Hilbert bases and feasible RHS for flow problems

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Abstract

In order to solve network design and multicommodity flow problems we want to be able to decompose them. The decomposition must be done in such a way that any solution of the original problem can be expressed as the sum of solutions of smaller problems, repetition been allowed. Conversely the sum of solutions of smaller problems arising in the decomposition is a solution of the original problem. Moreover any optimal solution with respect to a given objective function is the sum of optimal solutions over domains arising in the decomposition with respect to the objective function. For this approach we consider that the topology of the network is given. It is represented by vertex-arc incidence matrix of the supply graph. We characterize all the demand-capacity vectors that we need in order to be able to decompose over the domains involved by these vectors any routing or network design problem.

The approach that we propose is based on the notion of extended fibers, we call extended fiber of a given matrix A in b the following discrete set $Q_b^I = \{x \in \mathbb{Z},\ Ax = b\}$. A fiber Q_b^I is said decomposable if there exist b_1 and b_2 such that $b = b_1 + b_2$ and $Q_b^I = Q_{b_1}^I + Q_{b_2}^I$, otherwise it is said atomic. We show that the atomic extended fibers are in bijection with the vector of an Hilbert basis of the cone generated by the columns of A. Then we define truncated fibers of a matrix A with respect to a set M. The elements of a truncated fiber can't be expressed as the sum of an element of the fiber and an element of M. We show that any truncated fiber can be expressed as the sum of truncated atomic fibers.

In order to apply this material to flow problem we define an appropriate truncating set. We show that the Hilbert basis of the cone generated by the columns of vertex-arc incidence matrix of a graph is reduced to a subset of its columns. This implies that this Hilbert is rather small. Then starting from this point we show how this Hilbert can be easily extended to the Hilbert basis of the cone generated by the columns of the constraint matrix of a capacited flow problem. In fact we prove that we have a linear time, in the size of the output, in order to compute such a basis. At last we show how the result related to the flow problem can be directly extended to the multicommodity flow problem.

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