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# ANISOTROPIC GOAL-ORIENTED MESH OPTIMISATION

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**Abstract.** *We present some examples of mesh adaptive simulations for Fluid Mechanics. The common features are (1) unstructured meshes (2) metric based mesh adaptation. Two families of mesh adaptive methods are applied. First the treatment of transient free boundary flows is addressed with a fixed point transient mesh adaptive algorithm. The simulation of a wave impacting a dam demonstrates the interest of the method. Second, a combination of Hessian-based and goal-oriented adaptation is applied to the propagation of a steady sonic-boom wave.*

## 1 Introduction

Many process to be simulated in environment studies involve large and small scales. Although many systems with very different scales can be addressed by modeling, many other systems will not be accurately modelled without multi-scale discretisation. An important class of multi-scale discretisation is the class of mesh adaptive methods. A considerable progress in mesh adaptation techniques has resulted from a more accurate definition of the adapted mesh. This evolution has taken two forms. First, in the case of the minimisation of an interpolation error, Hessian-based methods assume that mesh is modeled by a local metric and then optimal metrics are obtained by calculus of variation, see for example [1, 3]. Second, the error to reduce can be defined as the error on a scalar functional and adjoint-based methods deliver goal-oriented criteria, e.g. [2].

## 2 $L^2$ Optimal metric

We investigate the specification of a mesh $\mathcal{H}_{opt}$ that is optimal for the $P_1$-interpolation of a given function $u$ on a domain $\Omega$ in $R^n$.

Find $\mathcal{H}_{opt}$ containing $N$ vertices such that $\mathcal{H}_{opt}(u) = \text{Arg}\ \min_{\mathcal{H}} \|u - \Pi_h u\|_{\mathcal{H},\mathbf{L}^2(\Omega)}$ .

In [1, 2, 3] this is formulated in a completely continuous standpoint: a mesh $\mathcal{H}$ is modelled continuously by a $n \times n$ field $\mathcal{M}$ on $\Omega$ called **metric** and a continuous analog $\pi_{\mathcal{M}}$ of the linear interpolation error is defined for this metric, $\mathcal{H}$: $u - \pi_{\mathcal{M}} u$. Let us solve:

Find $\mathcal{M}_{opt}$ of complexity $\mathcal{C}(\mathcal{M}) = N$ such that $\mathcal{E}_{\mathcal{M}_{opt}}(u) = \min\limits_{\mathcal{M}} \|u - \pi_{\mathcal{M}} u\|_{\mathcal{M},\mathbf{L}^p(\Omega)}$ ,

where the metric complexity is given by $\mathcal{C}(\mathcal{M}) = \int_\Omega \sqrt{\det(\mathcal{M}(\mathbf{x}))}\, \mathrm{d}\mathbf{x}$. The local error model for such metric simplifies to:

$$(u - \pi_{\mathcal{M}} u)(\mathbf{x}) = \sum_{i=1}^{n} h_i^2(\mathbf{x}) \left| \frac{\partial^2 u}{\partial {\alpha_i}^2}(\mathbf{x}) \right| ,$$

where $\Lambda_{ii} = \dfrac{\partial^2 u}{\partial {\alpha_i}^2}$ stands for the eigenvalues of the Hessian $H_u = \mathcal{R}_u \Lambda^t \mathcal{R}_u$ in the direction of the $i^{th}$ Hessian's eigenvectors and $n$ is the space dimension. Now, we are looking for the function $\mathcal{M}$ that minimizes, for a given complexity $N$, the $\mathbf{L}^2$-norm of this error:

$$\min_{\mathcal{M}} \mathcal{E}(\mathcal{M}) \ = \ \min_{\mathcal{M}} \left( \int_\Omega (e_{\mathcal{M}}(\mathbf{x}))^2 \ \mathrm{d}\mathbf{x} \right)^{\frac{1}{2}} \text{ under the constraint } \mathcal{C}(\mathcal{M}) = N \ . \tag{1}$$

Problem (1) is solved by variational calculus which gives an optimum $\bar{\mathcal{M}}_N$ and the corresponding (minimal) error $\mathcal{E}(\bar{\mathcal{M}}_N)$. Imposing the error $\mathcal{E} = \varepsilon$ instead of the number of nodes gives the following optimal metric:

$$\hat{\mathcal{M}}(\varepsilon) = D_{\mathbf{L}^2} \left(\det |H_u|\right)^{-\frac{1}{4+n}} \mathcal{R}_u \ |\Lambda| \ {}^t\mathcal{R}_u \ \text{ with } D_{\mathbf{L}^2} = \frac{n}{\varepsilon} \left( \int_\Omega (\det |H_u|)^{\frac{2}{4+n}} \right)^{-\frac{1}{2}} . \tag{2}$$

## 3 Transient optimal metric

We need to choose a space-time norm of the error. Our option is $L^\infty(O,T;L^2(\Omega))$. To solve the non-linear problem of mesh adaptation for unsteady simulation, the Transient Fixed Point method proposed in [5] and improved in [4] is applied. It relies on a coarse subdivision of the time interval. On each subinterval a fixed point is applied to obtained a metric that is adapted to all the solutions of the interval. An illustration of this method is presented in Fig.1, where the floshing of water on a small cubic obstacle is computing accurately during a long time.

## 4 Goal-oriented optimal metric

We consider the steady Euler equations, written in short:

$$W \in V = \left[H^1(\Omega)\right]^5, \ \forall \phi \in V, \quad (\Psi(W)\,,\,\phi) = \int_\Omega \phi\, \nabla.\mathcal{F}(W)\ \mathrm{d}\Omega - \int_\Gamma \phi\, \hat{\mathcal{F}}(W).\mathbf{n}\ \mathrm{d}\Gamma = 0$$

where $\Gamma$ is the boundary of $\Omega$ and $\hat{\mathcal{F}}$ holds for boundary fluxes giving the various boundary conditions. We approximate this with a mixed-element-volume appoximation:

$$\forall \phi_h \in V_h, \quad \int_{\Omega_h} \phi_h \nabla.\Pi_h \mathcal{F}(W_h)\, \mathrm{d}\Omega_h - \int_{\Gamma_h} \phi_h \Pi_h \hat{\mathcal{F}}(W_h).\mathbf{n}\, \mathrm{d}\Gamma_h = - \int_{\Omega_h} \phi_h\, D_h(W_h) \mathrm{d}\Omega_h,$$

where $\Pi_h$ is the usual elementwise linear interpolation and where $D_h$ holds for a numerical dissipation term. Starting from the following **adjoint-based** error estimate:

$$(g, W - W_h) \approx ((\Psi_h - \Psi)(W), P), \text{ with } \left[\frac{\partial \Psi}{\partial W}\right]^* P = g,$$

the optimal mesh is obtain after some transformations by solving:

$$\text{Find } \mathcal{M}_{opt} = \text{Argmin}_{\mathcal{M}} \int_{\Omega} |\nabla P| |\mathcal{F}(W) - \pi_{\mathcal{M}} \mathcal{F}(W)| d\Omega + \int_{\Gamma} |P| |(\bar{\mathcal{F}}(W) - \pi_{\mathcal{M}} \bar{\mathcal{F}}(W)).\mathbf{n}|\, d\Gamma$$

under the constraint $\mathcal{C}(\mathcal{M}) = N$. We present in Figure 2 an application to sonic boom prediction. The left part shows an adaptation based on the Hessian of Mach field. The right part uses the proposed adjoint-based method with a $g$ function concentrated on the bottom of domain, in order to maximize the accuracy of prediction at ground level.

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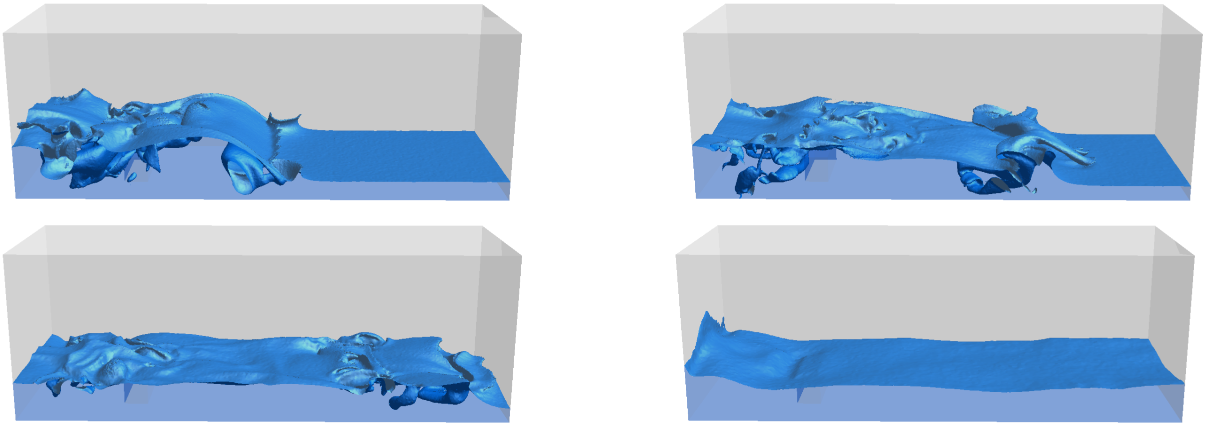

Figure 1: wave sloshing in a basin with cubic obstacle at different times.

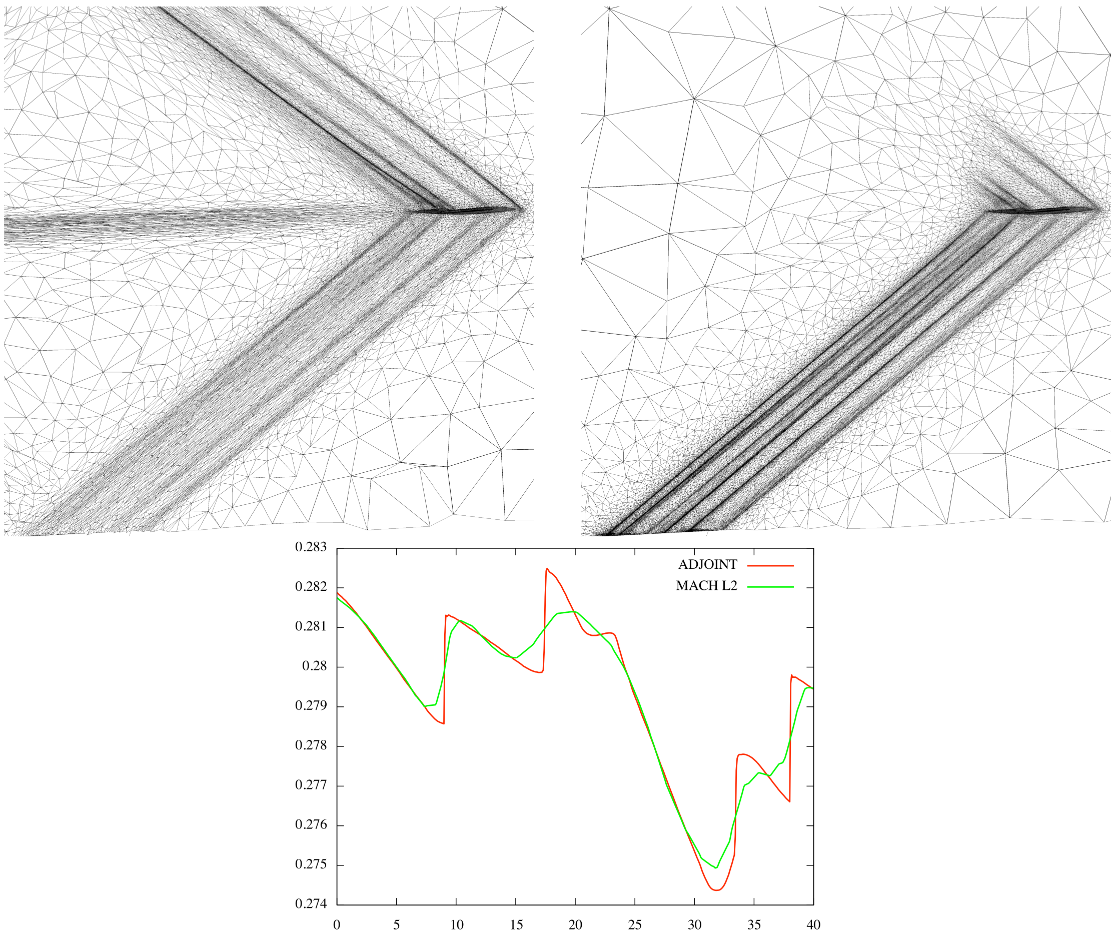


Figure 2: two strategies for sonic boom prediction and the pressures at ground level.