University of Zurich[UZH]
I·Math Institute of Mathematics

Schémas numériques et mécanique des fluides

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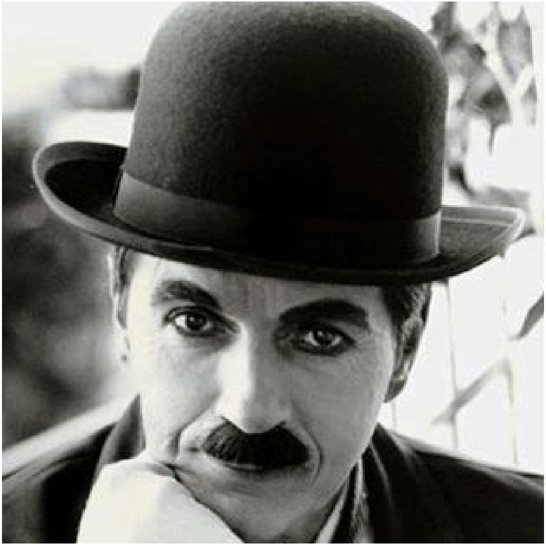
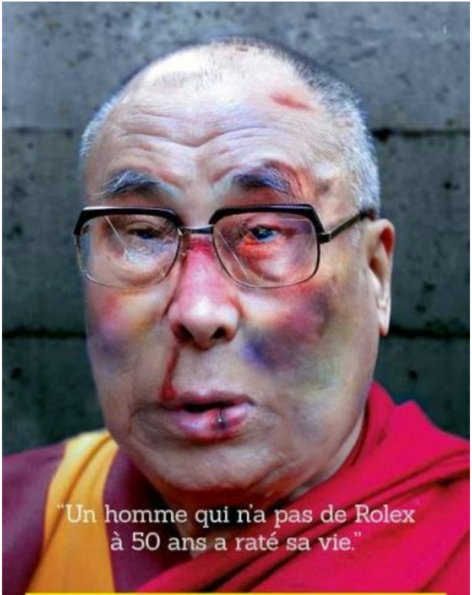
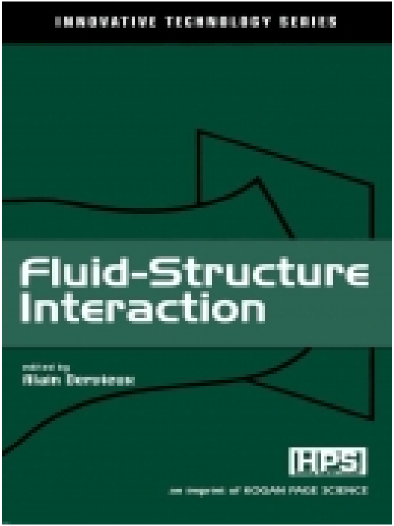
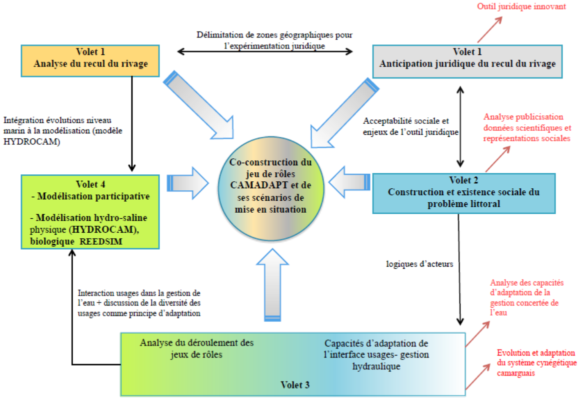
I-MATH
Universität Zürich

April 10, 2015



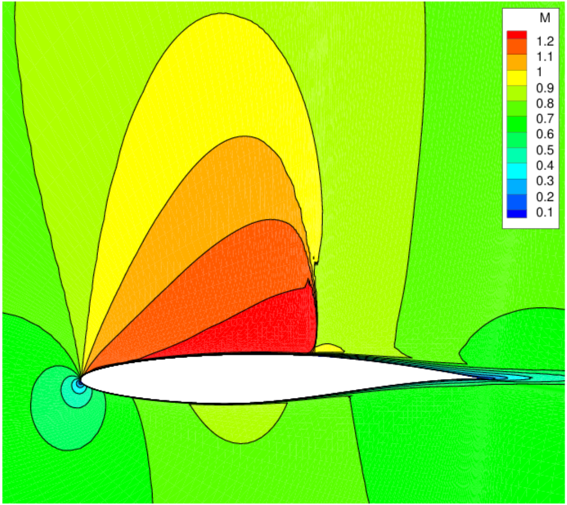

"Un homme qui n'a pas de Rolex
à 50 ans a raté sa vie."


INNOVATIVE TECHNOLOGY SERIES
Fluid-Structure Interaction
HPS


Outil juridique innovant
Délimitation de zones géographiques pour l'expérimentation juridique
Volet 1
Analyse du recul du rivage
Volet 1
Anticipation juridique du recul du rivage
Intégration évolutions niveau marin à la modélisation (modèle HYDROCAM)
Acceptabilité sociale et enjeux de l'outil juridique
Analyse publicisation données scientifiques et représentations sociales
Co-construction du jeu de rôles CAMADAPT et de ses scénarios de mise en situation
Volet 4
- Modélisation participative
- Modélisation hydro-saline physique (HYDROCAM), biologique REEDSIM
Volet 2
Construction et existence sociale du problème littoral
logiques d'acteurs
Interaction usages dans la gestion de l'eau + discussion de la diversité des usages comme principe d'adaptation
Analyse des capacités d'adaptation de la gestion concertée de l'eau
Analyse du déroulement des jeux de rôles
Capacités d'adaptation de l'interface usages- gestion hydraulique
Volet 3
Evolution et adaptation du système cynégétique camarguais

Merci Alain !

RAE2822 AIRFOIL, TURBULENT
M=0.734, RE=6.5 10^6, AOA=2.79°



Goals

$$\frac{\partial W}{\partial t} + \text{div}\,\big(F(W) - F_v(W, \nabla W)\big) = S(\mathbf{x}, W)$$

- Emphasis on the structure of the operators: multidimensional, symetries, structure of differential operators, equilibriums, etc: role of conservation in the large
- Easy implementation:
  very local structures (geometrical+memory-wise) $\longrightarrow$ compact stencil
- Truly high order
- Stable and parameter free, including for strong shocks

GOALS

$$\frac{\partial W}{\partial t} + \text{ div } \big(F(W) - F_v(W, \nabla W)\big) - S(\mathbf{x}, W) = 0$$

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Collaborative work

- INRIA: M. Ricchiuto, M. Mezine, A. Larat, D. de Santis, A. Froehly, L. Nouveau, P. Jacq, etc
- U. Bordeaux: K. Mer-Nkonga, H. Beaugendre
- VKI: H. Deconinck,
- U. Michigan: P.L. Roe
- Curved meshes: C. Dobrzynski (U. Bordeaux), A. Froehly (Inria)

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Outline

Formulations: conservation and accuracy issues

Residual distribution framework

Application to steady turbulent flow problems

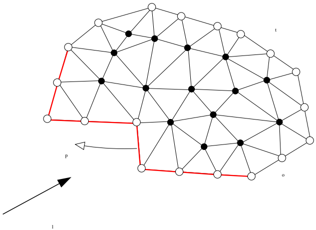
Extensions: shallow water

Conclusion

OVERVIEW

Model problem, Framework for steady scalar conservation laws.

$$\begin{aligned} \operatorname{div} f(u) &= 0 \qquad \text{in} \quad \Omega \\ u &= g \qquad \text{on} \quad \Gamma^- \end{aligned}$$



Some notations...

- Consider $\mathcal{T}_h$ triangulation of Ω (can do with quads...)
- Unknowns (Degrees of Freedom, DoF) : $u_i \approx u(M_i)$
- $M_i \in \mathcal{T}_h$ a given set of nodes (vertices +other dofs)
- Denote by u_h continuous piecewise approximation (for example P^k Lagrange triangles/quads)
- $u_h = \sum \varphi_i \, u_i$

Variational formulation of convected dominated problems,1

Continuous finite elements: Galerkin+stabilisation.: Choose $V^h = U^h = \bigoplus \{u^h_{|K} \in \mathbb{P}^k(K)$ and globally continuous$\}$

Streamline diffusion (Hughes et al.)

$$\sum_K \Big(-\int_K \nabla v^h \cdot f(u^h)dx + \int_{\partial K} v^h f(u^h)\cdot \mathbf{n} \\ + h_K \int_K (\nabla f_u(u^h)\cdot \nabla v^h) \mathcal{T} (\nabla f_u(u^h)\cdot \nabla u^h) dx \Big) = 0, \quad \mathcal{T} \geq 0.$$

Jump operator (Burman et al.)

$$\sum_K \Big(-\int_K \nabla v^h \cdot f(u^h)dx + \int_{\partial K} v^h f(u^h)\cdot \mathbf{n} \Big) \\ + \sum_{edges} \Gamma h_e^2 \int_e ||\nabla_u f(u^h)|| [\nabla u^h][\nabla v^h] = 0, \qquad \Gamma \geq 0$$

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REMARK

for all degree of freedom, $\sum_{K\ni} \Phi_i^K = 0$

SUPG

$$\sum_{i\in K}\Bigg(\overbrace{-\int_K \nabla\varphi_i\cdot f(u^h)d\mathbf{x}+\int_{\partial K}\varphi_i f(u^h)\cdot\mathbf{n}+h_K\int_K\left(\nabla f_u(u^h)\cdot\nabla\varphi_i\right)\mathcal{T}\left(\nabla f_u(u^h)\cdot\nabla u^h\right.}^{\Phi_i^K}$$

$$=\int_{\partial K} f(u^h)\cdot\mathbf{n}$$

Jump stabilisation

$$\sum_{i\in K}\Bigg(\overbrace{-\int_K \nabla\varphi_i\cdot f(u^h)d\mathbf{x}+\int_{\partial K}\varphi_i f(u^h)\cdot\mathbf{n}+\sum_{edges\subset K}\Gamma h_e^2\int_e\left|\left|\nabla_u f(u^h)\right|\right|[\nabla u^h][\nabla\varphi_i]}^{\Phi_i^K}\Bigg)$$

$$=\int_{\partial K} f(u^h)\cdot\mathbf{n}$$

because $\sum_{i\in K}\varphi_i = 1 \longrightarrow \sum_{i\in K}\nabla\varphi_i = 0$

Variational formulation of convected dominated problems,2

Discontinuous finite elements: Stabilisation via the jumps accross edges $V^h = U^h = \bigoplus \{u^h_{|K} \in \mathbb{P}^k(K)\}$

$$\sum_K \left(- \int_K \nabla v^h \cdot f(u^h) dx + \int_{\partial K} \hat{f}(u^h_+, u^h_-, \mathbf{n}) v^h dl \right) = 0$$

Choice of numerical flux $\hat{f}$: E-scheme implies entropy stability.

VARIATIONAL FORMULATION OF CONVECTED DOMINATED PROBLEMS,2

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DG

$$\sum_{i\in K}\left(\overbrace{-\int_K \nabla\varphi_i \cdot f(u^h)dx + \int_{\partial K} \hat{f}(u_+^h, u_-^h, \mathbf{n})\varphi_i dl}^{\Phi_i^K}\right)$$
$$= \int_{\partial K} \hat{f}(u_+^h, u_-^h, \mathbf{n})dl$$

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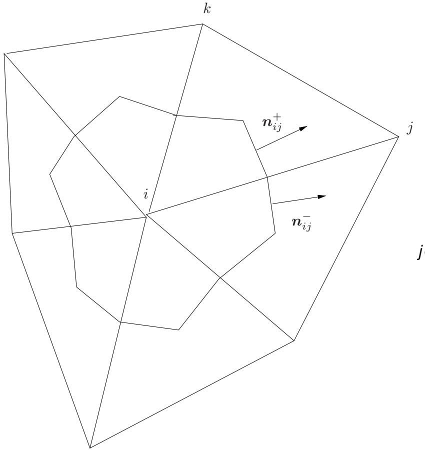
Case of finite volume formulation

$$\operatorname{div} \mathbf{f}(u) = 0$$



$$\sum_{j \in V(i)} \left[\hat{f}(u_i, u_j, \mathbf{n}_{ij}^{+}) + \hat{f}(u_i, u_j, \mathbf{n}_{ij}^{-}) \right] = 0$$

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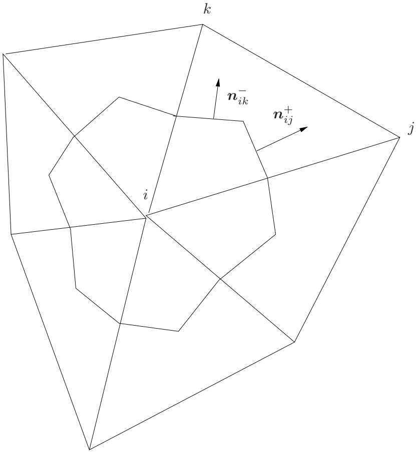

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CASE OF FINITE VOLUME FORMULATION

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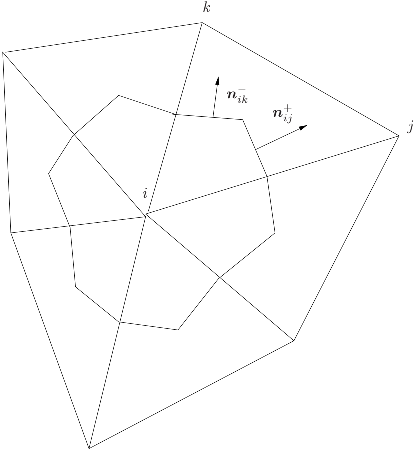

$$\sum_{K \ni i} \underbrace{\left[\hat{f}(u_i, u_j, \mathbf{n}_{ij}^+) + \hat{f}(u_i, u_k, \mathbf{n}_{ik}^-) - \mathbf{f}(u_i) \cdot \left(\mathbf{n}_{ij}^+ + \mathbf{n}_{ik}^- \right) \right]}_{\Phi_i^k} = 0$$

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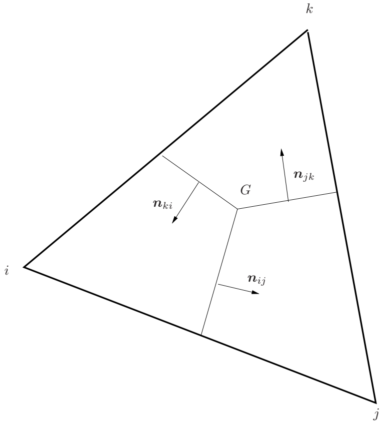

Again, we have

$$\sum_{K \ni i} \Phi_i^K = 0$$

with

$$\begin{aligned}
\Phi_i := {} & \hat{f}(u_i, u_j, \mathbf{n}_{ij}^+) + \hat{f}(u_i, u_k, \mathbf{n}_{ik}^-) \\
& - \mathbf{f}(u_i) \cdot \left(\mathbf{n}_{ij}^+ + \mathbf{n}_{ik}^-\right) \\
= {} & \hat{f}(u_i, u_j, \mathbf{n}_{ij}^+) + \hat{f}(u_i, u_k, \mathbf{n}_{ik}^-) \\
& - \mathbf{f}(u_i) \cdot \frac{\mathbf{n}_i}{2}
\end{aligned}$$

Case of finite volume formulation

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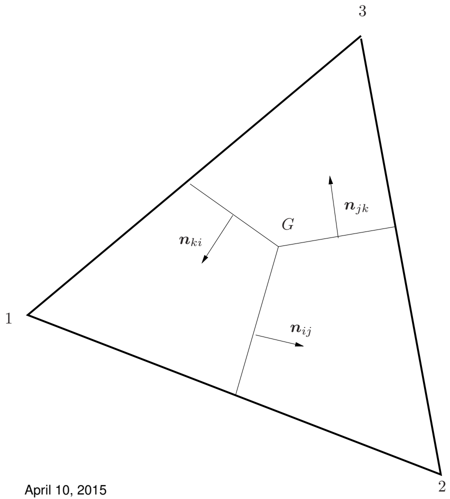

$$\Phi_i := \hat{f}(u_i, u_j, \mathbf{n}_{ij}^+) + \hat{f}(u_i, u_k, \mathbf{n}_{ik}^-) - \mathbf{f}(u_i) \cdot \frac{\mathbf{n}_i}{2}$$

$$\sum_{i \in K} \mathbf{\Phi}_i = -\sum_i \mathbf{f}(u_i) \cdot \frac{\mathbf{n}_i}{2} = \int_{\partial K} \mathbf{f}^h \cdot \mathbf{n}$$

PARTIAL CONCLUSION:

We can rephrase many/all known schemes as:

$$\sum_{K \ni i} \Phi_i^K(u^h) = 0$$

where

- The $\Phi_i^K(u^h)$ are residuals, i.e. basicaly difference of fluxes,
- They all satisfy a conservation relation:

$$\sum_{i \in K} \Phi_i^K(u^h) = \int_{\partial K} \hat{f}(u^+, u^-, \mathbf{n})$$

$\hat{f}$ numerical flux, take into account continuous/discontinuous element

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Can we exploit this to design schemes: yes !

FURTHER REMARKS ON THE TRUNCATION ERROR

Consider for example the residuals of SUPG for Steady problem

- $\text{div}\,\mathbf{f}(u^{ex}) = 0$ and assume u^{ex} smooth enough.
- Call u^h some interpolant of u^{ex}, $u^h - u^{ex} = O(h_K^{r+1})$, $u^h \in \mathbf{\Phi}^r(K)$
- Denote $\delta\mathbf{f} = \mathbf{f}(u^h) - \mathbf{f}(u^{ex})$

$$
\begin{aligned}
\Phi_i^K(u^h) = & -\int_K \nabla\varphi_i \cdot \mathbf{f}(u^h) d\mathbf{x} + \int_{\partial K} \varphi_i \mathbf{f}(u^h) \\
& h_K \int_K \left(\nabla f_u(u^h) \cdot \nabla\varphi_i\right) \mathcal{T} \left(\nabla f_u(u^h) \cdot \nabla u^h\right) d\mathbf{x} \\
= & -\int_K \nabla\varphi_i \cdot \delta\mathbf{f} d\mathbf{x} + \int_{\partial K} \varphi_i \delta\mathbf{f} \\
& h_K \int_K \left(\nabla f_u(u^h) \cdot \nabla\varphi_i\right) \mathcal{T} \left(\text{div}\delta\mathbf{f}\right) d\mathbf{x} \\
= & \; O(h^{k+1+d-1}) + O(h^{k+1+d-1}) + O(h^{1+d-1+k} = O(h^{k+d})
\end{aligned}
$$

FURTHER REMARKS ON THE TRUNCATION ERROR

ASSUMPTIONS AND FACTS:

Under steady problem+ smooth solution approximated in $\mathbf{\Phi}^k(K)$

$$\Phi_i^K(u^h) = O(k^{k+d}).$$

- Same true for stabilisation with jumps
- Same true for DG (thanks to flux consistency) BUT violated when limiting: extrema problem.
- Wrong in general for FV, or difficult to achieve (very large stencils)
- Can be shown as the basis of a systematic truncation error analysis.

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NEXT STEP

Dual situation

- Combine the conservation relation

$$\sum_{i \in K} \Phi_i^K(u^h) = 0$$

- and the residual property

$$\Phi_i^K(u^h) = O(k^{k+d}).$$

to construct compact, stable, accurate, non oscillatory schemes: Residual distribution scheme

Overview

Residual distribution schemes

History

- Ni scheme, 1981. Engineer at Bombardier
- Roe, 1981($-x$)-today
- Deconinck, Ricchiuto, Nishikawa, Caraeni, ...
- Strong connections with stabilized FEM methods for convection-diffusion problems.
- ...

Aims

- Combine ideas from finite volume schemes (non oscillatory, L^∞ stability, upwinding
  with finite element methods
- Simple implementation: no fancy limiters, no Riemann solvers, compact stencil, no tunable parameters
- Try something else than DG and high order finite volume

RESIDUAL DISTRIBUTION SCHEMES

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MODEL EQUATION: SCALAR STEADY CONVECTION-DIFFUSION

$$\operatorname{div} \mathbf{f}(u) - \operatorname{div}\left(\mathbb{K}\nabla u\right) = 0 \qquad \text{on } \Omega \subset \mathbb{R}^d \text{ boundary conditions on } \partial\Omega$$

- $\mathbf{f}(u) = \left(f_1(u), \cdots, f_d(u)\right)$, f_i smooth enough.
- Boundary conditions: Dirichlet or inflow/outflow depending on $\mathbb{K}$
- Scalar problem

Model equation: scalar steady convection-diffusion

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Analysis done for non viscous problems first

Model problem, Framework for scalar conservation laws.

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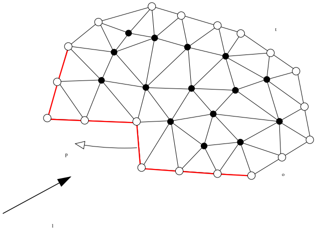

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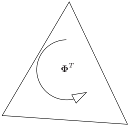
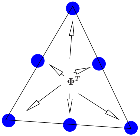
- $u_h = \sum_i \varphi_i \, u_i$

PRINCIPLE FOR HIGHER ORDER

1. $\forall K \in \mathcal{T}_h$ compute : $\Phi^K = \int_{\partial K} f_h(u_h) \cdot \mathbf{n}$

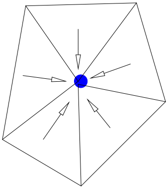


2. Distribution : $\Phi^K(u^h) = \sum_{i \in K} \Phi_i^K$



   Distribution coeff.s : $\Phi_i^K(u^h) =$ sub-residuals

3. Compute nodal values : solve algebraic system



   $\forall$ degree of freedom i $\sum_{K|i \in K} \Phi_i^K(u^h) = 0,$

Principle for higher order

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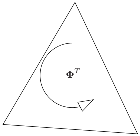

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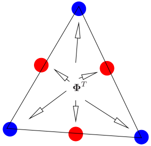

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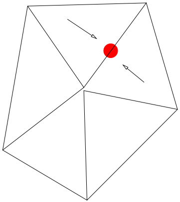
   $\forall$ degree of freedom i $\qquad \sum_{K|i \in K} \Phi_i^K(u^h) = 0,$



Solved by some iterative technique.

DESIGN PROPERTIES

STRUCTURAL CONDITIONS, BASIC PROPERTIES

Under which conditions on the Φ_i^Ks we get

- Correct weak solutions (if convergent with h)
- Formal k^{th} order of accuracy
- Monotonicity (discrete max principle)
- Convergence

Design properties

Structural conditions, basic properties

Under which conditions on the Φ_i^Ks we get

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- Formal k^{th} order of accuracy
- Monotonicity (discrete max principle)
- Convergence

Notation: DOF: σ_i or M_i or simply i

CONDITION 1 : CONSERVATION

CONSERVATION PRINCIPLE

If there is a f_h, continuous approximation of f such that

$$\Phi^K = \sum_{j \in K} \Phi_j^K = \oint_{\partial K} \mathbf{f}(u^h) \cdot \mathbf{n}$$

Implies convergence to a (weak) solution of the problem

$$\text{div } \mathbf{f}(u) = 0$$

under standard stability conditions

Condition 2 : accuracy.

$u^{ex,h}$ interpolant of exact sol. assumed smooth

Truncation error

$$\mathcal{E}(u^{ex,h}; v) := \sum_{i\in\mathcal{T}_h} v_i \Big(\sum_{K|\, i\in K} \Phi_i^K(u^{ex,h}) \Big)$$

Guiding principle

$$\mathcal{E}(u^{ex,h}; v) = \overbrace{\int_\Omega \nabla v_h \cdot f_h(u^{ex,h})}^{I \equiv \mathcal{E}^{\text{Galerkin}}} + \overbrace{\sum_{K\in\mathcal{T}_h} \frac{1}{N_K} \sum_{i,j\in K} (v_i - v_j)(\Phi_i^K - \Phi_i^{\text{Gal}})(u^{ex,h})}^{II}$$

$$\Phi_i^{Gal,K} = \int_K \Phi_i \text{div } f(u^h) dx = -\int_K \nabla\Phi_i \cdot \mathbf{f}(u^h) dx + \int_{\partial K} \Phi_i \mathbf{f}(u^h) \cdot \mathbf{n} d\sigma$$

CONDITION 2 : ACCURACY.

KEY REMARK

$\text{div } \mathbf{f}(w) = 0 \Longrightarrow \Phi_i^{Gal,K}(u^{ex,h}) =$
$\int_T \nabla\psi_i \cdot \mathbf{f}_h(u^{ex,h}) dx - \int_{\partial K} \Phi_i \mathbf{f}_h(u^{ex,h}) \cdot \mathbf{n} = O(h^{k+d})$

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FINAL RESULT

Truncation error : $|\mathcal{E}(u^{ex,h}; v)| \leq C'(\mathcal{T}_h, u^{ex})\|\nabla v\|_\infty \, h^{k+1}$

if (in d-D) $\quad |\Phi_i^K(u^{ex,h})| \leq C''(\mathcal{T}_h, u^{ex})h^{k+d} = \mathcal{O}(h^{k+d})$

Condition 2 : accuracy

"Linearity" (Accuracy) preserving schemes

Since $\Phi^K(u^h) = \int_{\partial K} f^h(u^h) \cdot \mathbf{n} = \mathcal{O}(h^{k+d})$ schemes for which

$\Phi_i^K = \beta_i^K \Phi^K$ with β_i^K uniformly bounded distribution coeff.s

are formally $k+1^{\text{th}}$ order accurate (for $k+1^{\text{th}}$ order spatial interpolation)

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HOWEVER: GODUNOV'S THEOREM

The β_i^K must depend on the solution : *A scheme cannot be both high order accurate and linear for a linear problem.*

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However: Godunov's theorem

The β_i^K must depend on the solution : *A scheme cannot be both high order accurate and linear for a linear problem.*

Solution method

– Start from a monotone scheme

– Use a 'limiter' that produce a set of residuals that enables the residual property for any elements and any degree of freedom

ONE EXAMPLE OF MONOTONE SCHEME: THE RUSANOV SCHEME (LOCAL LAX FRIEDRICHS)

First order distribution :

$$\Phi_i^{\mathrm{Rv}} = \frac{\Phi^K}{n_K} + \frac{\alpha}{n_K} \sum_{\substack{j \in K \\ j \neq i}} (u_i - u_j), \quad \alpha \geq \max_{j \in K} \left| \int_K \nabla_u f(u^h) \cdot \nabla \psi_j \right|$$

- n_K number of DoF per element
- φ_j Lagrange basis fcn. relative to node j

WHY THIS SCHEME ?

1. The Rv scheme is cheap and has general formulation
2. The Rv scheme is monotone and energy stable in the P^1 case.
3. By far one of the most dissipative ones

One example of monotone scheme: the Rusanov scheme (Local Lax Friedrichs)

Choice of Rusanov : not essential at all !
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Solution method
Summary

- For any K
- Start from Rusanov' residuals,
- Use Struijs' limiter

$$\beta_i^H = \frac{\max(0, \Phi_i^{Rv}/\Phi^K)}{\sum\limits_{j\in K} \max(0, \Phi_j^{Rv}/\Phi^K)}$$

- Define: $\Phi_i^H = \beta_i^H \Phi^K$.
- $\theta = 0$ is u_i is a local extrema, $u_i = 1$ else
- This scheme satisfies a local maximum/minimum property and is formaly k-th order accurate

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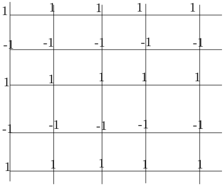
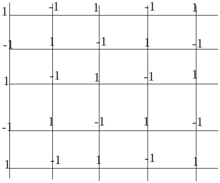
Motivation for this term

Solve $\dfrac{\partial u}{\partial x} = 0$ on $[0,1]^2$





- In both cases, $\Phi^K = 0$: these are steady solutions when $\Phi_i^H = \beta_i^K \Phi^K$.
- Cure :

$$\Phi_i^{H,K} = \beta_i^K \Phi^K \longrightarrow \beta_i^K \Phi^K + \theta(u^h) \times h_K \int_K (\nabla f_u(u^h) \cdot \nabla \varphi_i) \tau (\nabla f_u(u^h) \cdot \nabla u^h) dx$$

MOTIVATION FOR THIS TERM

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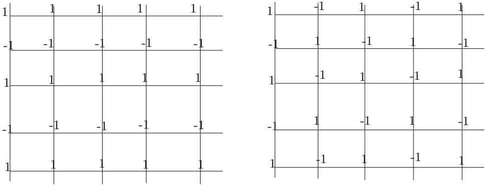

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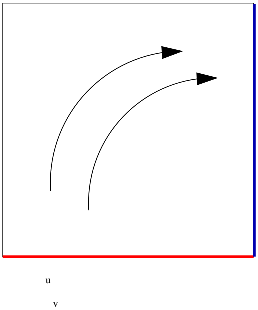
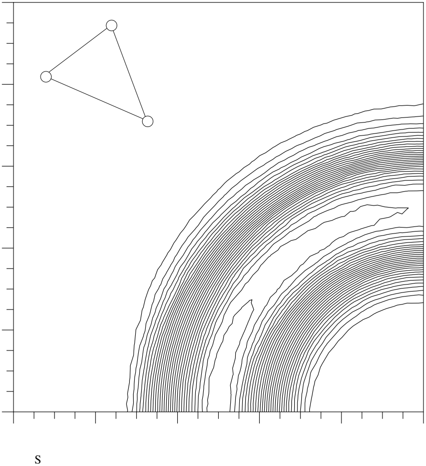
NUMERICAL EXAMPLE : ROTATION





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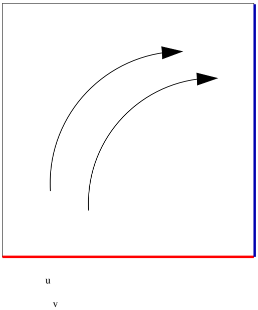
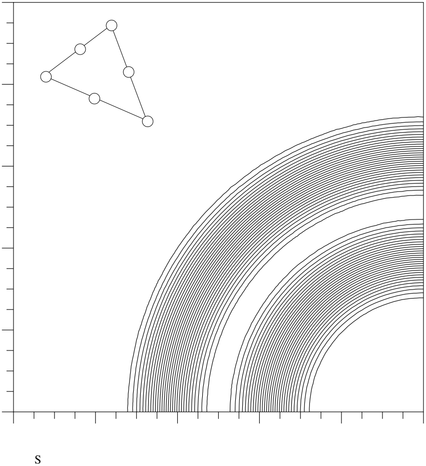
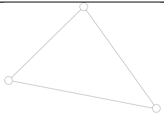
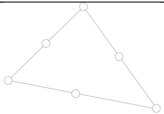
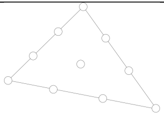


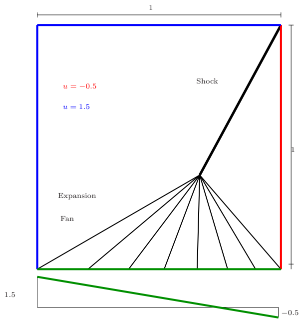


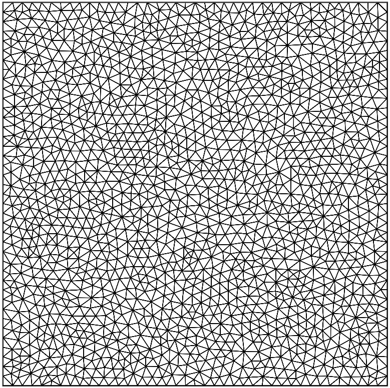
Grid convergence

			
h	$\epsilon_{L^2}(P^1)$	$\epsilon_{L^2}(P^2)$	$\epsilon_{L^2}(P^3)$
1/25	0.50493E-02	0.32612E-04	0.12071E-05
1/50	0.14684E-02	0.48741E-05	0.90642E-07
1/75	0.74684E-03	0.13334E-05	0.16245E-07
1/100	0.41019E-03	0.66019E-06	0.53860E-08
	$\mathcal{O}^{\mathbf{ls}}_{L^2} = \mathbf{1.790}$	$\mathcal{O}^{\mathbf{ls}}_{L^2} = \mathbf{2.848}$	$\mathcal{O}^{\mathbf{ls}}_{L^2} = \mathbf{3.920}$

NUMERICAL EXAMPLE : BURGER'S EQ.N

$$\nabla \cdot \left(\frac{u^2}{2}, u\right) = 0$$





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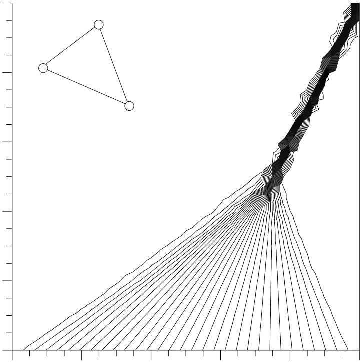
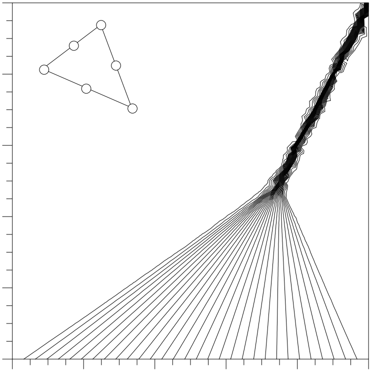


LxF+PSI+Filter scheme, P^1 interpolation



LxF+PSI+filter scheme, P^2 interpolation

Shock captured in 1 or 2 cells

ALGORITHM

The scheme consists in 4 steps :

1. Evaluate the total residual, local (continuous interpolant)
2. Evaluate monotone residual (Rusanov) : local,
3. Evaluate high order residual : local
4. Gather residual : indirections, importance of good numering of the degrees of freedom

The scheme is local and easy to parallelise

SOLUTION METHOD

Jacobian free + LUSGS-ILU

RD WITH VISCOUS TERMS: WHAT ARE THE PROBLEMS?

$$\text{div } \mathbf{f}^a(\mathrm{u}) - \text{ div } (\mathbb{K}(\mathrm{u}).\nabla \mathrm{u}) = \text{ div } \big(\mathbf{f}^a(\mathrm{u}) - \mathbb{K}(\mathrm{u}).\nabla \mathrm{u}\big) = 0$$

- Accuracy: coupling of convection and diffusion: one single operator
- Total residual:

$$\Phi^K(u^h) = \int_{\partial K} (\mathbf{f}^a(\mathrm{u}) - \mathbb{K}(\mathrm{u}^h)\nabla \mathrm{u}^h) \cdot \mathbf{n} d\partial K.$$

- Major issue: $\nabla \mathrm{u}^h$ not single valued on edges.
- Reconstruct the gradients using out-of-element information, and keeping the compacness of the stencil
- experimental fact: ∇u^h should be reconstructed with the same accuracy as u^h...

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RD with viscous terms: what are the problems?

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GRADIENT RECOVERY

IDEA

- Obtained from super-convergent patch recovery introduced by O. C. Zienkiewicz and J. Z. Zhu, *Int. J. Numer. Meth. Eng., 33, 1992*: Use of superconvergent points: $\nabla u^h(x_q) - \nabla u(x_q) = O(h^{k+1})$ instead of $O(h^k)$ as it should be
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LINEAR ADVECTION-DIFFUSION EQUATION

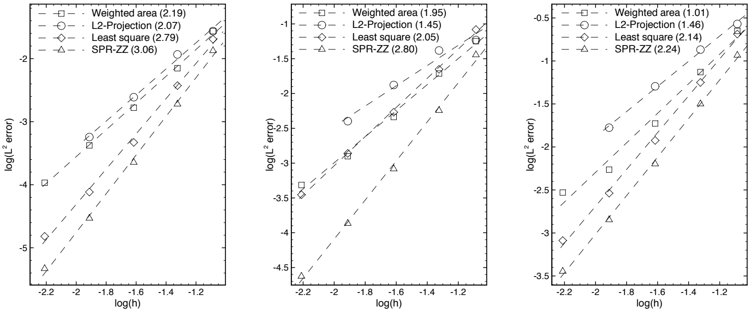
$$\boldsymbol{a}\cdot\nabla\boldsymbol{u} = \nu \operatorname{div}\left(\nabla\boldsymbol{u}\right), \quad \text{on} \quad \Omega = [0,1]^2,$$

the exact solution of the problem reads

$$u = -\cos(2\pi\eta)\exp\left(\frac{\xi\left(1-\sqrt{1+16\pi^2\nu^2}\right)}{2\nu}\right),$$

with $\eta = a_y x - a_x y$ and $\xi = a_x x + a_y y$. Here $\boldsymbol{a} = (0,1)^{\mathrm{K}}$ and $\nu = 0.01$

LINEAR ADVECTION-DIFFUSION EQUATION



L^2 error in the solution of the linear advection-diffusion problem on triangular girds with quadratic elements. Error of the solution (first column), error of the x-component of the gradient (second column) error of the y-component of the gradient (third column).

ANISOTROPIC PURE DIFFUSION

$$-\mathrm{div}\,\left(\mathbb{K}\nabla \boldsymbol{u}\right) = 0, \quad \text{on } \Omega = [0,1]^2,$$

with

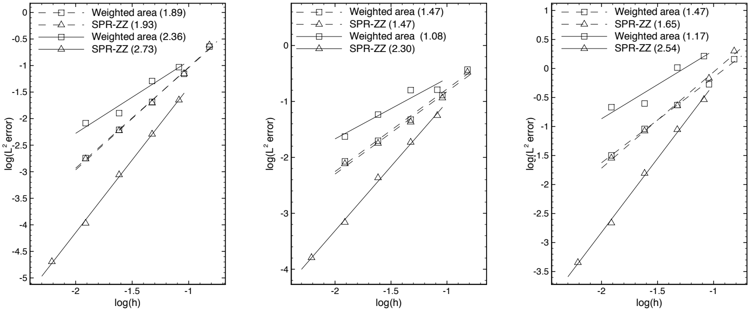
$$\mathbb{K} = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix},$$

the problem has the following exact solution

$$\boldsymbol{u} = \sin(2\pi x)\, e^{-2\pi y\sqrt{1/\delta}},$$

and in the numerical simulations $\delta = 10^3$.

Anisotropic diffusion

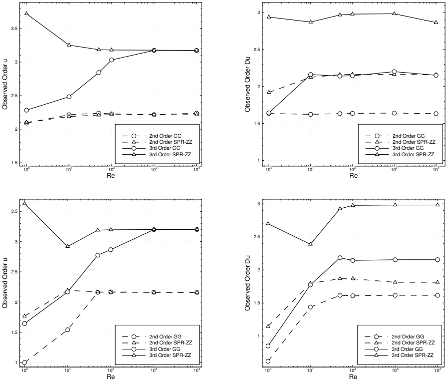


L^2 error in the solution of the anisotropic diffusion problem on triangular girds with linear (dashed lines) and quadratic (solid lines) elements. Error of the solution (first column), error of the x-component of the gradient (second column) error of the y-component of the gradient (third column).

Overview

MANUFACTURED SOLNS: ACCURACY TEST

SOL MADE OF TRIGS FUNCTIONS



Observed order

– Top: linear scheme;

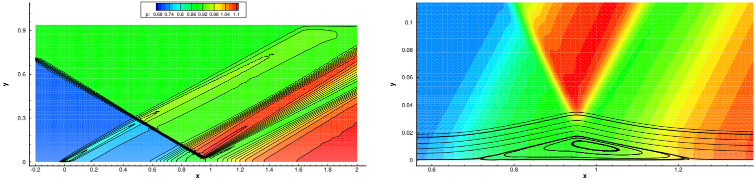
– Bottom: non linear scheme

– Left: error on solution;

– Right: error on gradients

SHOCK-WAVE/LAMINAR BOUNDARY LAYER INTERACTION

M=2.15, $\theta = 30, 8^{\circ}$, $Re = 10^5$

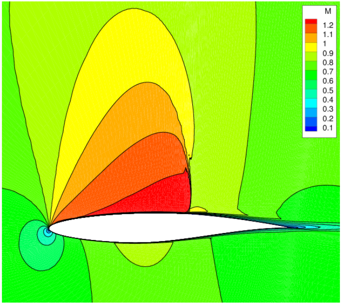


(a) (b)

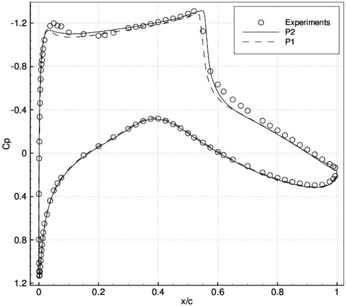
FIGURE : Left: contours of the pressure obtained with the third order scheme for the shock/boundary layer interaction. Right: zoom of the solution near the impinging point of the shock with the boundary layer, streamlines are also reported to show the separation bubble.

RAE2822 AIRFOIL, TURBULENT

M=0.734, RE=6.5 10^6, AOA=2.79°



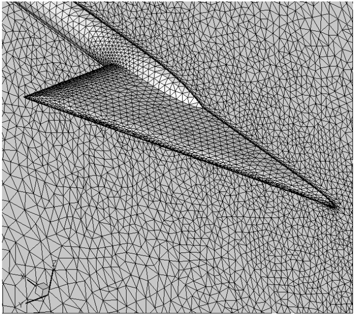
Mach



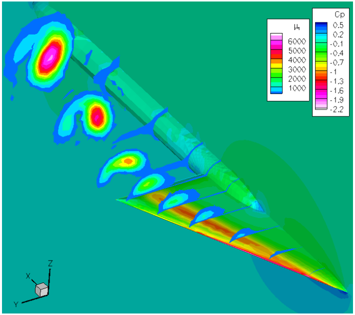
Pressure

DELTA WING, TURBULENT, IDIHOM PROJECT

M=0.734, RE=6.5 10^6, AOA=2.79°



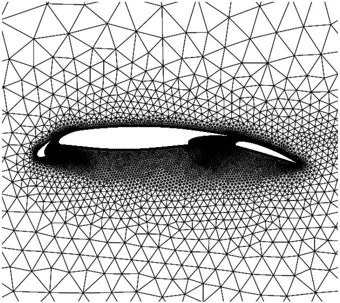
Mesh



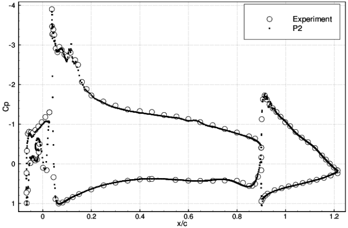
Pressure

L1T2 AIRFOIL, TURBULENT

M=0.197, RE=3.52 10^6, AOA=4.01°

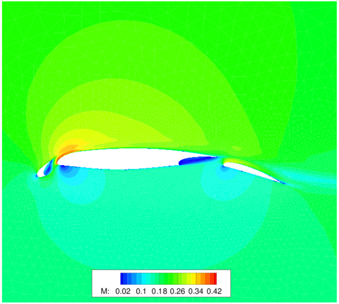


Mesh

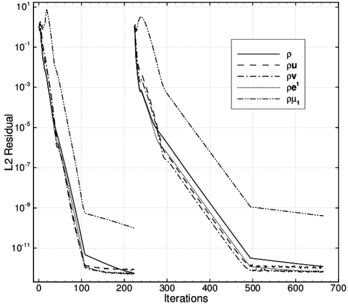


Mach

L1T2 airfoil, turbulent
M=0.197, Re=3.52 10^6, AoA=4.01°



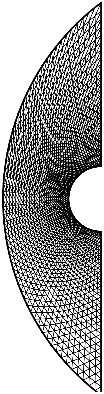
Mach

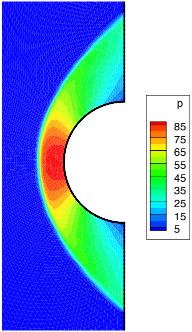


convergence history

HYPERSONICS

$M = 10$, $Re = 3 \times 10^5$, T_{wall}, $\mathbb{P}^2$

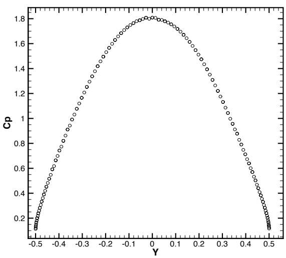




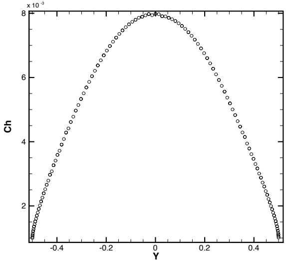
Pressure contour

Hypersonics

$M = 10$, $Re = 3 \times 10^5$, T_{wall}, $\mathbb{P}^2$



C_p



heat transfert

Overview

Formulations: conservation and accuracy issues

Residual distribution framework

Application to steady turbulent flow problems

Extensions: shallow water

Conclusion

SHALLOW WATER, MODEL

$$\frac{\partial W}{\partial t} + \mathrm{div}\mathbf{f}(W) - S(x, W) = 0$$

- $W = (h, h\mathbf{u})$, $S(x, W)$ depends on the bathymetry and W
- issues:
  - “lake at rest problem”, i.e and coupling between div **f** and S
  - $h \geq 0$: dry bed

Shallow water: Mario Ricchiuto (INRIA)

Shallow water, model

$$\frac{\partial W}{\partial t} + \text{div}\mathbf{f}(W) - S(x, W) = 0$$

- $W = (h, h\mathbf{u})$, $S(x, W)$ depends on the bathymetry and W
- issues:
  - "lake at rest problem", i.e and coupling between div $\mathbf{f}$ and S
  - $h \geq 0$: dry bed

Total residual to be distributed:

$$\Phi^K = \int_{\partial K} \mathbf{f}(u^h) \cdot \mathbf{n} - \int_K S(x, u^h)$$

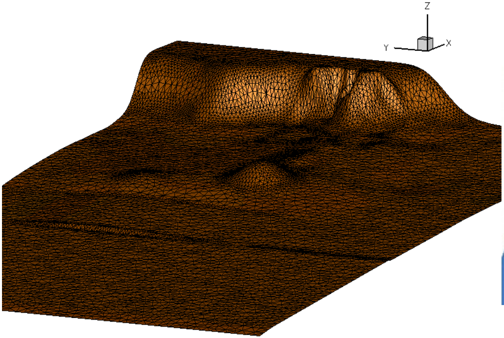
plus coherent integration formula for accuracy

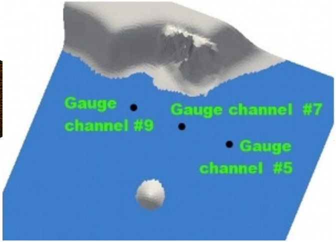
Shallow water: Mario Ricchiuto (INRIA)

Discussion of numerical results

OKUSHIRI TSUNAMI EXPERIMENT

3rd Int. workshop on long-wave run-up models. Bathymetry, inlet wave profile, experimental data on the workshop web.





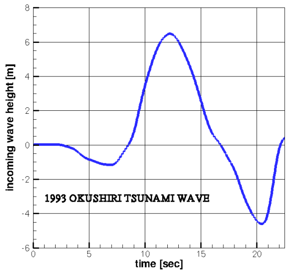
Discussion of numerical results

Okushiri tsunami experiment

3rd Int. workshop on long-wave run-up models. Bathymetry, inlet wave profile, experimental data on the workshop web.

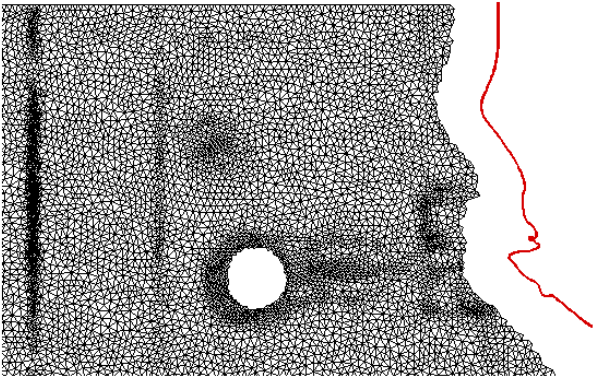
Parameters :

- Amplitude : $\pm 10\%$, uniform PDF
- Manning coefficient : $\pm 50\%$, uniform PDF
- Wave phase : $\pm 10\%$, uniform PDF



UQ: Pietro-Marco Congédo (INRIA)

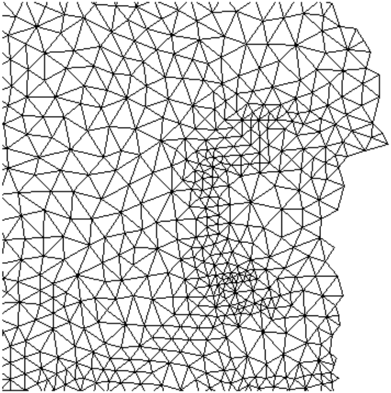
OKUSHIRI TSUNAMI EXPERIMENT



Deterministic result (parameters prescribed in the workshop)

Discussion of numerical results

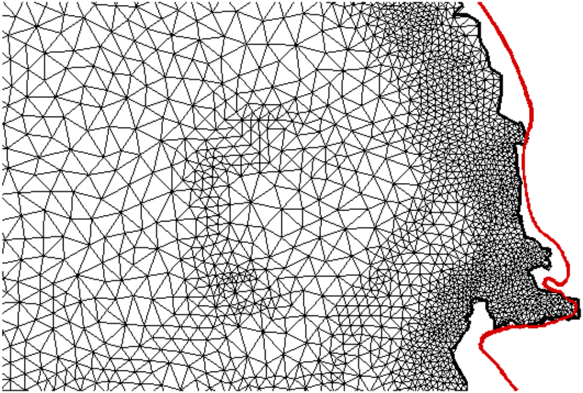
Okushiri tsunami experiment



Deterministic result (parameters prescribed in the workshop)

Discussion of numerical results

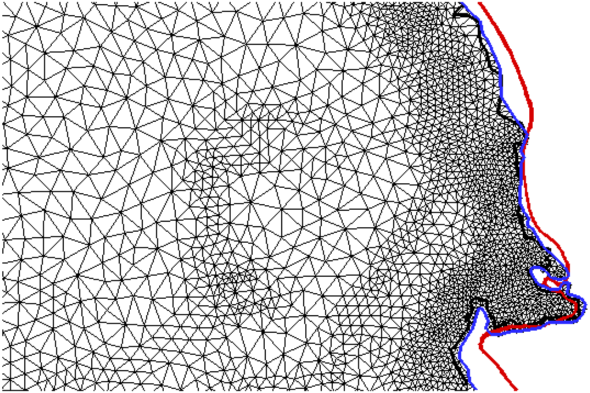
OKUSHIRI TSUNAMI EXPERIMENT



Deterministic result : run-up plot

Discussion of numerical results

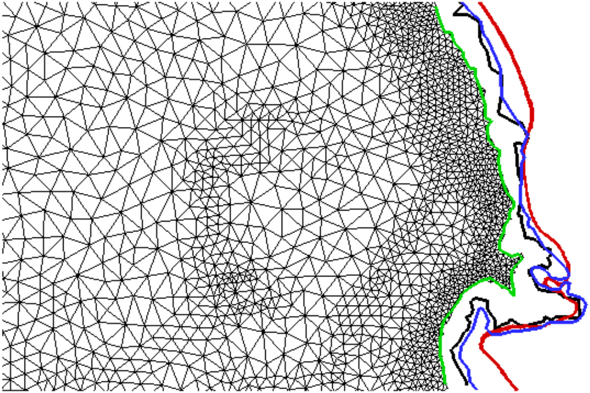
OKUSHIRI TSUNAMI EXPERIMENT



Deterministic run-up *vs* statistical average (mesh) plus deviation

Discussion of numerical results

OKUSHIRI TSUNAMI EXPERIMENT



Deterministic run-up *vs* statistical average (mesh) minus deviation

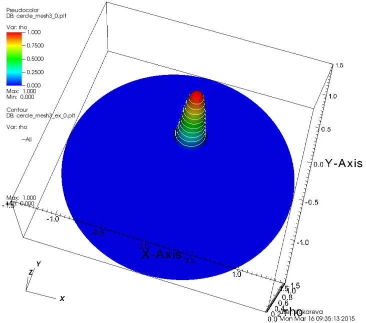
UNSTEADY SHOCK CALCULATION

Image !

Overview

CONCLUSIONS

- High order finite element like method
- All see from the discrete point of view: no natural variational formulation
- in the scalar case: L^∞ and L^2/entropy stable, and still formaly high order
- Applications for compressible flows, turbulent, shallow water, ...
- Some perspective on $h-p$ adaption (in progress)
- Unsteady high order: in progress.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Y-Axis
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