THE DOMAIN DECOMPOSITION :
A solution method

Colloque en l'honneur d'Alain DERVIEUX
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Marina Vidrascu
Project team REO
Inria Paris-Rocquencourt & Laboratoire J-L Lions
marina.vidrascu@inria.fr
& Collaborators!!!!

What is Domain Decomposition ?

Data decomposition

- ★ Parallel computing: distribute data among processors
- ★ Do not change the solution algorithm

Homogenous domain decomposition

- ★ New solution method for large scale linear systems
- ★ Well adapted for coarse grain parallel methods
- ★ Principle : solve independently on each domain and glue solutions at interface

Heterogenous domain decomposition

- ★ Separate the physical domain in regions
- ★ Use a different model in each region
- ★ Glue solutions at interface

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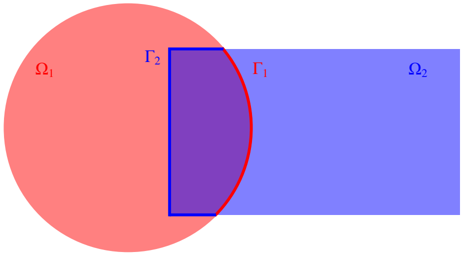
FSI

Un peu d'histoire

Schwarz alterné

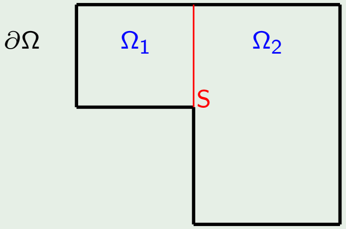
- décomposition avec recouvrement
- calcul **analytique** de fonctions harmoniques



Sous-structuration

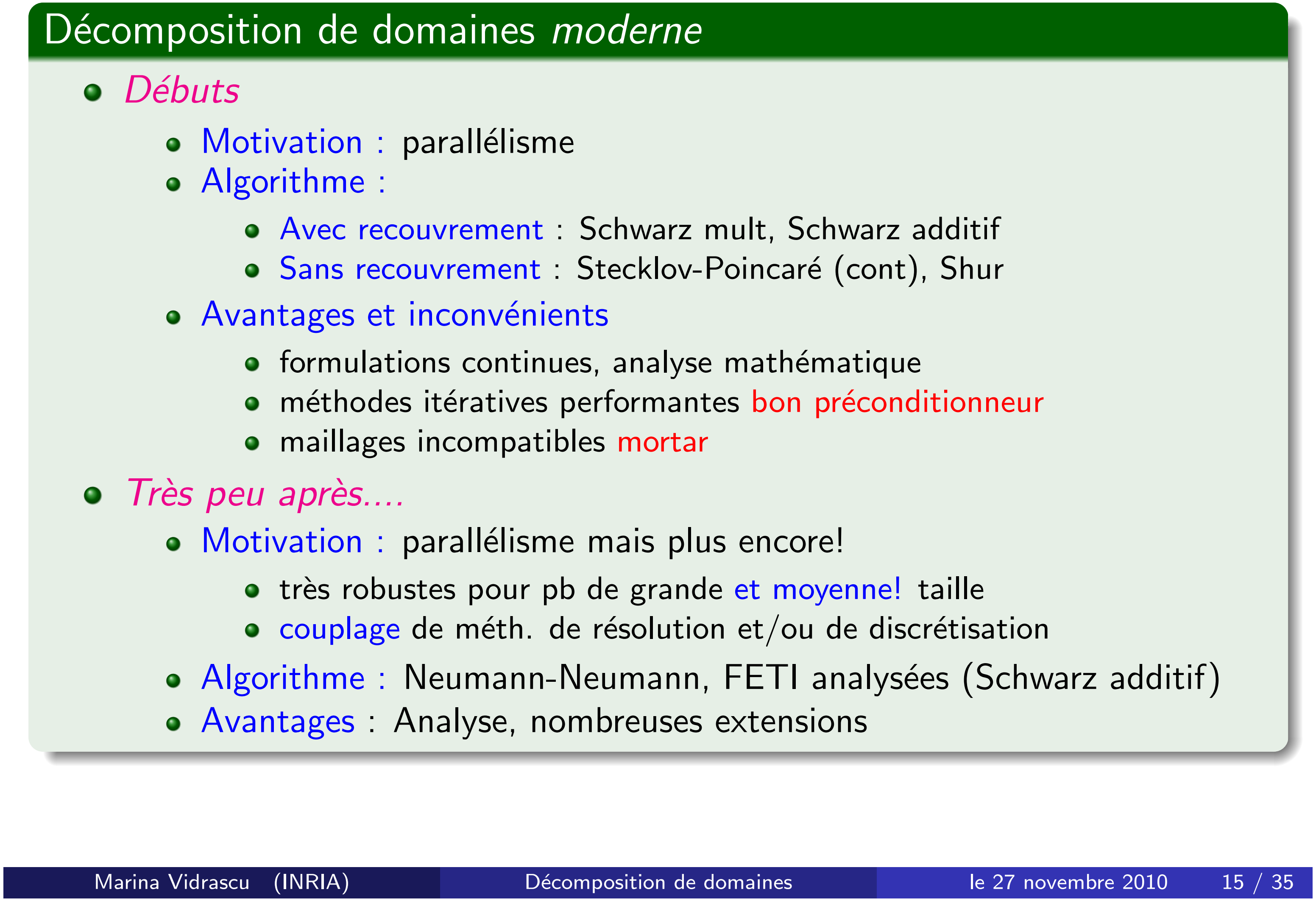
Numérotation : premier sous-domaine, deuxième , la frontière S

$$\begin{bmatrix} A_{11} & 0 & A_{1S} \\ 0 & A_{22} & A_{2S} \\ A_{1S}^T & A_{2S}^T & A_{SS} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_S \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_S \end{bmatrix}$$



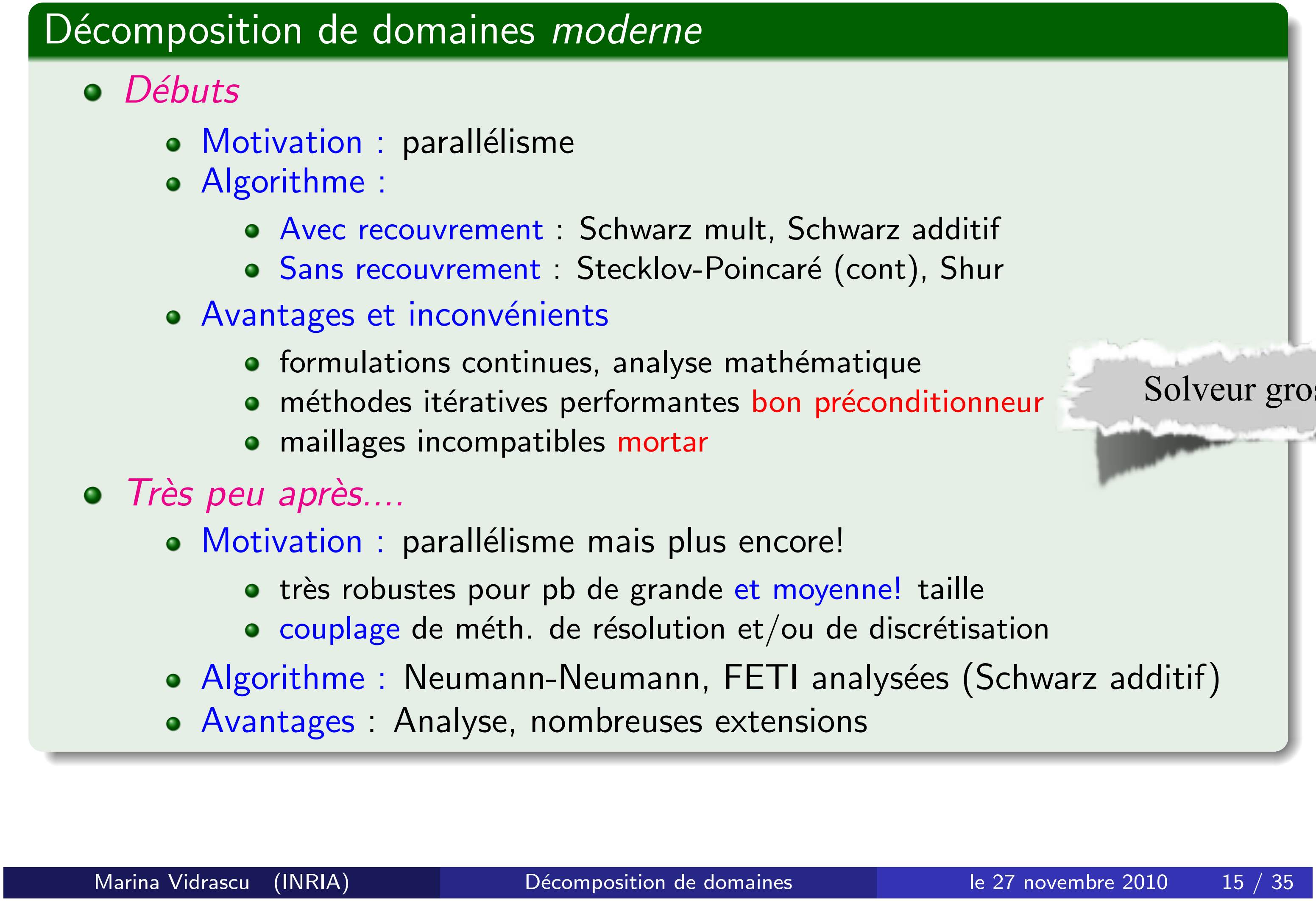
$$X_i = A_{ii}^{-1}\,(B_i - A_{iS}\,X_S)$$

$$\underbrace{(A_{SS} - \sum_{i=1}^{2} A_{iS}^T A_{ii}^{-1} A_{iS})}_{S\text{(Shur complement)}} X_S = B_S - \sum_{i=1}^{2} A_{iS}^T A_{ii}^{-1} B_i$$



Décomposition de domaines *moderne*

- *Débuts*
  - Motivation : parallélisme
  - Algorithme :
    - Avec recouvrement : Schwarz mult, Schwarz additif
    - Sans recouvrement : Stecklov-Poincaré (cont), Shur
  - Avantages et inconvénients
    - formulations continues, analyse mathématique
    - méthodes itératives performantes bon préconditionneur
    - maillages incompatibles mortar
- *Très peu après....*
  - Motivation : parallélisme mais plus encore!
    - très robustes pour pb de grande et moyenne! taille
    - couplage de méth. de résolution et/ou de discrétisation
  - Algorithme : Neumann-Neumann, FETI analysées (Schwarz additif)
  - Avantages : Analyse, nombreuses extensions



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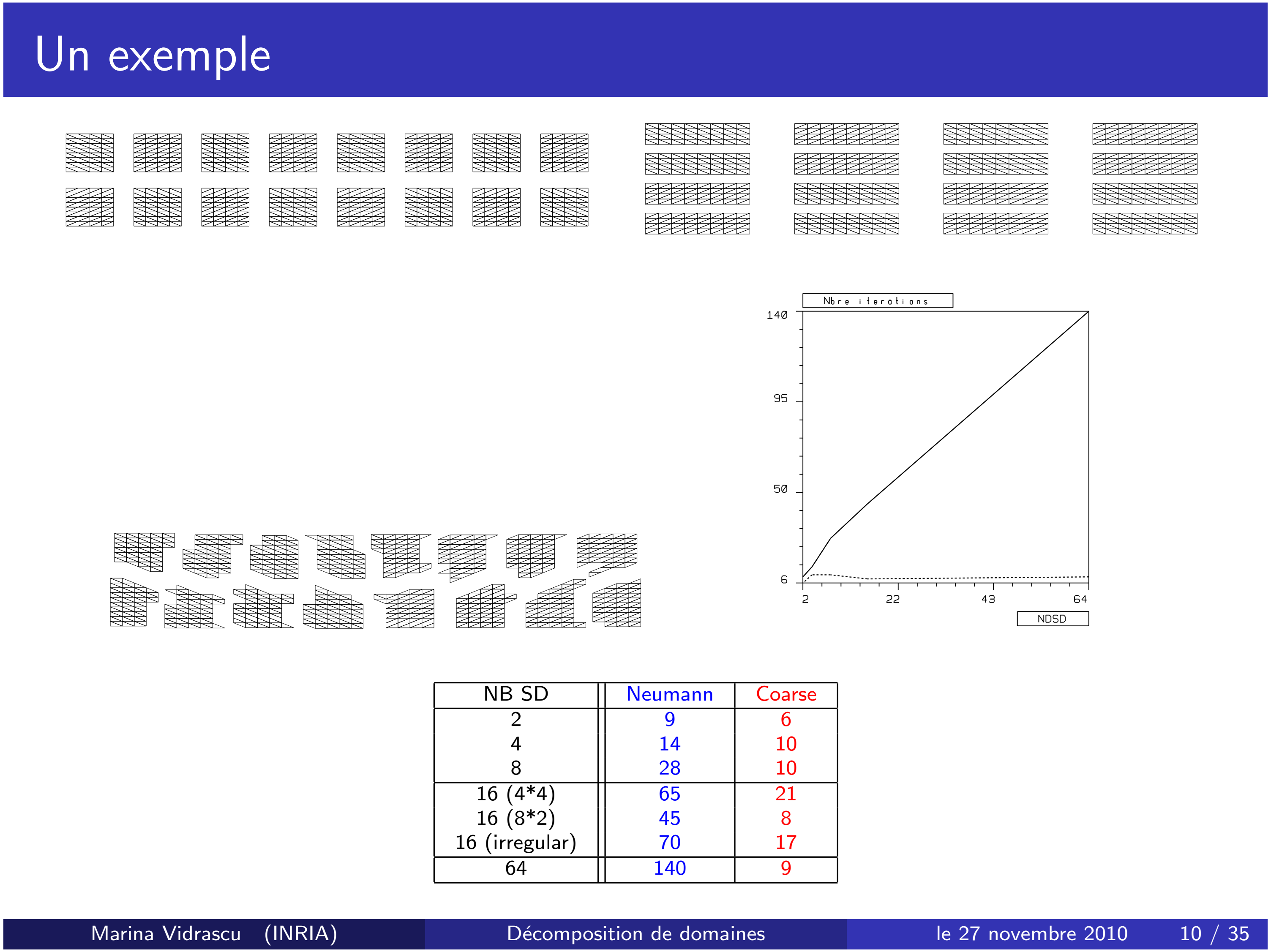
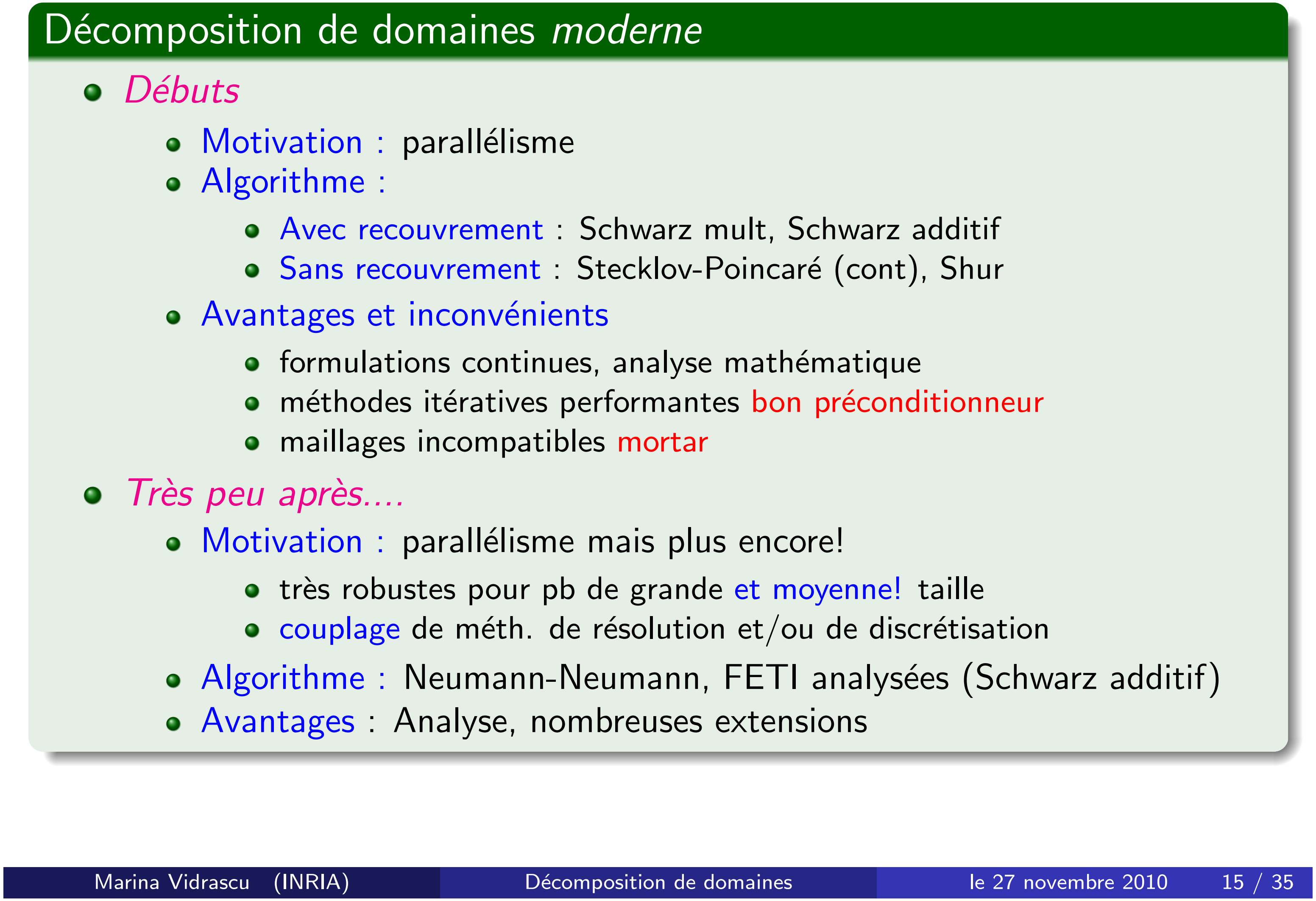
Marina Vidrascu (INRIA) Décomposition de domaines le 27 novembre 2010 15 / 35

Solveur grossier



Un exemple

NB SD	Neumann	Coarse
2	9	6
4	14	10
8	28	10
16 (4*4)	65	21
16 (8*2)	45	8
16 (irregular)	70	17
64	140	9



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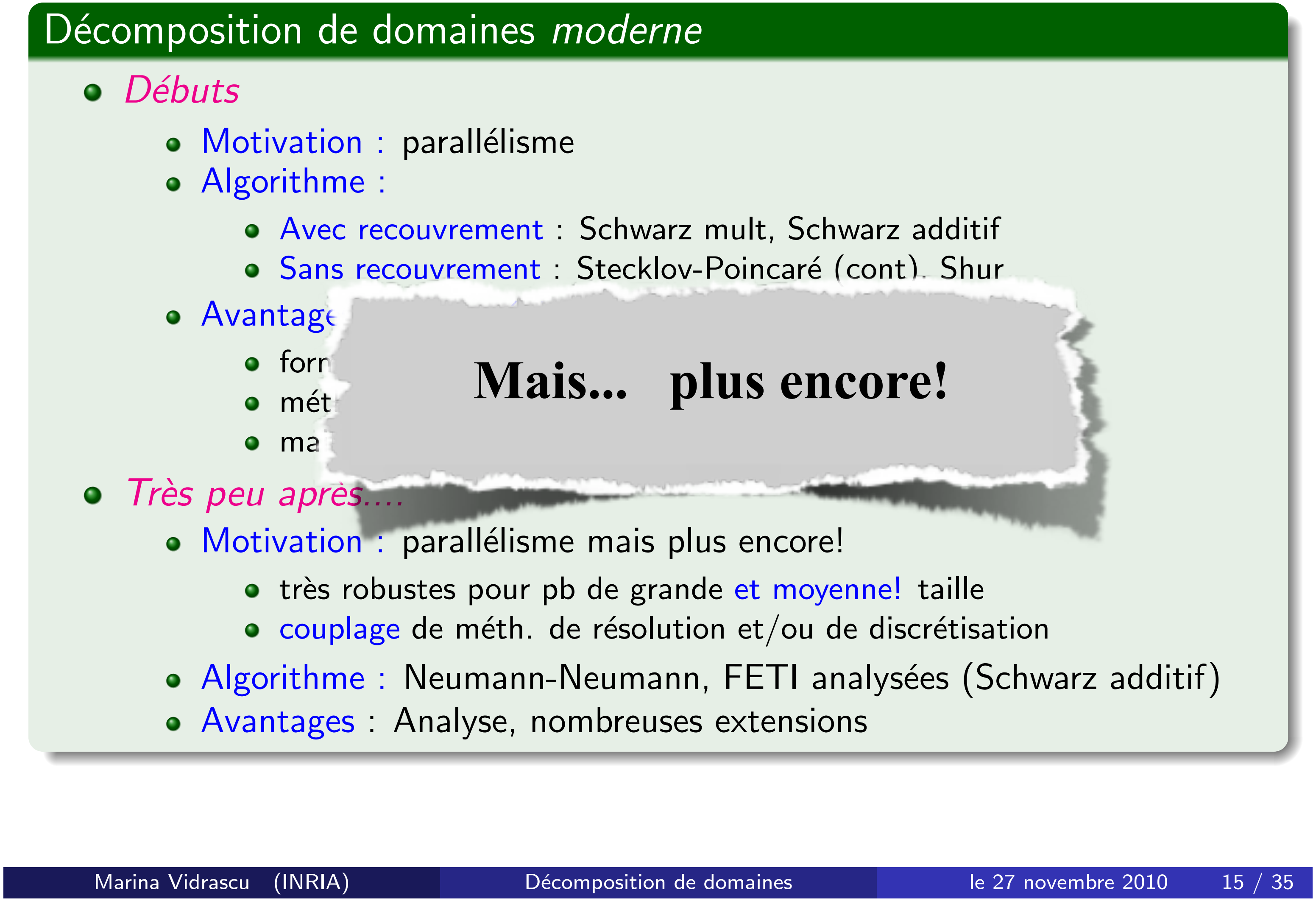

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    - forr
    - mét
    - ma

Mais... plus encore!

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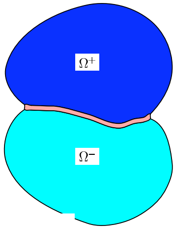
DD a tool for practical solution of asymptotic problems

Multi-scale elasticity problems

★A *sandwich* structure with a thin layer. High ratio in material properties

- $E_l << E_{3d}$ **glue**
- $E_l >> E_{3d}$ **reinforcement sheets**



★ With heterogeneities

Modeling

direct solution of the problem is not efficient

★poor conditioning
★large size (the element size same order as the thickness of the layer)

asymptotic study : **eliminate** the thin layer and replace it with ad hoc transition conditions on the interface

Soft layer

(G.Geymonat, F. Krasucki, D. Marini et M. V, 1996)

Goland and Reissner conditions

$$\left\{\begin{array}{lcl} \sigma^+ n^+ & = & -\sigma^- n^- \\ \sigma^+ n^+ & = & -\frac{K^s}{h}\,[u] \end{array}\right.$$

Fourier Robin conditions

$$\left\{\begin{array}{lcl} \sigma^+ n^+ 2\frac{K^s}{h} u^+ & = & \sigma^- n^- + 2\frac{K^s}{h} u^+ \\ \sigma^+ n^+ & = & -\sigma^- n^- \end{array}\right.$$

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Reinforcement sheet

(A.L Bessoud, F. Krasucki, M. Serpilli, 2008) (M. Serpilli, MV)

Add a **membrane** energy

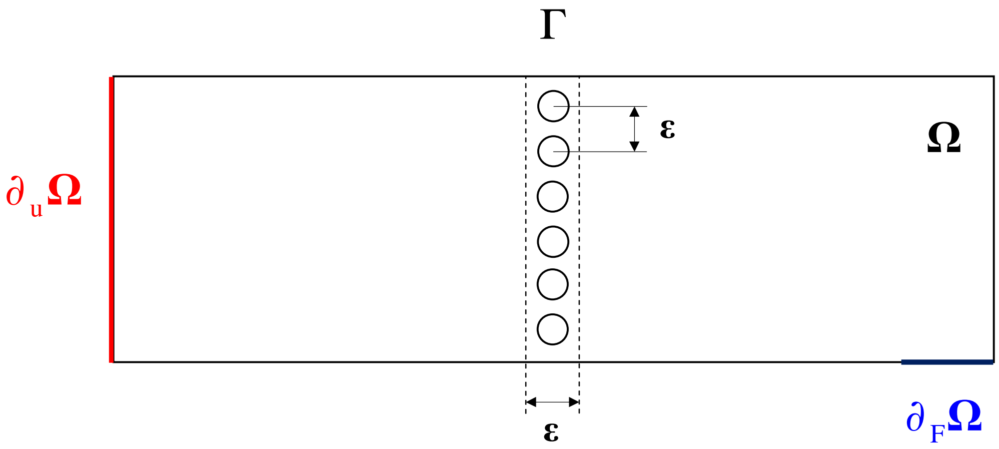
$$A_m(\bar{u}, \bar{v}) + A_{3D}(\bar{U}, \bar{V}) = F(V)$$

Interface problem

$$\mathcal{A}(\lambda) = S_1(\lambda) + S_2(\lambda) + A_m(\lambda) = 0$$

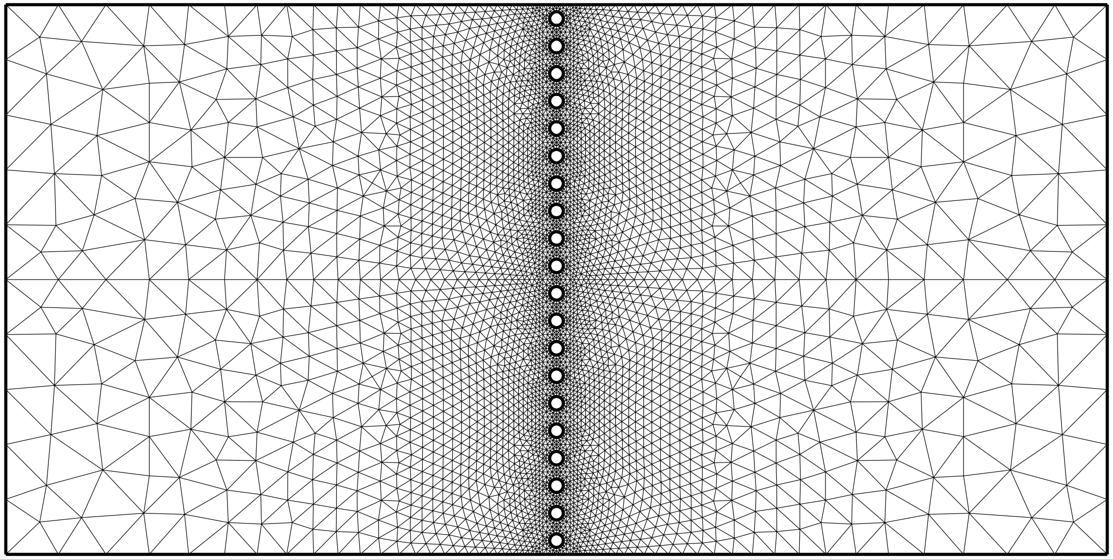
Heterogeneities: a model problem

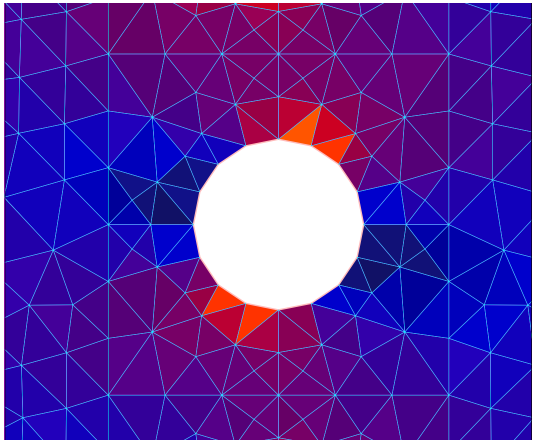
(G.Geymonat, S. Hendili, F. Krasucki, et M. V, 2012)



$$\begin{cases} -\boldsymbol{div}\boldsymbol{\sigma}^\varepsilon = \boldsymbol{g} & \text{in } \Omega \\ \boldsymbol{\sigma}^\varepsilon = \boldsymbol{A}\gamma(\boldsymbol{u}^\varepsilon) & \text{in } \Omega\backslash\mathcal{I}^\varepsilon \\ \boldsymbol{\sigma}^\varepsilon = \boldsymbol{A}^I\gamma(\boldsymbol{u}^\varepsilon) & \text{in } \mathcal{I}^\varepsilon \\ \boldsymbol{\sigma}^\varepsilon\boldsymbol{n} = \boldsymbol{F} & \text{on } \partial\Omega_F \\ \boldsymbol{u}^\varepsilon = \boldsymbol{0} & \text{on } \partial\Omega_u \end{cases}$$

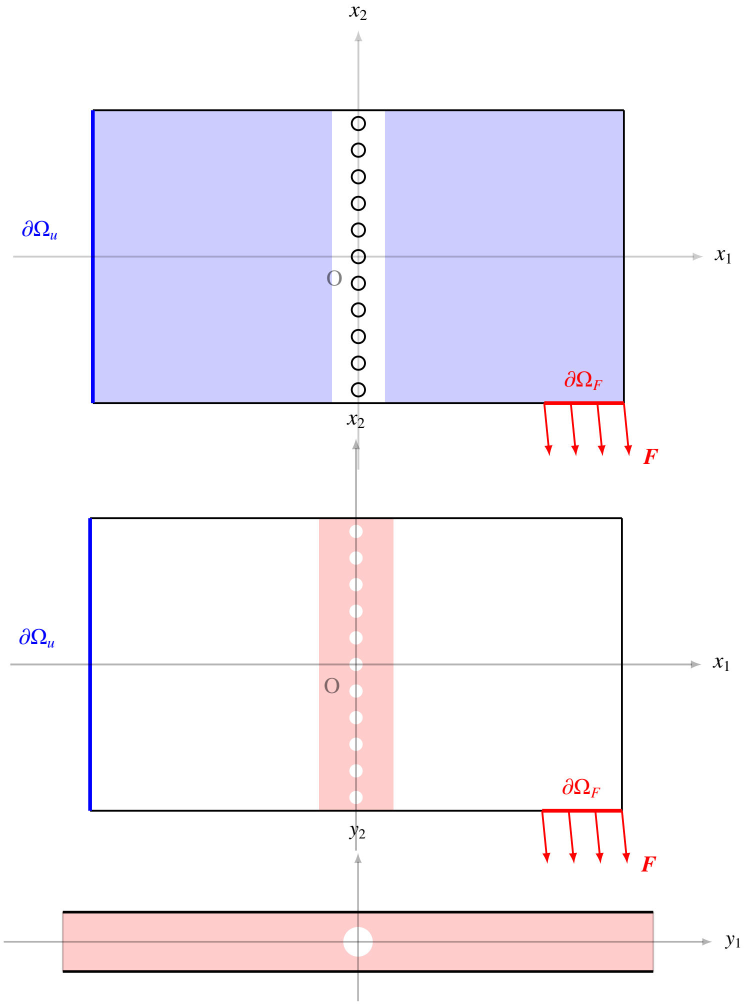
Difficulties to solve the problem





- Large number of heterogeneities
- Computational cost **increase** with the number of heterogeneities
- Difficult to obtain a correct mesh

A two scale problem



Outer expansion

$$\boldsymbol{u}^{\epsilon}(x_1,x_2) = \sum_{i=0}^{\infty} \epsilon^i \boldsymbol{u}^i(x_1,x_2)$$

$$\boldsymbol{\sigma}^{\epsilon}(x_1,x_2) = \sum_{i=0}^{\infty} \epsilon^i \boldsymbol{\sigma}^i(x_1,x_2)$$

Inner expansion

$$\boldsymbol{u}^{\epsilon}(x_1,x_2) = \sum_{i=0}^{\infty} \epsilon^i \boldsymbol{v}^i(x_2,y_1,y_2)$$

$$\boldsymbol{\sigma}^{\epsilon}(x_1,x_2) = \sum_{i=0}^{\infty} \epsilon^i \boldsymbol{\tau}^i(x_2,y_1,y_2)$$

with $y_1 = \dfrac{x_1}{\epsilon}$, $y_2 = \dfrac{x_2}{\epsilon}$

Matched asymptotic method : Results

Zero order approximation

- outer approximation

$$\left\{\begin{array}{llll} -\boldsymbol{div}\boldsymbol{\sigma}^0 & = & 0 & \text{in } \Omega\backslash\Gamma \\ \boldsymbol{\sigma}^0 & = & \boldsymbol{A}\gamma(\boldsymbol{u}^0) & \text{in } \Omega\backslash\Gamma \\ \boldsymbol{\sigma}^0\boldsymbol{n} & = & F & \text{on } \partial_F\Omega \\ \boldsymbol{u}^0 & = & \boldsymbol{u}^d & \text{on } \partial_u\Omega \\ \left[\boldsymbol{\sigma}^0\right]\boldsymbol{e}_1 & = & \left[\boldsymbol{u}^0\right] = 0 & \text{on } \Gamma \end{array}\right.$$

- inner approximation

independent of y

$$\boldsymbol{v}^0\left(\boldsymbol{x},\boldsymbol{y}\right) = \boldsymbol{v}^0\left(\boldsymbol{x}\right) = \boldsymbol{u}^0\left(0,\hat{\boldsymbol{x}}\right)$$

Remark : Zero order problem is **independent** of the heterogeneities

Matched asymptotic method : Results(cont)

First order approximation

- outer approximation :

$$\left\{\begin{array}{lcll} -\boldsymbol{div}\boldsymbol{\sigma}^{\mathbf{1}} & = & \mathbf{0} & \text{in } \Omega\backslash\Gamma \\ \boldsymbol{\sigma}^{\mathbf{1}} & = & \boldsymbol{A}\gamma(\boldsymbol{u}^1) & \text{in } \Omega\backslash\Gamma \\ \boldsymbol{\sigma}^{\mathbf{1}}\boldsymbol{n} & = & \mathbf{0} & \text{on } \partial\Omega_F \\ \boldsymbol{u}^1 & = & \mathbf{0} & \text{on } \partial\Omega_u \\ \left[\boldsymbol{u}^1\right](\hat{\boldsymbol{x}}) & = & \mathcal{G}_d\left(\boldsymbol{u}^0(0,\hat{\boldsymbol{x}})\ ;\ \left[\boldsymbol{V}^{ij}\right]^{\infty}\right) & \\ \left[\boldsymbol{\sigma}^1\boldsymbol{e}_1\right](\hat{\boldsymbol{x}}) & = & \mathcal{G}_{nS}\left(\boldsymbol{u}^0(0,\hat{\boldsymbol{x}})\ ;\ \int_{Y^\star}\boldsymbol{T}^{ij}(\boldsymbol{y})\,d\boldsymbol{y}\right) & \end{array}\right.$$

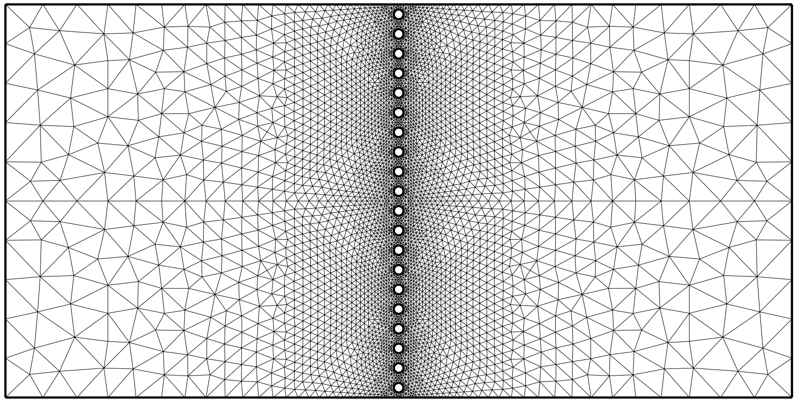
- $\mathcal{G}_d, \mathcal{G}_{nS}$ depend on $\boldsymbol{u}^0$ and its first derivative

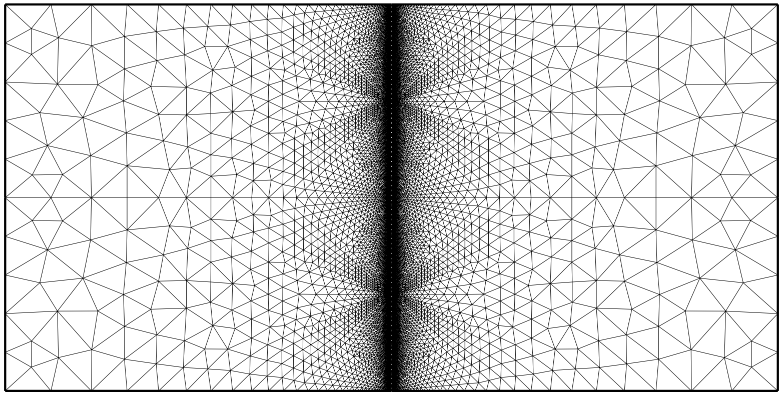
☑ This is a **non standard** problem which will be solved by a ***domain decomposition type*** algorithm

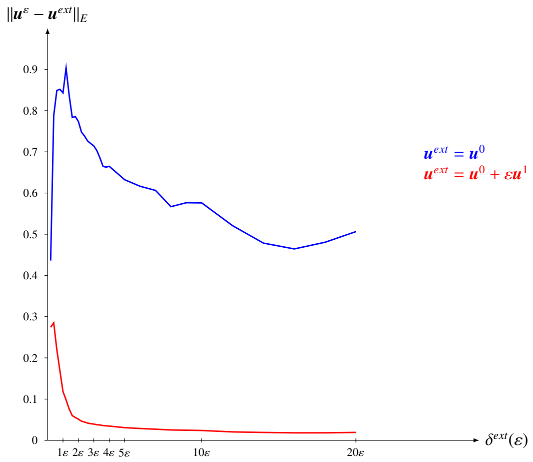
Results at a glance

$\epsilon = \frac{1}{20}$ 9836 P_2 *elements* 20125 *nodes*

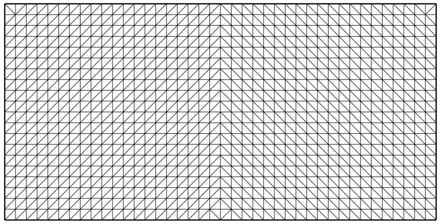
$\epsilon = \frac{1}{80}$ 41332 P_2 *elements* 85313 *nodes*



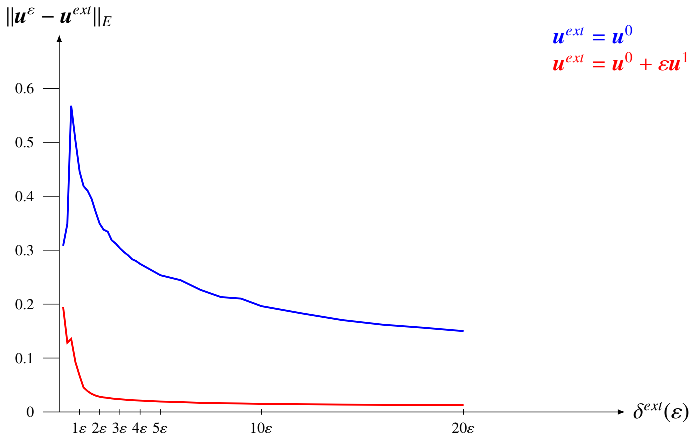




$\|u^\varepsilon - u^{ext}\|_E$
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
$u^{ext} = u^0$
$u^{ext} = u^0 + \varepsilon u^1$
1ε 2ε 3ε 4ε 5ε
10ε
20ε
$\delta^{ext}(\varepsilon)$



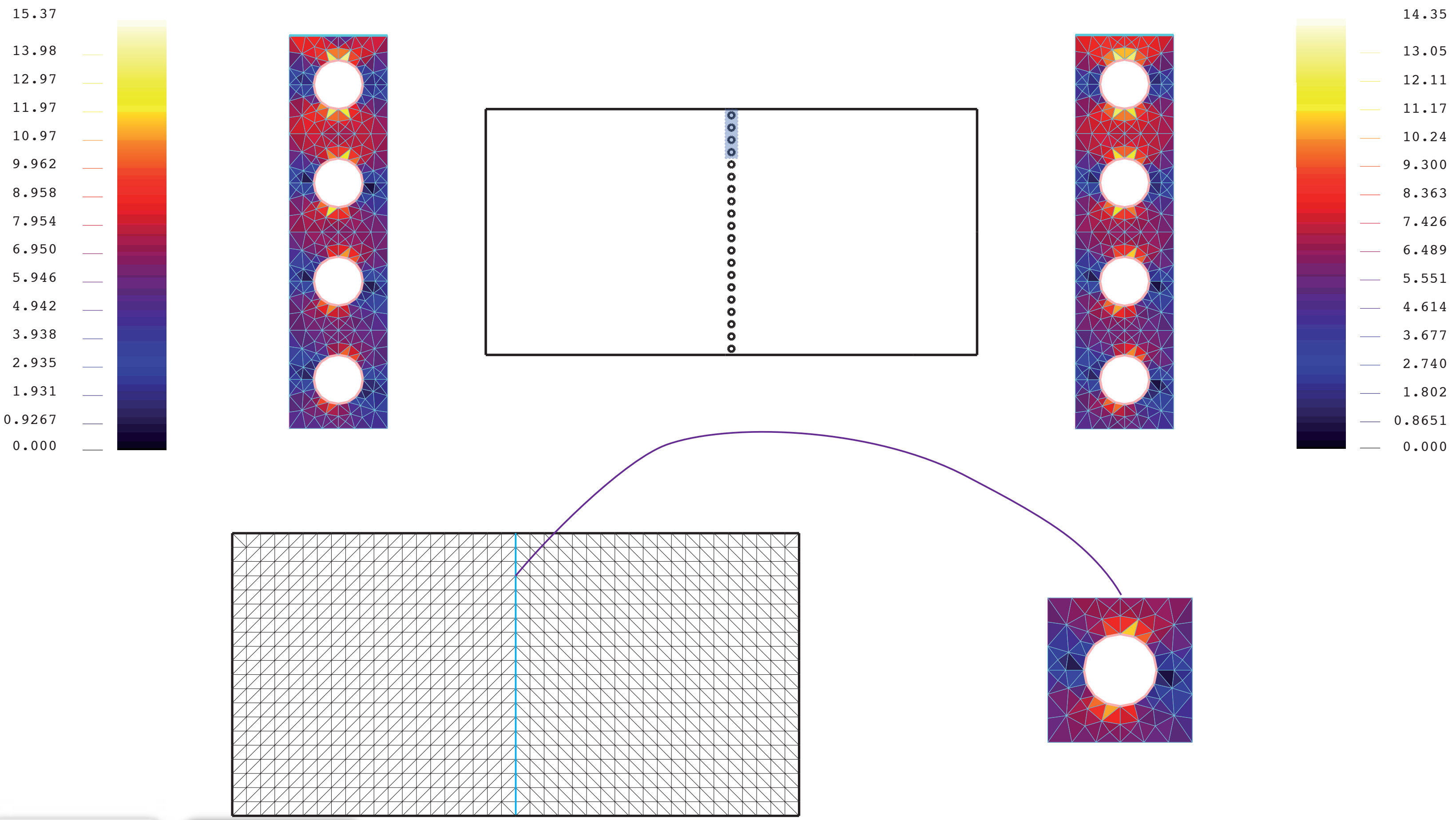
1600 elements P_2 3321 nodes



Inner problem : comparison of stresses

Solution de référence

Solution du modèle asymptotique



Heterogeneous Domain Decomposition

FLUID STRUCTURE INTERACTION

Collaboration with: Miguel Fernández, Mikel Landajuela

Incompressible Fluid-Structure Interaction

- Framework: coupling of
  - Fluid: incompressible (viscous,...)
  - Structure: elastic (non-linear,...)
- Widespread multi-physic problem:
  - Aeroelasticity (bridge, parachute, etc.), naval hydrodynamics,...
  - Mechanics of bio-fluid flow: blood, cerebrospinal fluid, air,...

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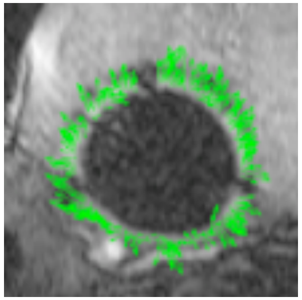
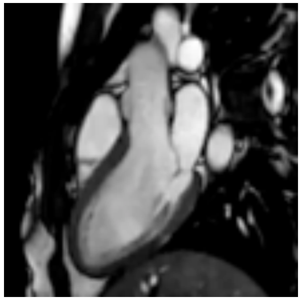
- Targeted application: arterial and ventricular blood flow simulation





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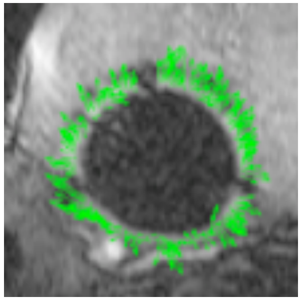
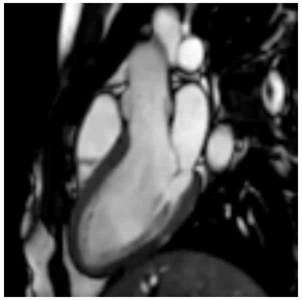




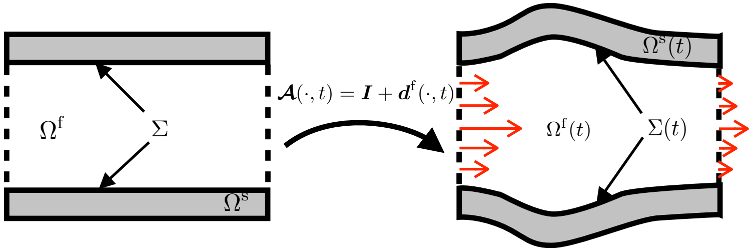
Motivations:

- Improve diagnosis (via data assimilation), therapy planing, medical devices
- Major issues in modeling, scientific computing and numerical analysis

Standard 3D Model of Blood Flow in Arteries

- Structure: non-linear elastodynamics

$$\begin{cases} \rho^{\mathrm{s}}\partial_t \dot{\boldsymbol{d}} - \mathbf{div}\left(\boldsymbol{\Pi}(\boldsymbol{d})\right) = \boldsymbol{0} & \text{in} \quad \Omega^{\mathrm{s}} \\ \dot{\boldsymbol{d}} = \partial_t \boldsymbol{d} & \text{in} \quad \Omega^{\mathrm{s}} \end{cases}$$



reference configuration

current configuration

- Fluid: Navier-Stokes *(ALE formalism)*

$$\begin{cases} \rho^{\mathrm{f}}\partial_t \boldsymbol{u}|_{\boldsymbol{\mathcal{A}}} + \rho^{\mathrm{f}}(\boldsymbol{u}-\boldsymbol{w})\cdot\boldsymbol{\nabla}\boldsymbol{u} - \mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u},p) = \boldsymbol{0} & \text{in} \quad \Omega^{\mathrm{f}}(t) \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in} \quad \Omega^{\mathrm{f}}(t) \end{cases}$$

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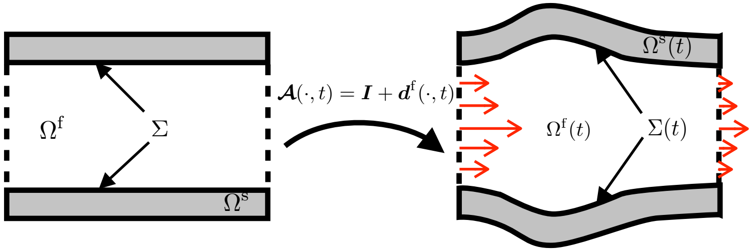

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- Coupling conditions:

$$\begin{cases} \boldsymbol{d}^{\mathrm{f}} = \operatorname{Ext}\left(\boldsymbol{d}|_{\Sigma}\right), \quad \boldsymbol{w} = \partial_t \boldsymbol{d}^{\mathrm{f}} & \text{in} \quad \Omega^{\mathrm{f}} & \longleftarrow \text{geometric compatibility} \\ \boldsymbol{u} = \partial_t \boldsymbol{d} & \text{on} \quad \Sigma & \longleftarrow \text{kinematic continuity} \\ \boldsymbol{\Pi}(\boldsymbol{d}) \boldsymbol{n}^{\mathrm{s}} = -J \boldsymbol{\sigma}(\boldsymbol{u}, p) \boldsymbol{F}^{-\mathrm{T}} \boldsymbol{n} & \text{on} \quad \Sigma & \longleftarrow \text{kinetic continuity} \end{cases}$$

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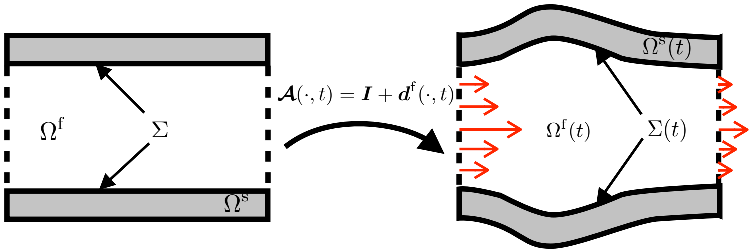

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⟵ geometric compatibility

⟵ kinematic continuity

⟵ kinetic continuity

Major issue:

Computational complexity: efficient partitioning extremely difficult

Why FSI in a DD framework?

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Revisit vocabulary

Coupling schemes

- **Explicit (weak)** coupling vs **Implicit (strong)** and **semi-implicit** coupling

Wikipedia

Two main approaches exist for the simulation of fluid–structure interaction problems:

- **Monolithic** approach: the equations governing the flow and the displacement of the structure are solved simultaneously, with a single solver
- **Partitioned** approach: the equations governing the flow and the displacement of the structure are solved separately, with two distinct solvers

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Domain decomposition framework

- Formulation : monolithic
- Solution algorithm : DD (partitioned...)
- Weak coupling : one iteration of a given DD algorithm (with an appropriate initialization)
- Strong coupling : iterate till convergence

FSI = Heterogenous domain decomposition
(Not a new idea!)

DD5 (1991) *Quarteroni, Pasquarelli,Valli*

In introduction FSI is mentioned,

Focus on problems homogenous in nature can be faced in a heterogeneous fashion after reducing the given problem to a simplified one in a subregion ex in fluid dynamics NS and Boltzmann Kinetic models

(1999) *Le Tallec, Mouro* *Fluid structure interaction with large structural displacements*

(2001---> today) a lot of people!

Design efficient and reliable parallel methods

✱ Methodology :

- Take advantage of modularity and use robust well validated components
- Numerical methods:

  □ Fluid : ALE Navier-Stokes
  □ Structure : Non-linear elasto-dynamic (shells)
  ☑ FSI : Explicit coupling

- Design algorithms **well suited for parallel computing** : **use domain decomposition**

  □ Additive Schwarz for the fluid (PETSCI)
  ☑ Balanced domain decomposition method for the solid solver

✱ Implementation

- ● Use different solvers for fluid and solid validation platform
- ● Use the **most appropriate** parallelization technique (PVM, MPI....)

✱ Performance

☑ **Robustness and efficiency** of the algorithms, ***numerical scalability***

□ Optimize the decomposition, level of parallelism....

Monolithic or partitioned approach?

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Domain decomposition technique

- **Monolithic** approach (formulation view point)
- **Partitioned** approach (implementation view point)
- Possible to use *state of the art* **different solvers**

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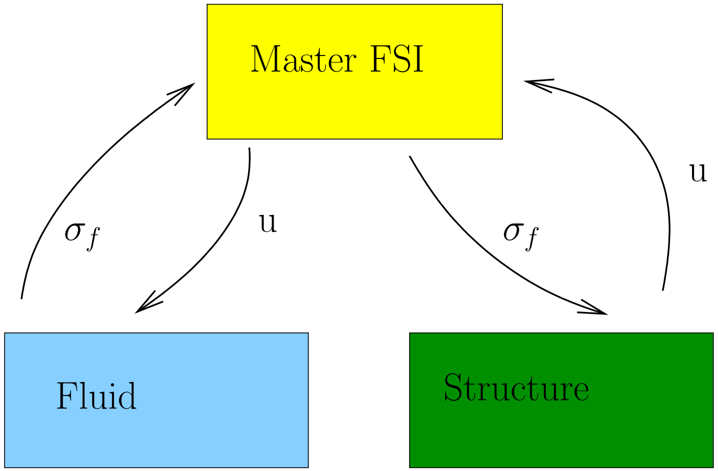

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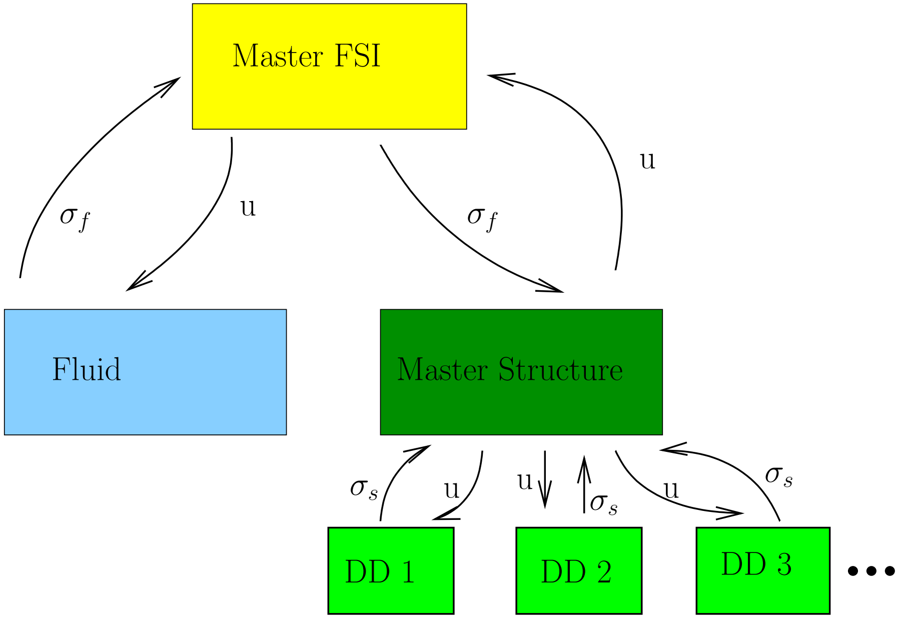

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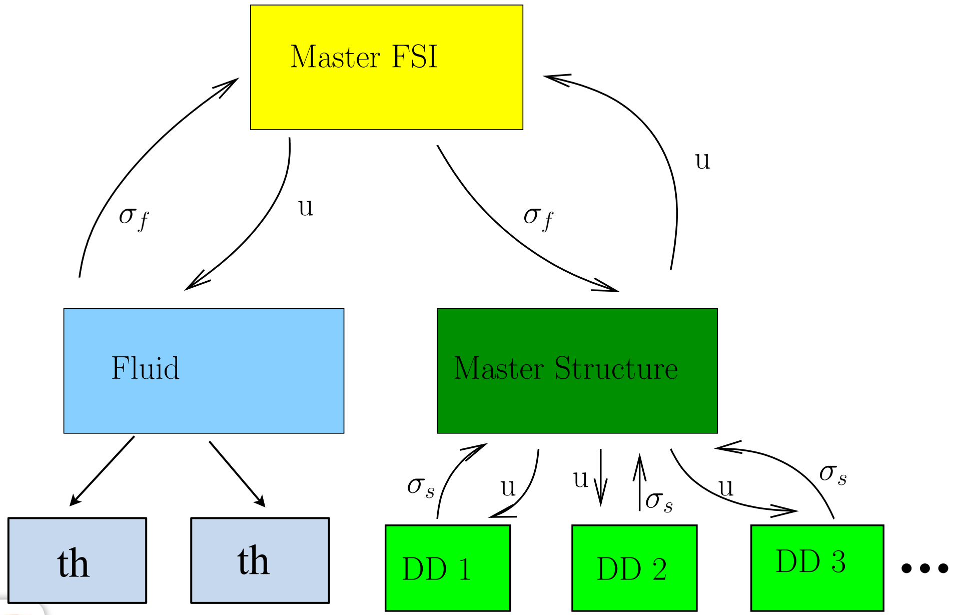

Difficulties

Scientific limitations:

Software limitations:

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- **pertinence** of the models used
- **efficiency** and **reliability** of the numerical methods
- how to validate ?

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- how to validate ? ← use benchmarks from literature

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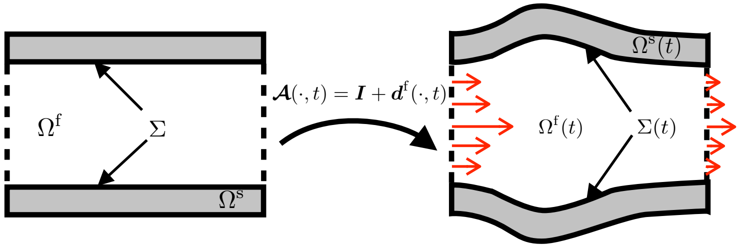
Standard 3D Model of Blood Flow in Arteries

- Structure: non-linear elastodynamics

$$\begin{cases} \begin{pmatrix}\rho^{\mathrm{s}}\epsilon\partial_t\dot{\boldsymbol{d}}\\ 0\end{pmatrix} + \begin{pmatrix}\boldsymbol{L}^{\mathrm{e}}_{\boldsymbol{d}}((\boldsymbol{d},\boldsymbol{\theta}))\\ L^{\mathrm{e}}_{\boldsymbol{\theta}}((\boldsymbol{d},\boldsymbol{\theta}))\end{pmatrix} = \begin{pmatrix}-J\boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{n}\\ \boldsymbol{0}\end{pmatrix} & \text{on} \quad \Sigma \\ \dot{\boldsymbol{d}} = \partial_t \boldsymbol{d} & \text{on} \quad \Sigma \end{cases}$$

- Fluid: Navier-Stokes *(ALE formalism)*

$$\begin{cases} \rho^{\mathrm{f}}\partial_t\boldsymbol{u}|_{\boldsymbol{\mathcal{A}}} + \rho^{\mathrm{f}}(\boldsymbol{u}-\boldsymbol{w})\cdot\boldsymbol{\nabla}\boldsymbol{u} - \mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u},p) = \boldsymbol{0} & \text{in} \quad \Omega^{\mathrm{f}}(t) \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in} \quad \Omega^{\mathrm{f}}(t) \\ \boldsymbol{d}^{\mathrm{f}} = \operatorname{Ext}(\boldsymbol{d}|_{\Sigma}), \quad \boldsymbol{w} = \partial_t\boldsymbol{d}^{\mathrm{f}} & \text{in} \quad \Omega^{\mathrm{f}} \\ \boldsymbol{u} = \partial_t\boldsymbol{d} & \text{on} \quad \Sigma \end{cases}$$



reference configuration

current configuration

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← geometric compatibility

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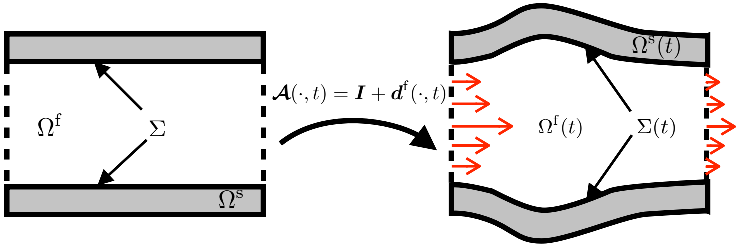

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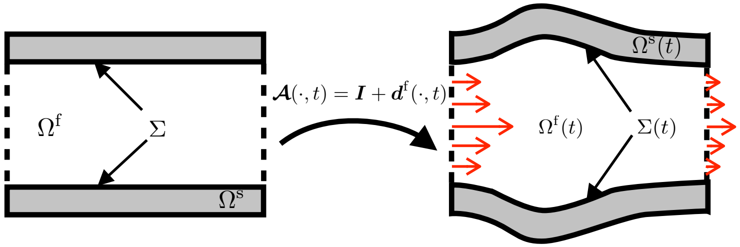

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Major issue:

Computational complexity: efficient partitioning extremely difficult

The Failure of (D-N) Explicit Coupled Schemes

- Explicit treatment of the geometric and kinematic compatibility:

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Notation: backward difference

$$\partial_\tau x^n \overset{\mathrm{def}}{=} \frac{x^n - x^{n-1}}{\tau}$$

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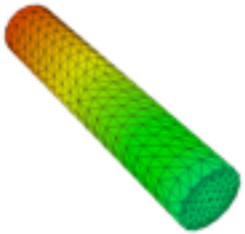
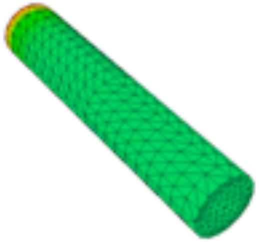


D-N loosely coupled scheme



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- Added-mass INStability condition (simplified model): $\dfrac{\rho^{\mathrm{s}} \epsilon}{\rho^{\mathrm{f}} \lambda_{\mathrm{add}}} < 1$

  (Causin, Gerbeau, Nobile '05)

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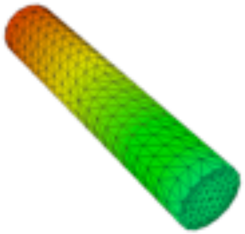

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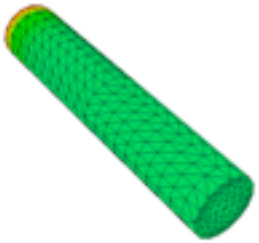

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  (Causin, Gerbeau, Nobile '05)

Standard Cures, Trends and Alternatives

- Implicit the coupling schemes:
  - Unconditionally energy stable, but computationally demanding
  - Vast literature (partitioned, monolithic,...)

(Mok et al. ’01,Heil ’04, Fernández, Moubachir ’05, Dettmer, Peric ’06, Badia et al.’08, Gee et al. ’11,...)

Domain decomposition approach

Option 1 : decompose first then linearize

- Dirichlet-Neumann
  - Fixed-point *Le Tallec-Mouro ’99,Wall-Ramm ’01....*
  - Newton *Fernàndez-Moubachir ’03....*
  - Inexact Newton *Matthies-Steindorf ’03, Gerbeau,MV. ’03,Mischler-van Brummelen-de- Borst ’05*
- Neumann-Neumann *Deparis-Discacciati-Quarteroni ’05*
- Robin-Neumann *Badia-Nobile-Vergara ’07*

Option 2 : linearize first then decompose *Fernàndez-Gerbeau-Cloria,MV*

- Dirichlet-Neumann
- Neumann-Neumann
  - Does not work $M = \frac{1}{2}S_f^{-1} + \frac{1}{2}S_s^{-1}$
  - Seams to work $M = \alpha_1 S_f^{-1} + \alpha_2 S_s^{-1}$ but $\alpha_1 \approx 0$

Standard Cures, Trends and Alternatives (cont)

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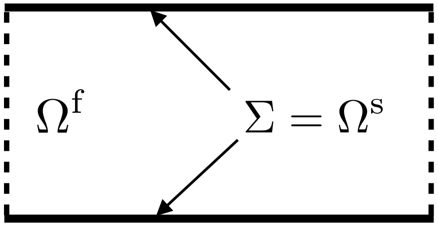
Main issue:

Stable and optimally accurate loosely coupled schemes (& mathematically sound)

Linear Model Problem with thin-structure

- Fluid: Stokes flow

$$\left\{\begin{aligned} \rho^{\mathrm{f}}\partial_t\boldsymbol{u} - \mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u},p) &= \mathbf{0} \quad \text{in} \quad \Omega^{\mathrm{f}}\\ \mathrm{div}\,\boldsymbol{u} &= 0 \quad \text{in} \quad \Omega^{\mathrm{f}}\\ \boldsymbol{u} &= \dot{\boldsymbol{d}} \quad \text{on} \quad \Sigma \end{aligned}\right.$$



- Thin-solid: shell

$$\left\{\begin{aligned} \begin{pmatrix}\rho^{\mathrm{s}}\epsilon\partial_t\dot{\boldsymbol{d}}\\ 0\end{pmatrix} - \begin{pmatrix}\boldsymbol{L}^{\mathrm{e}}_{\boldsymbol{d}}(\boldsymbol{d},\boldsymbol{\theta})\\ L^{\mathrm{e}}_{\boldsymbol{\theta}}(\boldsymbol{d},\boldsymbol{\theta})\end{pmatrix} &= \begin{pmatrix}-\boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{n}\\ \mathbf{0}\end{pmatrix} \quad \text{on} \quad \Sigma\\ \dot{\boldsymbol{d}} &= \partial_t\boldsymbol{d} \quad \text{on} \quad \Sigma \end{aligned}\right.$$

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☑ Splitting via displacement extrapolation:

$$\boldsymbol\sigma(\boldsymbol u^n,p^n)\boldsymbol n + \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\boldsymbol u^n = \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\dot{\boldsymbol d}^{n-1} - \boldsymbol L^{\mathrm{e}}_{\boldsymbol d}(\boldsymbol d^\star,\boldsymbol\theta^\star) \quad \text{on} \quad \Sigma\,, \qquad \boldsymbol d^\star = \begin{cases}\boldsymbol d^{n-1}\\ \boldsymbol d^{n-1} + \tau\dot{\boldsymbol d}^{n-1}\end{cases}$$

Robin-Neumann Loosely Coupled Schemes

1) Solve fluid:

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\rho^{\mathrm{f}} \partial_\tau \boldsymbol{u}^n - \mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{u}^n, p^n) = \mathbf{0} & \text{in} \quad \Omega^{\mathrm{f}} \\
\mathrm{div}\, \boldsymbol{u}^n = 0 & \text{in} \quad \Omega^{\mathrm{f}} \\
\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} + \dfrac{\rho^{\mathrm{s}}\epsilon}{\tau}\boldsymbol{u}^n = \dfrac{\rho^{\mathrm{s}}\epsilon}{\tau}\dot{\boldsymbol{d}}^{n-1} - \boldsymbol{L}^{\mathrm{e}}_{\boldsymbol{d}}(\boldsymbol{d}^\star, \boldsymbol{\theta}^\star) & \text{on} \quad \Sigma
\end{cases}
$$

2) Solve solid:

$$
\begin{cases}
\begin{pmatrix} \rho^{\mathrm{s}}\epsilon\partial_\tau \dot{\boldsymbol{d}}^n \\ 0 \end{pmatrix} - \begin{pmatrix} \boldsymbol{L}^{\mathrm{e}}_{\boldsymbol{d}}(\boldsymbol{d}^n, \boldsymbol{\theta}^n) \\ L^{\mathrm{e}}_{\boldsymbol{\theta}}(\boldsymbol{d}^n, \boldsymbol{\theta}^n) \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} \\ \mathbf{0} \end{pmatrix} & \text{on} \quad \Sigma \\
\dot{\boldsymbol{d}}^n = \partial_\tau \boldsymbol{d}^n & \text{on} \quad \Sigma
\end{cases}
$$

Robin-Neumann Loosely Coupled Schemes

1) Solve fluid:

$$\begin{cases} \rho^{\mathrm{f}}\partial_\tau \boldsymbol{u}^n - \mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n) = \mathbf{0} \quad \text{in} \quad \Omega^{\mathrm{f}} \\ \operatorname{div} \boldsymbol{u}^n = 0 \quad \text{in} \quad \Omega^{\mathrm{f}} \\ \boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} + \dfrac{\rho^{\mathrm{s}}\epsilon}{\tau}\boldsymbol{u}^n = \dfrac{\rho^{\mathrm{s}}\epsilon}{\tau}\dot{\boldsymbol{d}}^{n-1} - \boldsymbol{L}^{\mathrm{e}}_{\boldsymbol{d}}(\boldsymbol{d}^\star, \boldsymbol{\theta}^\star) \quad \text{on} \quad \Sigma \end{cases}$$

2) Solve solid:

$$\begin{cases} \begin{pmatrix} \rho^{\mathrm{s}}\epsilon\partial_\tau\dot{\boldsymbol{d}}^n \\ 0 \end{pmatrix} - \begin{pmatrix} \boldsymbol{L}^{\mathrm{e}}_{\boldsymbol{d}}(\boldsymbol{d}^n, \boldsymbol{\theta}^n) \\ L^{\mathrm{e}}_{\boldsymbol{\theta}}(\boldsymbol{d}^n, \boldsymbol{\theta}^n) \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} \\ \mathbf{0} \end{pmatrix} \quad \text{on} \quad \Sigma \\ \dot{\boldsymbol{d}}^n = \partial_\tau \boldsymbol{d}^n \quad \text{on} \quad \Sigma \end{cases}$$

Remarks:

- Semi-implicit coupling scheme which becomes explicit (thin-solid model)

Alternative Formulations (same method)

- Explicit Robin-Neumann coupling:

$$\left\{\begin{aligned} \boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} + \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\boldsymbol{u}^n &= \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\dot{\boldsymbol{d}}^{n-1} - \boldsymbol{L}^{\mathrm{e}}\boldsymbol{d}^{\star} \quad \text{on} \quad \Sigma \\ \rho^{\mathrm{s}}\epsilon\partial_\tau\dot{\boldsymbol{d}}^n + \boldsymbol{L}^{\mathrm{e}}\boldsymbol{d}^n &= -\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} \quad \text{on} \quad \Sigma \end{aligned}\right.$$

Alternative Formulations (same method)

- Explicit Robin-Neumann coupling:

$$\left\{\begin{aligned} \boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} + \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\boldsymbol{u}^n &= \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\dot{\boldsymbol{d}}^{n-1} - \boldsymbol{L}^{\mathrm{e}}\boldsymbol{d}^{\star} \quad \text{on} \quad \Sigma \\ \rho^{\mathrm{s}}\epsilon\partial_\tau\dot{\boldsymbol{d}}^{n} + \boldsymbol{L}^{\mathrm{e}}\boldsymbol{d}^n &= -\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} \quad \text{on} \quad \Sigma \end{aligned}\right.$$

omit rotations

Alternative Formulations (same method)

- Explicit Robin-Neumann coupling:

$$\left\{\begin{aligned} \boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} + \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\boldsymbol{u}^n &= \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\dot{\boldsymbol{d}}^{n-1} - \boldsymbol{L}^{\mathrm{e}}\boldsymbol{d}^{\star} \quad \text{on} \quad \Sigma\\ \rho^{\mathrm{s}}\epsilon\partial_{\tau}\dot{\boldsymbol{d}}^{n} + \boldsymbol{L}^{\mathrm{e}}\boldsymbol{d}^{n} &= -\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} \quad \text{on} \quad \Sigma \end{aligned}\right.$$

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- Robin based kinematic relaxation:

$$\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} + \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\boldsymbol{u}^n = \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\left(\dot{\boldsymbol{d}}^{n-1} + \tau\partial_\tau\dot{\boldsymbol{d}}^\star\right) + \boldsymbol{\sigma}(\boldsymbol{u}^\star, p^\star)\boldsymbol{n} \quad \text{on} \quad \Sigma$$

Alternative Formulations (same method)

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$$\left\{\begin{aligned} \boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} + \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\boldsymbol{u}^n &= \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\dot{\boldsymbol{d}}^{n-1} - \boldsymbol{L}^{\mathrm{e}}\boldsymbol{d}^{\star} \quad \text{on} \quad \Sigma \\ \rho^{\mathrm{s}}\epsilon\partial_\tau\dot{\boldsymbol{d}}^n + \boldsymbol{L}^{\mathrm{e}}\boldsymbol{d}^n &= -\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} \quad \text{on} \quad \Sigma \end{aligned}\right.$$

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- *Incremental* displacement-correction:

$$\frac{\rho^{\mathrm{s}}\epsilon}{\tau}\left(\dot{\boldsymbol{d}}^n - \boldsymbol{u}^n\right) + \boldsymbol{L}^{\mathrm{e}}\left(\boldsymbol{d}^n - \boldsymbol{d}^{\star}\right) = 0 \quad \text{on} \quad \Sigma$$

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Remark:

For $\boldsymbol{d}^{\star} = \boldsymbol{0}$ (*non-incremetal* displacement-correction) and membrane we retrieve the kinematically coupled scheme *(Guidoboni et al. '09)*

Stability and Accuracy: Main Principle

(M. Fernández '11, '12)

- These loosely coupled schemes enforce:

$$\begin{cases} \rho^{\mathrm{s}}\epsilon\partial_\tau \dot{\boldsymbol{d}}^n + \boldsymbol{L}^{\mathrm{e}}\boldsymbol{d}^n = -\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} & \text{on} \quad \Sigma \\ \dfrac{\rho^{\mathrm{s}}\epsilon}{\tau}\left(\dot{\boldsymbol{d}}^n - \boldsymbol{u}^n\right) + \boldsymbol{L}^{\mathrm{e}}\left(\boldsymbol{d}^n - \boldsymbol{d}^\star\right) = 0 & \text{on} \quad \Sigma \end{cases}$$

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$\Big\}$ **implicit kinetic coupling**

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$$\Big\} \text{ implicit kinetic coupling}$$

$$\Downarrow$$

$$\boldsymbol{u}^n = \dot{\boldsymbol{d}}^n + \frac{\tau}{\rho^{\mathrm{s}}\epsilon}\boldsymbol{L}^{\mathrm{e}}\big(\boldsymbol{d}^n - \boldsymbol{d}^\star\big) \quad \text{on} \quad \Sigma$$

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$$\Downarrow$$

$$\boldsymbol{u}^n = \dot{\boldsymbol{d}}^n + \frac{\tau}{\rho^{\mathrm{s}}\epsilon}\boldsymbol{L}^{\mathrm{e}}\big(\boldsymbol{d}^n - \boldsymbol{d}^\star\big) \quad \text{on} \quad \Sigma \Big\}$$

kinematic perturbation of implicit coupling

Stability and Accuracy: Main Principle

(M. Fernández '11, '12)

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$$\left\{\begin{aligned} \rho^{\mathrm{s}}\epsilon\partial_\tau\dot{\boldsymbol{d}}^n + \boldsymbol{L}^{\mathrm{e}}\boldsymbol{d}^n &= -\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} \quad \text{on} \quad \Sigma \;\Big\}\ \textbf{implicit kinetic coupling}\\ \boldsymbol{u}^n &= \dot{\boldsymbol{d}}^n + \frac{\tau}{\rho^{\mathrm{s}}\epsilon}\boldsymbol{L}^{\mathrm{e}}\left(\boldsymbol{d}^n - \boldsymbol{d}^\star\right) \quad \text{on} \quad \Sigma \;\Big\}\ \begin{array}{c}\textbf{kinematic perturbation of}\\ \textbf{implicit coupling}\end{array} \end{aligned}\right.$$

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- Key issue: how does this kinematic perturbation affect the stability and accuracy of the 'underlying' implicit coupling scheme?

Stability and Accuracy: Main Principle

(M. Fernández '11, '12)

- These loosely coupled schemes enforce:

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- Key issue: how does this kinematic perturbation affect the stability and accuracy of the 'underlying' implicit coupling scheme?

Remark:

The size of the perturbation depends on the displacement extrapolation

$$\boldsymbol{d}^\star = \boldsymbol{0} \qquad \text{(sub-optimal?)}$$

$$\boldsymbol{d}^\star = \left\{\begin{array}{l} \boldsymbol{d}^{n-1} \\ \boldsymbol{d}^{n-1} + \tau\dot{\boldsymbol{d}}^{n-1} \end{array}\right. \qquad \text{(optimal?)}$$

Stability: A Priori Energy Estimates

- Energy-norm:

$$E^n \overset{\text{def}}{=} \frac{\rho^{\mathrm{f}}}{2}\|\boldsymbol{u}^n\|^2_{0,\Omega^{\mathrm{f}}} + \frac{\rho^{\mathrm{s}}\epsilon}{2}\|\dot{\boldsymbol{d}}^n\|^2_{0,\Sigma} + \frac{1}{2}\|\boldsymbol{d}^n\|^2_{\mathrm{e}}$$

Proposition:

For $n \geq 1$, there holds

$$E^n \lesssim E^0 \quad \begin{cases} \text{if} \quad \boldsymbol{d}^\star = \boldsymbol{0} \\ \text{if} \quad \boldsymbol{d}^\star = \boldsymbol{d}^{n-1} \\ \text{if} \quad \begin{cases} \boldsymbol{d}^\star = \boldsymbol{d}^{n-1} + \tau\dot{\boldsymbol{d}}^{n-1} \\ \tau\omega_{\mathrm{e}}^{\frac{6}{5}} = \mathcal{O}\big(h^{\frac{6}{5}}\big) \end{cases} \end{cases}$$

with $\omega_{\mathrm{e}} \overset{\text{def}}{=} \sqrt{\beta_{\mathrm{e}}/(\rho^{\mathrm{s}}\epsilon)}$.

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with $\omega_{\text{e}} \overset{\text{def}}{=} \sqrt{\beta_{\text{e}}/(\rho^{\text{s}}\epsilon)}$.

Remarks:

- Incremental 1st-order extrap. unconditionnally stable
- Incremental 2nd-order extrap. stable under 6/5-CFL condition
- Stability independent of the added-mass effect

Convergence: A Priori Error Estimates

- Energy-norm error:

$$e^n \overset{\text{def}}{=} \sqrt{\frac{\rho^{\mathrm{f}}}{2}\|\boldsymbol{u}^n - \boldsymbol{u}(t_n)\|^2_{0,\Omega^{\mathrm{f}}} + \frac{\rho^{\mathrm{s}}\epsilon}{2}\|\dot{\boldsymbol{d}}^n - \dot{\boldsymbol{d}}(t_n)\|^2_{0,\Sigma} + \frac{1}{2}\|\boldsymbol{d}^n - \boldsymbol{d}(t_n)\|^2_{\mathrm{e}}}$$

Proposition:
For smooth enough solutions and $n \geq 1$, there holds:

$$e^n \lesssim h^k + \tau + \frac{\beta_{\mathrm{e}}}{\sqrt{\rho^{\mathrm{s}}\epsilon}} \cdot \begin{cases} \tau^{\frac{1}{2}} & \text{if} \quad \boldsymbol{d}^\star = \boldsymbol{0} \\ \tau & \text{if} \quad \boldsymbol{d}^\star = \boldsymbol{d}^{n-1} \\ \tau^2 & \text{if} \quad \boldsymbol{d}^\star = \boldsymbol{d}^{n-1} + \tau\dot{\boldsymbol{d}}^{n-1} \end{cases}$$

with $k \geq 1$ the convergence order of the Stokes-projection.

Convergence: A Priori Error Estimates

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with $k \geq 1$ the convergence order of the Stokes-projection.

Remarks:

- Non-incremental: expected sub-optimal time accuracy
- Incremental: overall optimal accuracy
- Splitting error constant depends on physical parameters

Generalization to non-linear

☑ Fluid: Navier-Stokes (ALE formalism)

☑ Solid: non-linear shell (complete strain tensor)

Ambiguity in the computation of the **second order extrapolation**

$$\begin{aligned} \boldsymbol{d}^\star &= \boldsymbol{d}^{n-1} + \tau \dot{\boldsymbol{d}}^{n-1} = 2\boldsymbol{d}^{n-1} - \boldsymbol{d}^{n-2} \\ \boldsymbol{\theta}^\star &= 2\boldsymbol{\theta}^{n-1} - \boldsymbol{\theta}^{n-2} \end{aligned}$$

$$\boldsymbol{L}^{\mathrm{e}}_{\boldsymbol{d}}((\boldsymbol{d}^\star, \boldsymbol{\theta}^\star)) = \begin{cases} \boldsymbol{L}^{\mathrm{e}}_{\boldsymbol{d}}((2\boldsymbol{d}^{n-1} - \boldsymbol{d}^{n-2}, 2\boldsymbol{\theta}^{n-1} - \boldsymbol{\theta}^{n-2})) \\ 2\boldsymbol{L}^{\mathrm{e}}_{\boldsymbol{d}}((\boldsymbol{d}^{n-1}, \boldsymbol{\theta}^{n-1})) - \boldsymbol{L}^{\mathrm{e}}_{\boldsymbol{d}}\boldsymbol{d}((\boldsymbol{d}^{n-2}, \boldsymbol{\theta}^{n-2})) \end{cases}$$

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$$\boldsymbol{L}^{\mathrm{e}}_{\boldsymbol{d}}((\boldsymbol{d}^\star, \boldsymbol{\theta}^\star)) = \begin{cases} \boldsymbol{L}^{\mathrm{e}}_{\boldsymbol{d}}((2\boldsymbol{d}^{n-1} - \boldsymbol{d}^{n-2}, 2\boldsymbol{\theta}^{n-1} - \boldsymbol{\theta}^{n-2})) & \longleftarrow \text{does not work} \\ 2\boldsymbol{L}^{\mathrm{e}}_{\boldsymbol{d}}((\boldsymbol{d}^{n-1}, \boldsymbol{\theta}^{n-1})) - \boldsymbol{L}^{\mathrm{e}}_{\boldsymbol{d}}\boldsymbol{d}((\boldsymbol{d}^{n-2}, \boldsymbol{\theta}^{n-2})) \end{cases}$$

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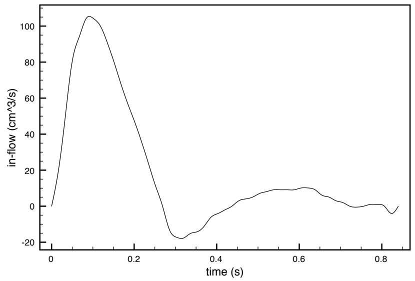
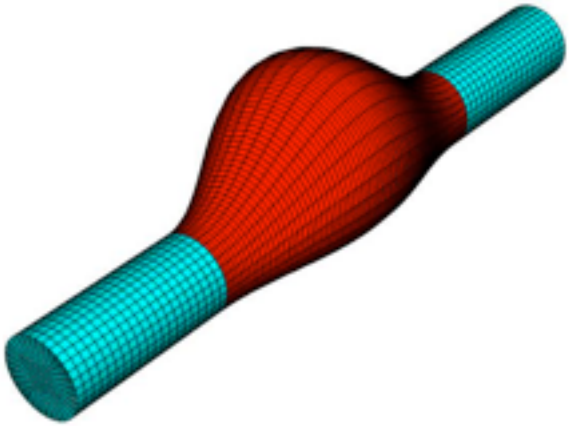
does not work

Our choice

ALE Navier-Stokes/Linear-Shell: In-vitro Abdominal Aortic Aneuvrism

- In-vitro abdominal aortic aneurysm:





$\boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{n} = -R_{\text{out}}Q_{\text{out}}\boldsymbol{n}$

$\boldsymbol{u} = \boldsymbol{u}_{in}$

(Salsac et al. ‘05)

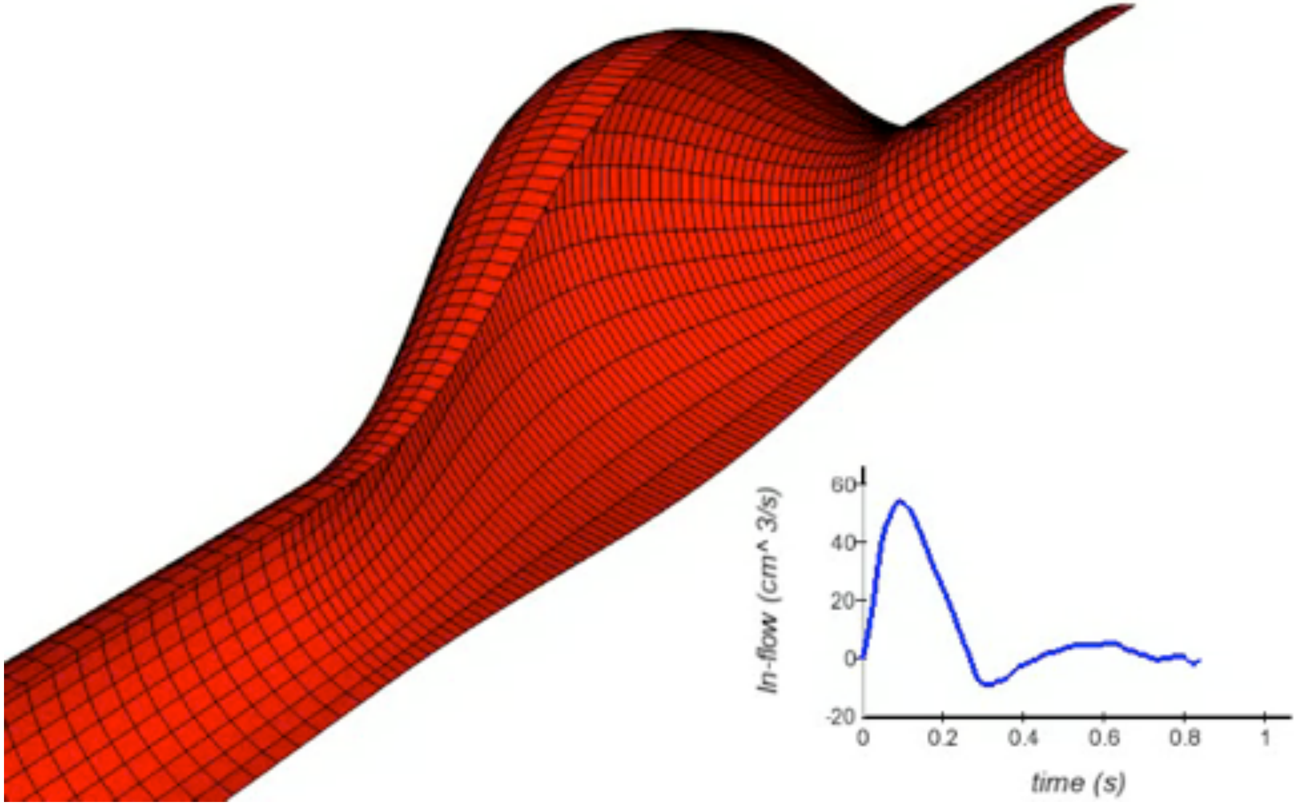
- Thin-solid: based on general shell elements
  (Chapelle, Bathe ’01)

- Physical data:

$$
\begin{aligned}
\epsilon &= 0.17\,\text{cm} & \rho^{\text{f}} &= 1.1\,\text{g}/\text{cm}^3\\
\rho^{\text{s}} &= 1.2\,\text{g}/\text{cm}^3 & \mu &= 0.035\,\text{P}\\
E &= 6\times 10^6\,\text{dyne}/\text{cm}^2 & R_{\text{out}} &= 300\,\text{dyne}\,\text{s}/\text{cm}^5\\
\nu &= 0.3
\end{aligned}
$$

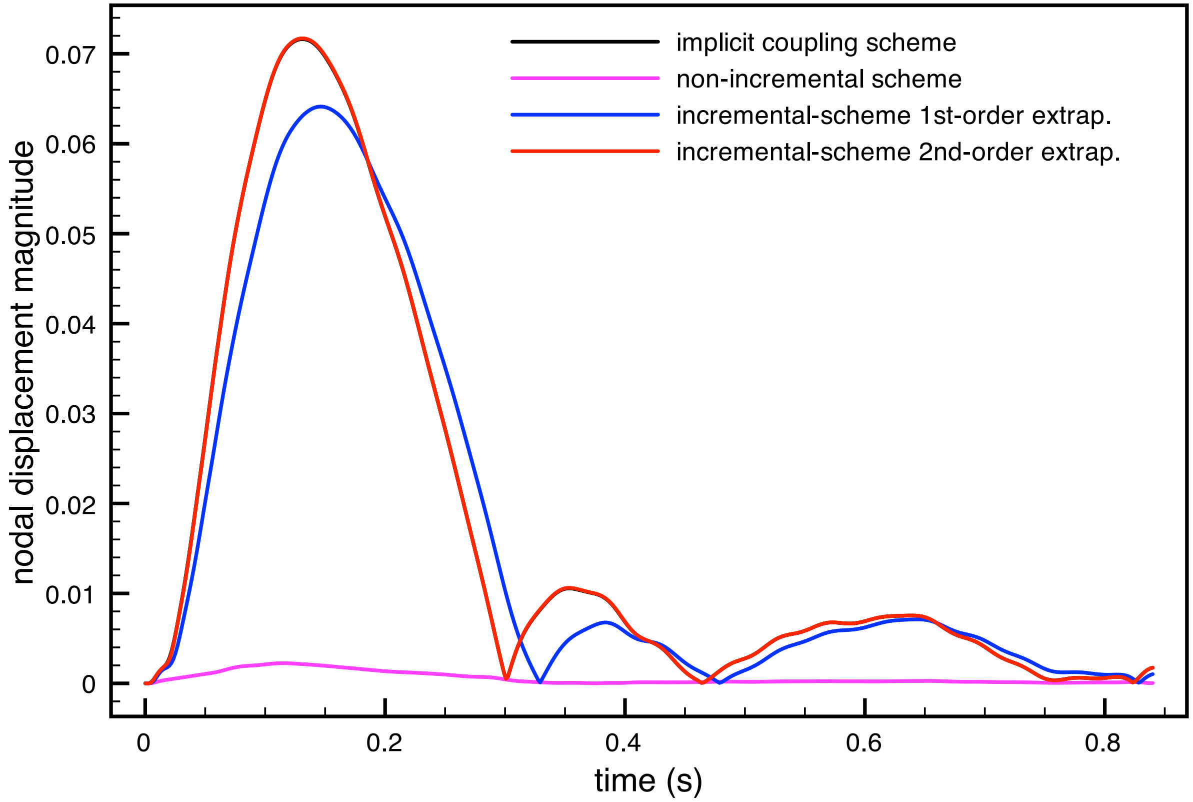
- Space discretization: MITC4 for the solid, $\mathbb{Q}_1/\mathbb{Q}_1$ stabilized for the fluid

ALE Navier-Stokes/Linear-Shell (cont.)



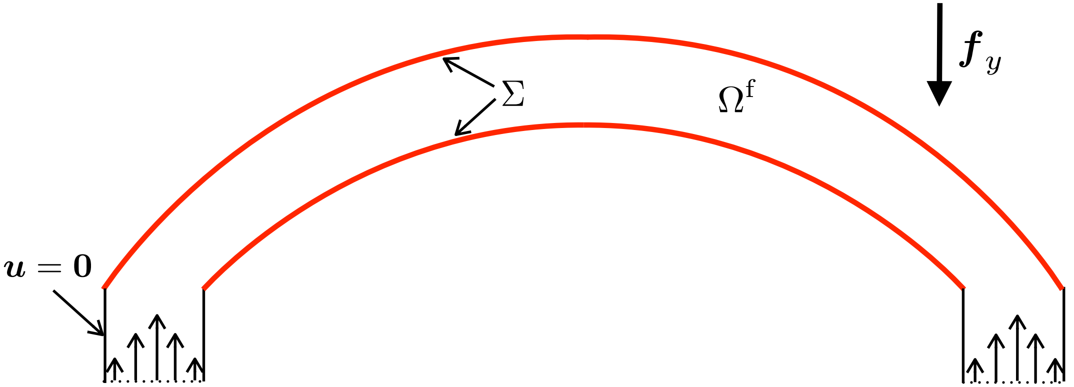
Accuracy

- $\tau = 4.2 \times 10^{-4}$



ALE Navier-Stokes/Non-Linear Shell: Inflating balloon

- Inflating balloon problem (incompressible ‘dilemma’):



(Wall et al. ’06)

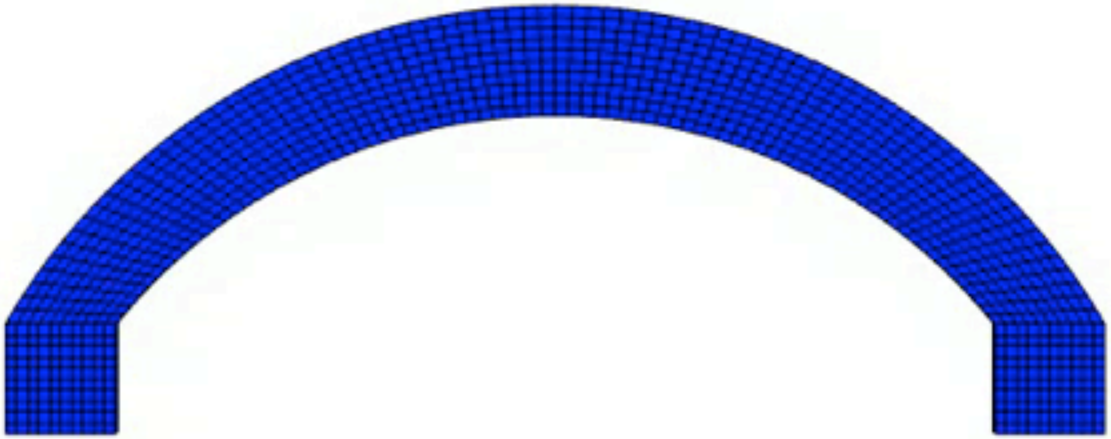
- Thin-solid: based on general shell elements
  (Chapelle, Bathe ’01)
- Physical data:

$$\begin{aligned} \epsilon &= 0.1\,\mathrm{m} & E_{\mathrm{top}} &= 9\times 10^5 \mathrm{N/m^2} & \rho^{\mathrm{f}} &= 1\,\mathrm{kg/m^3} \\ \rho^{\mathrm{s}} &= 500\,\mathrm{kg/m^3} & E_{\mathrm{bottom}} &= 9\times 10^8 \mathrm{N/m^2} & \mu &= 9\,\mathrm{Pa\,s} \\ \nu &= 0.3 \end{aligned}$$

- Space discretization: MITC4 for the solid, $\mathbb{Q}_1/\mathbb{Q}_1$ stabilized for the fluid

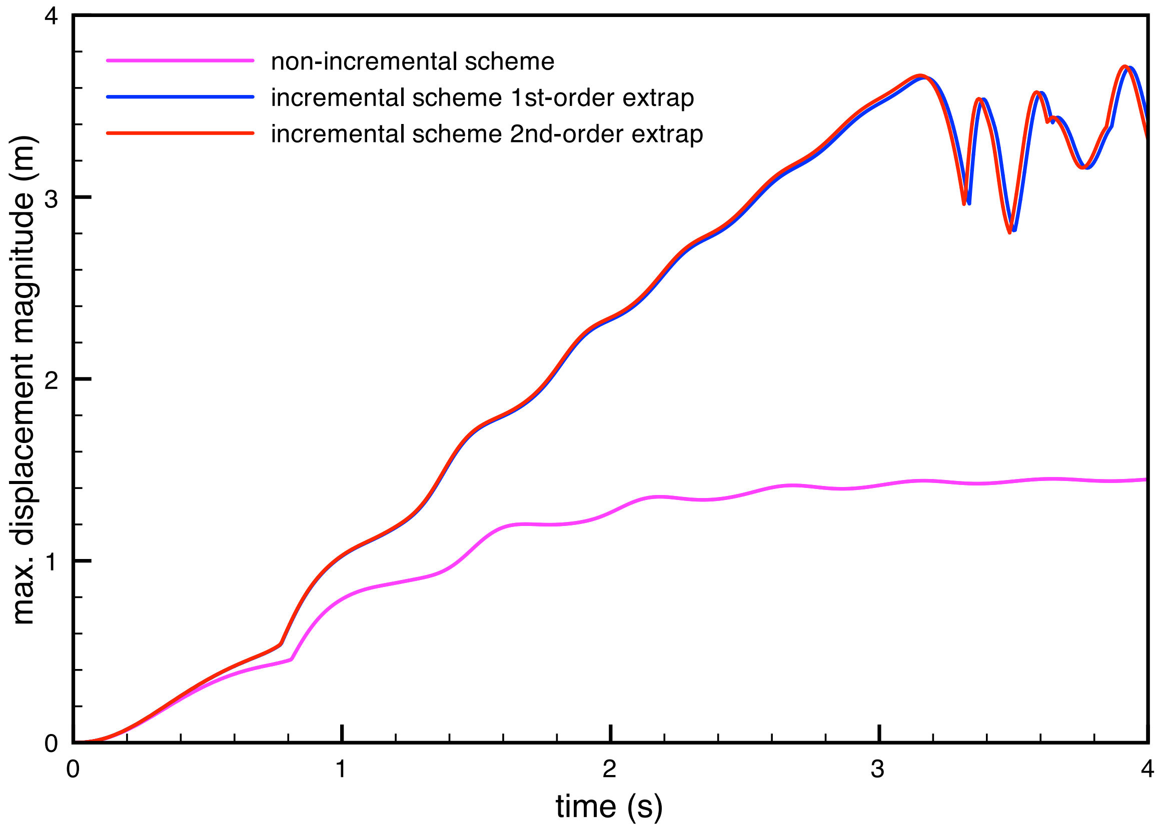
ALE Navier-Stokes/Non-Linear Shell (cont.)

ALE Navier-Stokes/Non-Linear Shell (cont.)



Accuracy

- $\tau = 0.1$



Solid Damping Effects (Viscoelasticity)

$$\rho^{\mathrm{s}}\epsilon\partial_t\dot{\boldsymbol{d}} + \boldsymbol{L}^{\mathrm{e}}\boldsymbol{d} + \boldsymbol{L}^{\mathrm{v}}\dot{\mathrm{d}} = -\boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{n} \quad \text{on} \quad \Sigma$$

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Proposition: *(Fernández, Mullaert, MV '13)*

For $n \geq 1$, there holds

$$E^n \lesssim E^0 + \tau^2\|\dot{\boldsymbol{d}}^0\|_{\mathrm{e}}^2 + \frac{\tau^2}{\rho^{\mathrm{s}}\epsilon}\|\boldsymbol{L}^{\mathrm{e}}\boldsymbol{d}^0 + \boldsymbol{L}^{\mathrm{v}}\dot{\boldsymbol{d}}^0\|_{0,\Sigma}^2.$$

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Remarks:

- Explicit treatment of damping does not compromise unconditional stability
- Implicit treatment of damping (kinematically coupled scheme)

$$\boldsymbol{\sigma}(\boldsymbol{u}^n,p^n)\boldsymbol{n} + \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\boldsymbol{u}^n + \boldsymbol{L}^{\mathrm{v}}\boldsymbol{u}^n = \frac{\rho^{\mathrm{s}}\epsilon}{\tau}\dot{\boldsymbol{d}}^{n-1} \quad \text{on} \quad \Sigma$$

yields a non-standard Robin condition *(Guidoboni et al. '09)*

Parallelization of the solid and fluid solvers

- Design algorithms **well suited for parallel computing** : **use domain decomposition**

  □ Additive Schwarz for the fluid (PETSCI)
  ☑ **BDD** Balanced domain decomposition method for the solid solver

Alternatives for the solid solver

- **FETI** (finite element tearing and interconnect) *Farhat, Roux*
- **BDDC** (balancing domain decomposition by constraints) *C. R. Dohrmann*

Revisit domain decomposition (linear elasticity)

Solve by an iterative method the primal Shur complement (interface problem)

$$\begin{aligned} S\bar{X} &= \bar{F} \\ S &= \sum_i \mathbf{R}^{i^t}\mathbf{S}^i\mathbf{R}^i \end{aligned}$$

Neumann-Neumann preconditioner (*Bourgat, Glowinski, Le Tallec, De Roeck, MV*)

$$\mathbf{M}^{-1} = (\sum_i \mathbf{D}^i\tilde{\mathbf{S}}_i^{-1}\mathbf{D}^{i^t})$$

Balanced domain decomposition brilliant idea *J. Mandel '92 '93*

Instead of tacking **arbitrary rigid bodies** in the solution of the Neumann problems choose them in order **to minimize residual of the next iteration**

Balanced Domain Decomposition Method

Particular case of the additive Schwarz method applied to interface problem

$$(\sum_i \mathbf{R}^{i^t}\mathbf{S}^i\mathbf{R}^i)\bar{X} = \bar{F}$$

Define:

- the space of global interface values $\mathbf{V} = \{\bar{v} = Tr\ v|_\Gamma, v \in \mathbf{H}(\Omega)\}$
- a partition of unity $\mathbf{D}^i : Tr\mathbf{V}_{|\Gamma_i} \to \mathbf{V}$
- an approximate local operator $\tilde{\mathbf{S}}^i$ st $\tilde{\mathbf{S}} = \sum \mathbf{R}^{i^t}\tilde{\mathbf{S}}^i\mathbf{R}^i.$
- a $\tilde{\mathbf{S}}^i$orthogonal decomposition $Tr\mathbf{V}_{|\Gamma_i} = \mathbf{V}_i \oplus \mathbf{Z}_i.$

Neumann-Neumann : **additive Schwarz** (solving S on V)

i) $\mathbf{V}_0 = \sum_{i=1}^N \mathbf{D}^i\mathbf{Z}_i \subset \mathbf{V}$, (scalar product $\tilde{\mathbf{S}}$), coarse space

ii) $\mathbf{V}_i$ (scalar product $\mathbf{B}_i = \tilde{\mathbf{S}}^i$) local spaces

iii) $I_i = (I - \mathbf{P})\mathbf{D}^i$ $\mathbf{P}$ the $\tilde{\mathbf{S}}$ orth projection of $\mathbf{V} \to \mathbf{V}_0$. (extensions)

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The BDD preconditioner

$$\mathbf{M}^{-1}\mathbf{S}\bar{u} = \mathbf{P}\bar{u} + (\mathbf{I}-\mathbf{P})(\sum_i \mathbf{D}^i \tilde{\mathbf{S}}_i^{-1} \mathbf{D}^{i^t})(\mathbf{I}-\mathbf{P})^t \mathbf{S}\bar{u}$$

Remarks :

- coarse space contains **local singularities** (rigid bodies....)
- standard N-N : S exact Schur interface operator $(\mathbf{I}-\mathbf{P})^t\mathbf{S}\bar{u} = \mathbf{S}\bar{u}$
- interest to use an **approximate local operator** easy to generalize the method

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Nonlinear elasticity

☑ use a Newton Algorithm
☑ use domain decomposition for each linearized problem
☑ construct the preconditioner once for the first linearised problem and reuse it

Time dependent problems

☑ use a Newmark or Euler time discretisation
☑ solve by domain decomposition at each time step
☑ construct the preconditioner using the rigid bodies of the linearized stiffness

Time dependent problem (cont)

After discretization in space, at each time step solve the **non-linear** problem

$$\frac{\rho M}{\Delta t^2} u + \mathcal{G}(u) = rhs$$

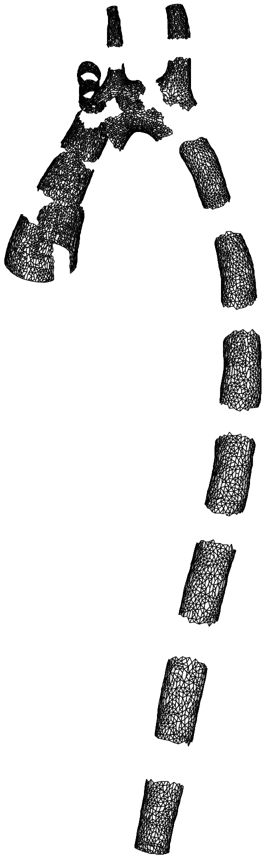
Use **Newton** algorithm, at each iteration solve

$$(\frac{\rho M}{\Delta t^2} + K^n) u = rhs$$

Remarks :

- the presence of the **mass-matrix** regularize the linear system
- thus the size of the **coarse space** based on **rigid bodies** is **zero**
- more **robust** approach **coarse space** based on the **stiffness matrix** only

Coarctated Aorta Blood Flow



Simulation by M. Landajuela

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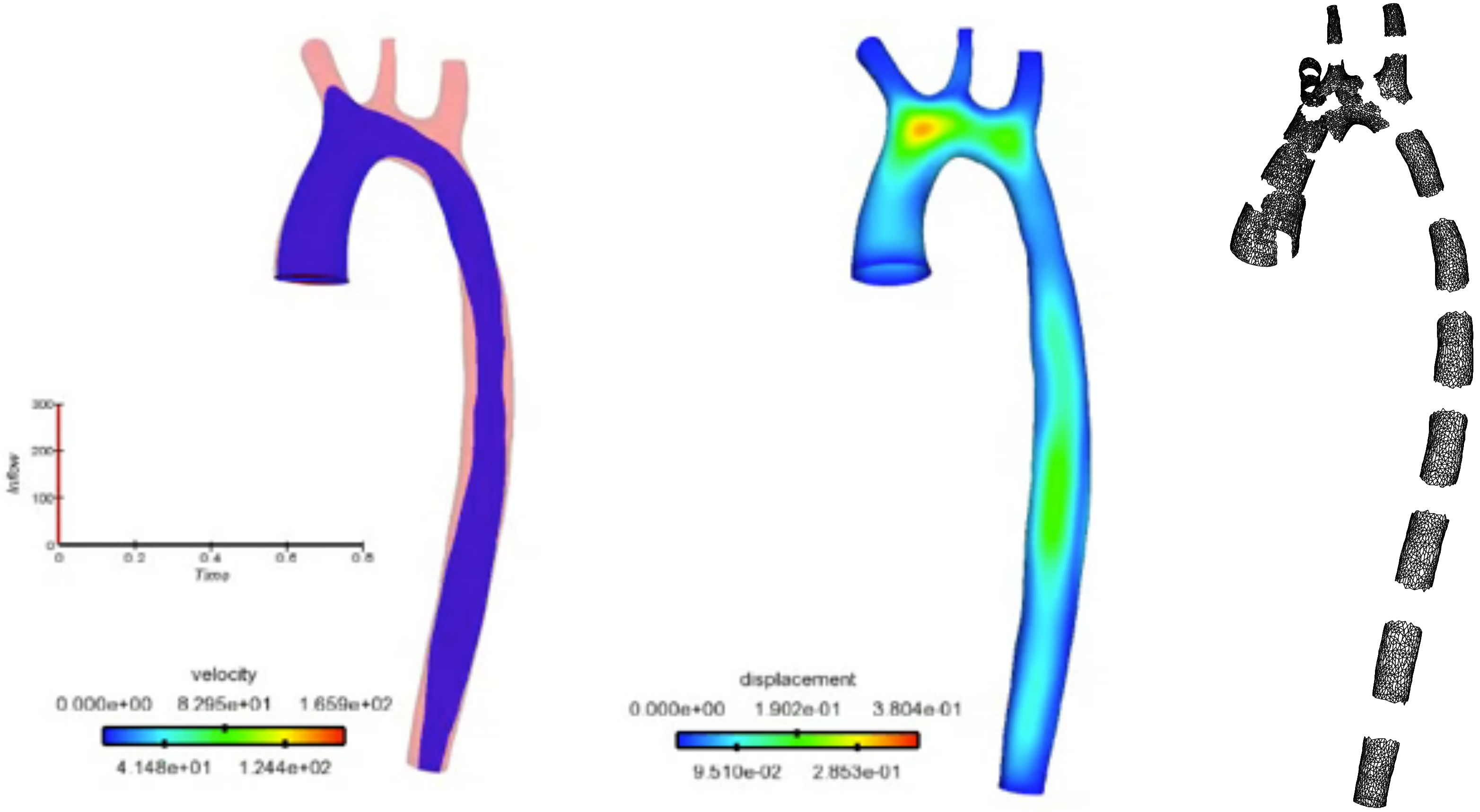

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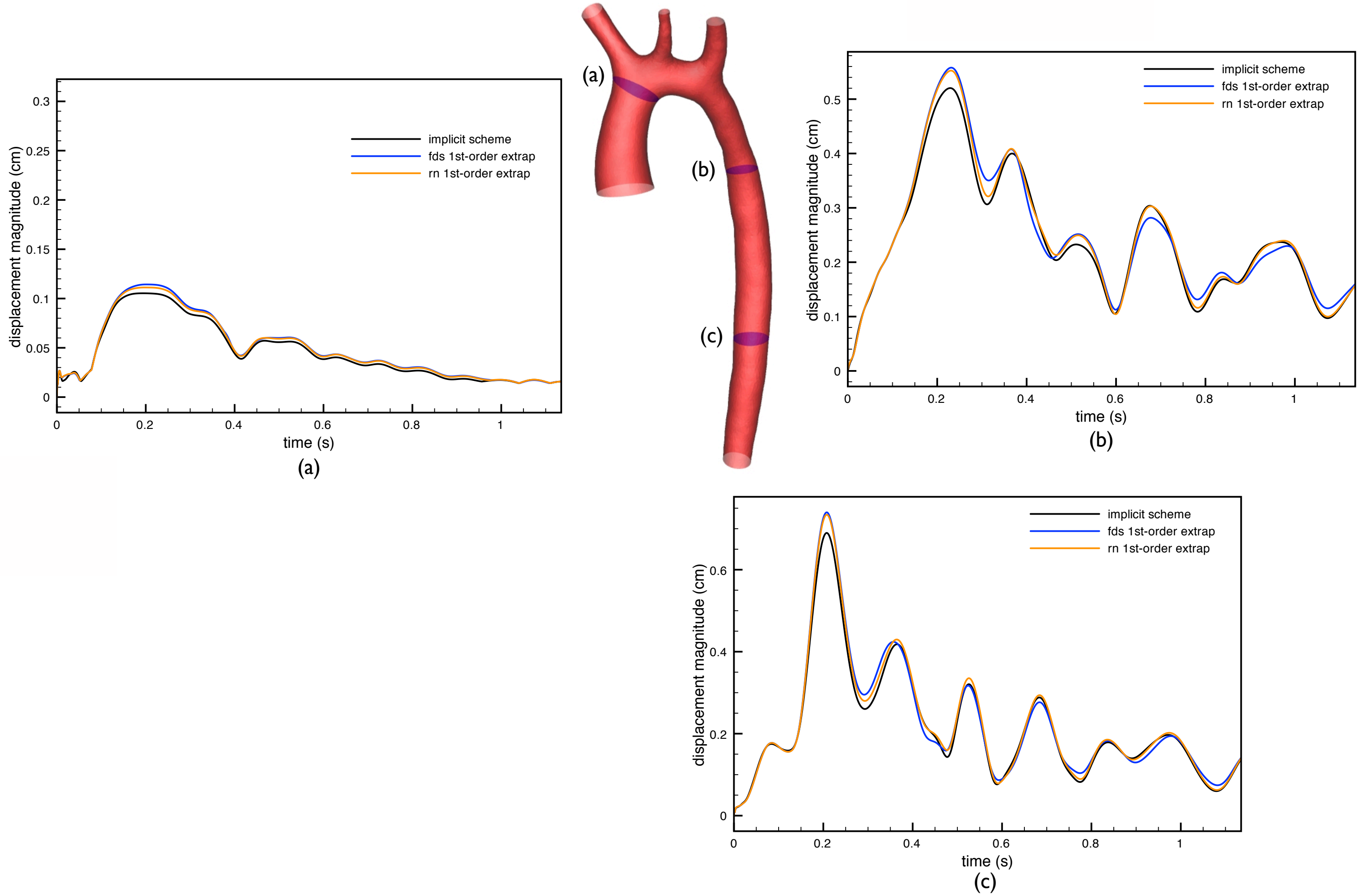
Accuracy: Explicit *vs.* Implicit



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Partitioned Solution Of Implicit Coupling

- Explicit Robin-Neumann coupling:

$$\begin{cases} \boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} + \dfrac{\rho^{\mathrm{s}}\epsilon}{\tau}\boldsymbol{u}^n = \dfrac{\rho^{\mathrm{s}}\epsilon}{\tau}\left(\dot{\boldsymbol{d}}^{n-1} + \tau\partial_\tau\dot{\boldsymbol{d}}^{\star}\right) + \boldsymbol{\sigma}(\boldsymbol{u}^{\star}, p^{\star})\boldsymbol{n} \quad \text{on} \quad \Sigma \\ \rho^{\mathrm{s}}\epsilon\partial_\tau\dot{\boldsymbol{d}}^{n} + \boldsymbol{L}^{\mathrm{e}}\boldsymbol{d}^n = -\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} \quad \text{on} \quad \Sigma \end{cases}$$

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- Single iteration of a Robin-Neumann implicit solution algorithm *(Badia, Nobile, Vergara '08)*:

$$\begin{cases} \boldsymbol{\sigma}(\boldsymbol{u}_k, p_k)\boldsymbol{n} + \alpha\boldsymbol{u}_k = \alpha\dot{\boldsymbol{d}}_{k-1} + \boldsymbol{\sigma}(\boldsymbol{u}_{k-1}, p_{k-1})\boldsymbol{n} \quad \text{on} \quad \Sigma \\ \rho^{\mathrm{s}}\epsilon\partial_\tau\dot{\boldsymbol{d}}_k + \boldsymbol{L}^{\mathrm{e}}\boldsymbol{d}_k = -\boldsymbol{\sigma}(\boldsymbol{u}_k, p_k)\boldsymbol{n} \quad \text{on} \quad \Sigma \end{cases}$$

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Proposition: *(Fernández, Mullaert, MV '13)*

$$\sum_{k=1}^{\infty} e_k \leq \tau\|\boldsymbol{d}_0 - \boldsymbol{d}^n_{\mathrm{imp}}\|^2_{\mathrm{e}} + \frac{\tau^2}{\rho^{\mathrm{s}}\epsilon}\|\boldsymbol{L}^{\mathrm{e}}(\boldsymbol{d}_0 - \boldsymbol{d}^n_{\mathrm{imp}})\|^2_{0,\Sigma}.$$

Concluding Remarks

☑ Domain Decomposition is a powerful tool to solve large multi-scale problems

- Heterogenous methods adapted to be design ***parallel scalable algorithms***
- Heterogenous methods well suited for ***FSI***

☑ Stable explicit coupling schemes based on a built-in Robin interface consistency

- Only solid-inertia needs to be implicitly coupled with the fluid
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☑ Robin-Neumann schemes with interesting features:

- Stability (added-mass free)
- Optimally first-order accurate (coupling with thin-solid)
- Kinematic perturbations of implicit coupling (fundamental for the analysis)
- Single iteration of a strong coupling solution procedure
- Parameter free

Joyeux anniversaire, Alain!!!

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Une retraite heureuse!

Joyeux anniversaire, Alain!!!

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Le travail c'est la santé

Joyeux anniversaire, Alain!!!

Le travail c'est la santé

Bonne chance dans ta nouvelle vie!