The effect of consistent coarse grid in Schwarz algorithms

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Abstract: Our midterm purpose is to develop 2-level Schwarz solution algorithms for solving the compressible Navier-Stokes equations. Before this we consider the introduction of an algebraic coarse grid based on deflation or balancing method on diffusion-convection models. We study the issue of the consistency of the coarse grid and its influence on scalability.

Keywords: Domain Decomposition, Schwarz, Computational Fluid Dynamics.

1 Introduction

The solution of steady and unsteady compressible Navier-Stokes equations is produced rather efficiently by applying an additive Schwarz algorithm combined with an ILU local preconditioner, see [1]. This algorithm is of rather good numerical scalability, *i.e.* a computation with $2n$ processors and $2N$ unknowns is run in about the same time as a computation with n processors and N unknowns. But with the availability of a large number of cores, it appears necessary to improve this scalability towards a quasi perfect one. Historically, the additive Schwarz method was early identified as a no-scalable method for elliptic problems, and S. Brenner shown [2] that adding a coarse finite element grid in the preconditioner can help recovering a perfect scalability. This idea is inspired by the Multigrid method and the control of the convergence of both grids to a common continuous limit. In fact, for Domain Decomposition Methods, building the coarse grid in such a way that it approximates the continuous solution is not always mandatory for scalability. Further, consistent coarse grids are difficult to build. An attempt is refered as smoothed aggregation methods.

The proposed study examines the influence of the coarse grid consistency on convergence and scalability for Poisson and convection-diffusion problems.

2 Model problems and baseline Schwarz method

The two test problem we concentrate on are the following ones. The first is inspired by a pressure-correction phase in Navier-Stokes, and expresses as a Neumann problem with strongly discontinuous coefficient and writes:

$$-\nabla\cdot\frac{1}{\rho}\nabla u = RHS \text{ in } \Omega \qquad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \qquad u(0) = 0.$$

in which the well-posedness is fixed with a Dirichlet condition on one cell.

The second is a classical diffusion-convection problem:

$$-\nu\nabla\cdot\nabla u + \frac{\partial u}{\partial x} = RHS \text{ in } \Omega \qquad \frac{\partial u}{\partial n} = 0 \text{ for } x = 0, x = 10 \qquad u = 0 \text{ for } y = 0, y = 1.$$

The third model (study in progress) is the compressible Navier-Stokes system.

Our discrete model relies on a vertex-centered formulation expressed on a triangulation. Let us assume that the computational domain Ω is split into two sub-domains, Ω_1 and Ω_2 , with an intersection $\overline{\Omega_1}\cap\overline{\Omega_2}$ with a thickness of at least one layer of elements. The *Additive Schwarz* algorithm is written in terms of preconditioning, as $M^{-1} = \sum_{i=1}^{2} A^{-1}_{|\Omega_i}$ where $A^{-1}_{|\Omega_i}$ holds for the Dirichlet problem on sub-domain Ω_i. The preconditioner M^{-1} can be used in a Krylov subspace method. In this paper, in order to keep some generality in our algorithms, we use a conjugate gradient for the symmetric cases and a GMRES for non-symmetric ones. In the *Additive Schwarz-ILU* version, the exact solution of the Dirichlet on each sub-domain is replaced by the less costly Incomplete Lower Upper (ILU) approximate solution.

3 Coarse grid system

We consider a fine-grid approximation space V_n of dimension n with basis functions $(\phi_i)_i$, giving a convergent approximation of the exact PDE solution u_{exact}, in short:

$$Au = b \;\;\Rightarrow\;\; \sum u_i\phi_i \rightarrow u_{exact} \text{ as } n\rightarrow\infty.$$

We assume we have a coarse basis $(\Phi_i)_i$ defining a coarse approximation space $V_N = [\Phi_1\cdots\Phi_N]$. The operator Z the column of which are the component of Φ_i in V_n is an extension operator from V_0 in V and Z^T a restriction operator from V in V_0. Then the so-called Galerkin-MG coarsening gives the following coarse system:

$$Z^TAZ\,U = Z^Tb.$$

An interesting question is the convergence of the coarse system: we assume that a constant ratio between n and N is maintained as $n\rightarrow\infty$. Does the coarse grid approximation produces or not a convergent approximation of the exact solution in the following sense:

$$\sum(ZU)_i\phi_i \rightarrow u_{exact} \text{ as } n\rightarrow\infty.$$

3.1 Smooth and non-smooth coarse grid

The coarse grid is then defined by set of basis functions. A central question is the smoothness of these functions. According to Galerkin-MG, smooth enough functions provide consistent coarse-grid solutions. Conversely, DDM methods preferably use the characteristic functions of the sub-domains, $\Phi_i(x_j) = 1 \;\; si \;\; x_j \in \;\; \Omega_i$. In the case of P^1 finite-elements, for example, the typical characteristic basis function corresponds to setting to 1 all degrees of freedom in sub-domain. According to [6], the resulting coarse system

$$U^H(x) = \Sigma_i U_i\Phi_i(x) \quad ; \quad \int \nabla U^H\nabla\Phi_i = \int f\Phi_i \quad \forall i$$

Figure 1: Left: characteristic coarse grid basis function. Right: smooth coarse grid basis function.

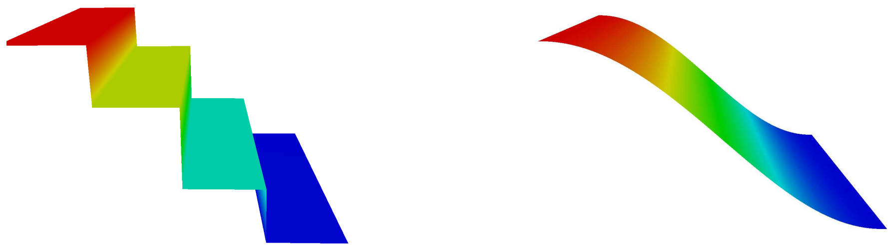

Figure 2: Accuracy of the coarse grid approximation for a Poisson problem with a *sin* function (of amplitude 2.) as exact solution. Left: coarse grid solution with the characteristic basis (amplitude is 0.06). Right: coarse grid solution with a smooth basis (amplitude is 1.8).

produces a solution U^H_{charac} which does not converge towards the continous solution U when H tends to 0.

In order to build a better basis, we need to introduce a hierarchical coarsening process from the fine grid to a coarse grid which will support the preconditioner. Level j is made of N_j macro-cells C_{jk}, *i.e.* $\mathcal{G}_j = \cup_{k=1}^{N_j} C_{jk}$. Transfer operators are defined between successive levels (from coarse to fine):

$$P_i^j \ : \ \mathcal{G}_j \rightarrow \mathcal{G}_i \qquad P_i^j(u)(C_{k'i}) = u(C_{kj}) \ \text{ with } \ C_{k'i} \subset C_{kj}$$

Following [6] we introduce the smoothing operator:

$$(L_k u)_i = \sum_{j \in \mathcal{N}(i) \cup \{i\}} \text{meas}(j)\ u_j / \{ \sum_{j \in \mathcal{N}(i) \cup \{i\}} \text{meas}(j) \}$$

where $\mathcal{N}(i)$ holds for the set of cells which are direct neigbors of cell i. The smoothing is applied at each level between the coarse level k defining the characteristic basis and the finest level.

$$\Psi_k = (L_1 P_1^2 L_2 \cdots P_{p-2}^{p-1} L_{p-1} P_{p-1}^{p}) \Phi_k.$$

The resulting smooth basis function is compared with the characteristic one in Figure 2. The inconsistency of the characteristic basis and the convergence of this new smooth basis is illustrated by the solution of a Poisson equation with a *sin* function as exact solution, Figure 3.

Conversely, first-order hyperbolic problems, like advection, allow both types of basis. This is illustrated by the solution of the diffusion convection problem with a Peclet of 100, and an upwind fine approximation. For the fine approximation the mesh numerical Peclet is $1/2$ and the approximation solution is free of oscillation, Fig.3a. The characteristic basis produces a not so bad approximation (Fig.3b) We force the smooth coarse basis to satisfy the Dirichlet boundary conditions. Since the mesh numerical

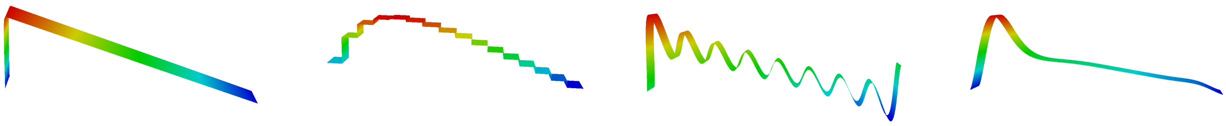

Figure 3: Accuracy of the coarse grid approximation for an advection-diffusion problem: (a) fine grid solution, (b) coarse solution with characteristic basis, (c) coarse solution with smooth basis,(d) coarse solution with smooth basis and numerical viscosity.

Peclet is now much larger, the solution oscillates (Fig.3c). We have tried to moderate the oscillation by means of a coarse-grid numerical viscosity, built with the difference between the coarse mass matrix and its lumped version (sum of each line concentrated on the diagonal term)(Fig.3d).

4 Introduction of coarse grid

There are many ways to introduce the coarse grid system for accelerating a Schwarz algorithm, see for example [4]. The deflation method [7] splits the problem into a coarse one solved once for all and an (ill-posed) complementary one. In the Balancing Decomposition (BD) as in [5, 3], the coarse and complementary problem are resolved iteratively, with some extra cost with respect to deflation. In the present paper we use the BD approach of [8]. Further, a coarse grid can be introduced accelerating the solution in each subdomain.

5 Some preliminary outputs

The smooth-coarse grid has been compared with the characteristic one inside a BD preconditioner. The smooth-coarse option is faster and more scalable (Tab.1) than the characteristic one for the elliptic model. The difference is not clear for the Peclet-100 advection-diffusion model. One difficulty is the insufficient local resolution by ILU which induces a plateau in the convergence. The introduction of coarse-grid dissipation did not carry any improvement.

The smooth-coarse grid has been introduced at the same time as (1) a coarse grid and (2) a domain-by domain medium grid. Only elliptic cases have been run yet. The improvement is important and compares well with the case of a two-level Schwarz with exact solution in subdomains (Tab.2). Further results are in progress, in particular with an Euler model, and will be presented at the conference.

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Table 1: Scalability of the two-level Additive-Schwarz-ILU method

# Cells	10K	20K	47K	94K
Domains	12	28	66	142
# Cells/domain	833	714	712	661
Char. basis (# it.)	480	546	750	810
Smooth basis (# it.)	400	391	444	491

Table 2: Scalability for the Schwarz, two-level Schwarz and three level Schwarz-ILU

Method	# cells	# sub-domains	# medium basis funct	Iterations
Schwarz	40,000	4		320
Schwarz	160,000	16		451
Two-level Schwarz	40,000	4		130
Two-level Schwarz	160,000	16		212
Three level ILU-Schwarz	40,000	4	64	164
Three level ILU-Schwarz	160,000	16	256	176

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