesh convergence down to less than first order accuracy.

We have analysed a particular family: discontinuities, steadiness.

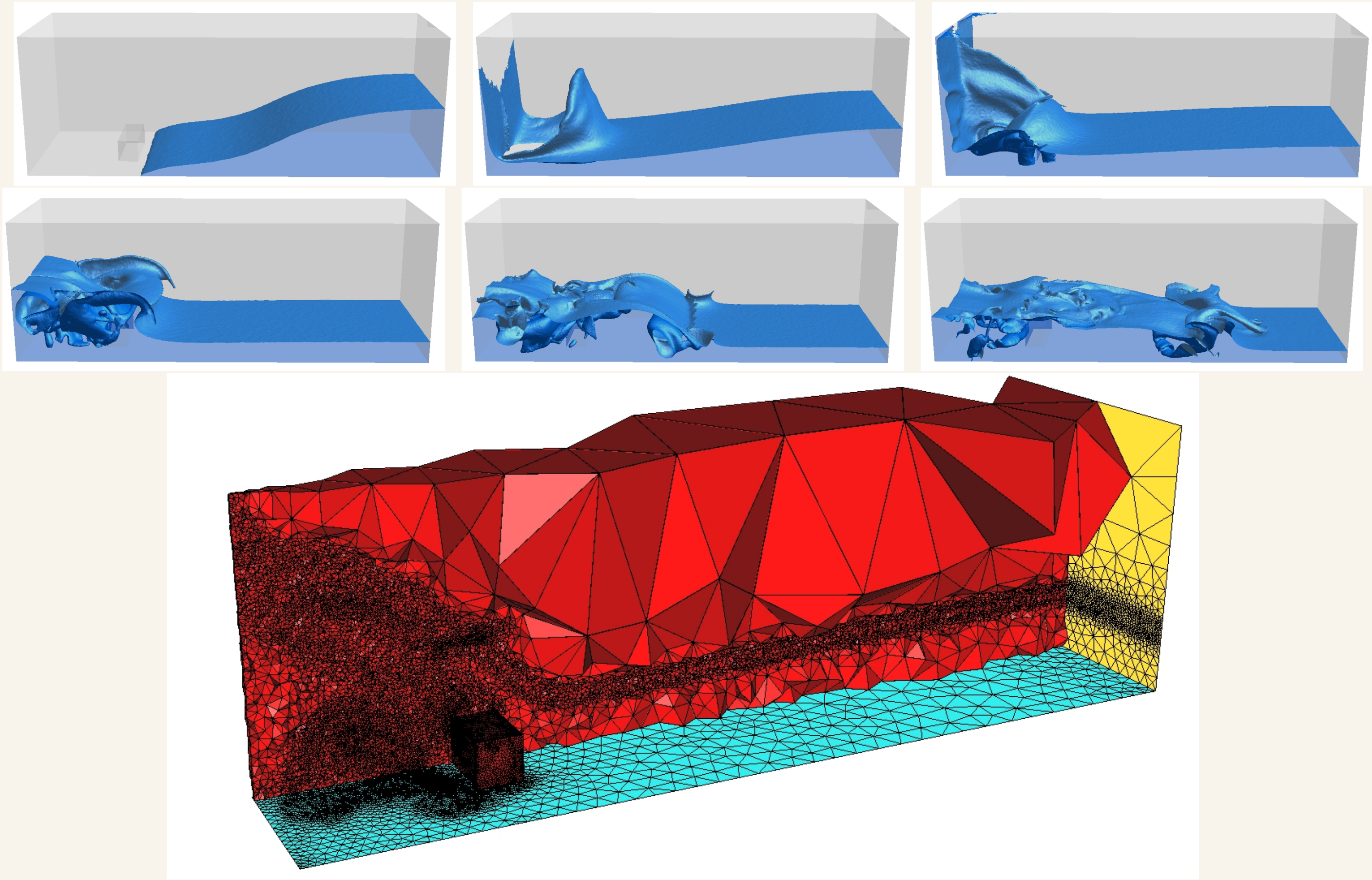
A: Initial best mesh setting is to obtain a minimal L^2 interpolation error.

B: In contrast with smooth case, mesh distribution and stratching is specified by a N-dependant metric.

C: Second order convergence is theoretically predicted and numerically obtained.

Current work concerns extension to unsteady flows with interfaces (with D. Guegan).

5. CONCLUDING REMARKS (end'd)



Thank you for your attention!

Retour plan