A Crash Course in Robust Optimization

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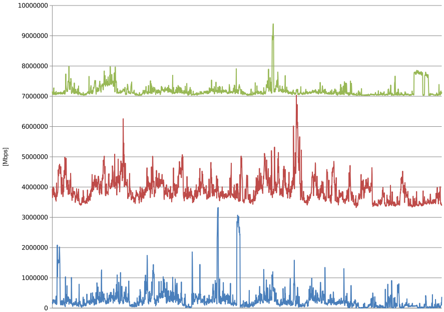
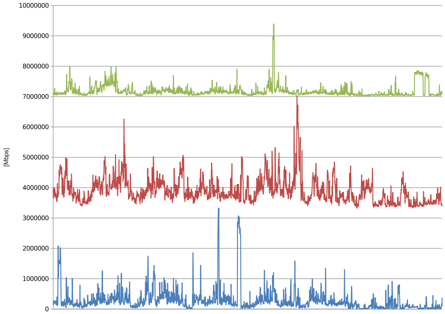
Lehrstuhl II für Mathematik

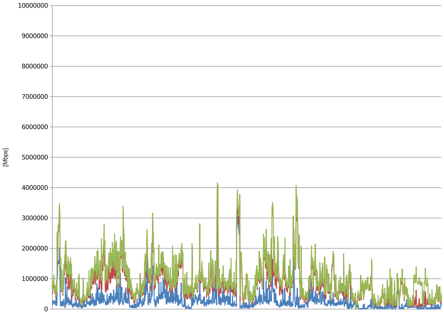


Demand uncertainties Traffic fluctuations in the US abilene Internet2 network in time intervals of 5 minutes during one week:

- Traffic fluctuates heavily between node-pairs

How robust is a network design?





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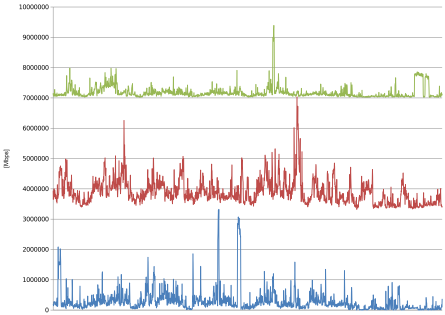

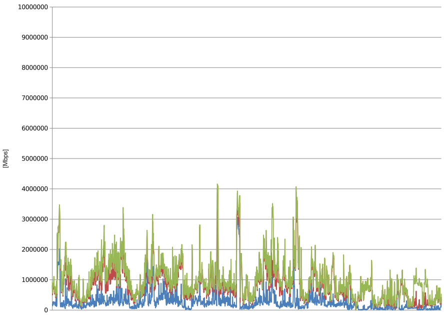

Demand uncertainties Traffic fluctuations in the US abilene Internet2 network in time intervals of 5 minutes during one week:

- Traffic fluctuates heavily between node-pairs
- Load of links will fluctuate alike
- To avoid congestion, demand is overestimated by, e.g., $\geq$ 300%
- Can we do better?

Lower overestimation

The network is designed such that capacities are as small as possible; traffic fluctuations might result in high network congestion

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Network design has to be computed for many scenarios; high computational effort

Alternative Approaches

Lower overestimation

The network is designed such that capacities are as small as possible; traffic fluctuations might result in high network congestion

Stochastic Programming

Network design has to be computed for many scenarios; high computational effort

Multi-period Network Design

Many traffic matrices have to be considered simultaneously; high computational effort

Chance-constrained Optimization Problem (COP)

Find among all solutions that satisfy all constraints with high probability a solution with optimal objective value.

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  - Quite challenging to solve resulting problems
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- Robust Optimization
  - Uncertainty comes from a known set, the *uncertainty set*
  - No information on probability distribution needed
  - Seeks for solution with best worst-case objective guarantee

Chance-Constrained Linear Programming

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

with Entries of A, b and/or c are not constant but random variables

Chance-Constrained Linear Programming with joint constraints

$$
\begin{aligned}
\min \quad & c^T x \\
\text{s.t.} \quad & \mathcal{P}\left(Ax \leq b\right) \geq 1 - \epsilon \\
& x \geq 0
\end{aligned}
$$

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Chance-Constrained Linear Programming with individual constraints

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & \mathcal{P}\left(A_i x \leq b_i\right) \geq 1 - \epsilon_i \qquad \forall i = 1, \ldots, m \\ & x \geq 0 \end{aligned}$$

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Knapsack with n Items, profits c_i, uncertain weights a_i, and capacity b

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How to solve this problem?

Assumption: Weights are independently and normally distributed with expectation m_i and standard deviation σ_i.

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Example (cont.)

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$$\mathcal{P}\left(\sum_{i=1}^{n} a_i x_i \leq b\right) = \mathcal{P}\left(\frac{\sum_{i=1}^{n}(a_i x_i - m_i x_i)}{\sqrt{\sum_{i=1}^{n}\sigma_i^2 x_i^2}} \leq \frac{b - \sum_{i=1}^{n} m_i x_i}{\sqrt{\sum_{i=1}^{n}\sigma_i^2 x_i^2}}\right)$$

$$=$$

Example (cont.)

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$$= \mathcal{P}\left(Z \leq \frac{b - \sum_{i=1}^{n} m_i x_i}{\sqrt{\sum_{i=1}^{n}\sigma_i^2 x_i^2}}\right)$$

with $Z = \frac{\sum_{i=1}^{n}(a_i x_i - m_i x_i)}{\sqrt{\sum_{i=1}^{n}\sigma_i^2 x_i^2}}$

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\end{aligned}$$

with $Z = \frac{\sum_{i=1}^{n}(a_i x_i - m_i x_i)}{\sqrt{\sum_{i=1}^{n}\sigma_i^2 x_i^2}}$

Let $\Phi(.)$ be the cumulative distribution function of the standard normal distribution. Then,

$$\frac{b - \sum_{i=1}^{n} m_i x_i}{\sqrt{\sum_{i=1}^{n}\sigma_i^2 x_i^2}} \geq \Phi^{-1}(1 - \epsilon)$$

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If $1-\epsilon > 0.5$, $\Phi^{-1}(1-\epsilon) > 0$

Example (cont.)

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If $1-\epsilon > 0.5$, $\Phi^{-1}(1-\epsilon) > 0$ and the chance constrained knapsack can be reformulated as

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} c_i x_i \\ \text{s.t.} \quad & \Phi^{-1}(1-\epsilon)\sqrt{\sum_{i=1}^{n} \sigma_i^2 x_i^2} + \sum_{i=1}^{n} a_i x_i \leq b \\ & x \in \{0,1\}^n \end{aligned}$$

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After relaxing the integrality of x, a second order cone problem remains, which can be solved in polynomial time.

Observation

In the example, normal distribution of the weights was assumed. What if, the weights are distributed differently, or unknown?

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Uncertain Linear Program

An Uncertain Linear Optimization problem (ULO) is a collection of linear optimization problems (instances)

$$\left\{\min\{c^T x : Ax \leq b\}\right\}_{(c,A,b)\in\mathcal{U}}$$

where all input data stems from an uncertainty set $\mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^{m\times n} \times \mathbb{R}^m$.

ULO $\left\{\min\{c^T x : Ax \leq b\}\right\}_{(c,A,b)\in\mathcal{U}}$

Robust feasible solution

A vector $x \in \mathbb{R}^n$ is robust feasible for ULO if

$$Ax \leq b \qquad \forall (c, A, b) \in \mathcal{U}$$

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Given a vector $x \in \mathbb{R}^n$, the robust solution value $\hat{c}(x)$ is defined as

$$\hat{c}(x) := \sup_{(c,A,b)\in\mathcal{U}} c^T x$$

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Robust Counterpart

The robust counterpart of an ULO is the optimization problem

$$\min\{\hat{c}(x) : x \text{ is robust feasible}\}$$

Let $\Big\{\min\{c^T x : Ax \leq b, x \geq 0\}\Big\}_{(c,A,b)\in\mathcal{U}}$ be an ULO with uncertain right-hand-side

uncertain matrix A,

but certain objective vector c.

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The robust counterpart can be written as

$$\min\{c^T x : (\bar{A} + \hat{A})x \leq \bar{b}, x \geq 0\}$$

Observation

If the objective is certain, the robust counterpart can be constructed row-wise, i.e.,

- keep the objective
- replace every constraint $a_i^T x \leq b_i$ by its robust counterpart

$$a_i^T x \leq b_i \qquad \forall (a_i, b_i) \in \mathcal{U}_i$$

where

$$\mathcal{U}_i := \left\{ (\tilde{a}_i, \tilde{b}_i) \in \mathbb{R}^{n+1} : \exists (A, b) \in \mathcal{U} \text{ with } A_{i\cdot} = \tilde{a}_i, b_i = \tilde{b}_i \right\}$$

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Note: the robust counterpart does not change if $\hat{\mathcal{U}} = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_m$ instead of $\mathcal{U}$ is used.

Corollary

If only the right hand side b is uncertain, the robust counter part reads

$$Ax \leq \bar{b}$$

with $\bar{b}_i = \min\{b_i : (A, b, c) \in \mathcal{U}\}$.

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Challenge: Find another way to handle right-hand-side uncertainty.

- Minoux [16] considers

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instead of

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- Problem is NP-hard for commonly used uncertainty sets [17, 18]
- Intermediate solutions required!

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  equivalent: set of discrete scenarios (extreme points of polyhedron)
  special case: Γ-Robustness

Robust Optimization Approach

Simplifying assumption: b and c are certain

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Uncertainty Set by Bertsimas & Sim [5, 6]: Let $\bar{a}_{ij} \in \mathbb{R}$, $\hat{a}_{ij} \geq 0$ be given, and $\Gamma \in \mathbb{R}_+$ a parameter.

$$\mathcal{U}_i(\Gamma) = \{a_i \in \mathbb{R}^n : a_{ij} = \bar{a}_{ij} + \hat{a}_{ij} z_{ij} \quad \forall j = 1, \ldots, n, \quad z_i \in \mathcal{Z}_i(\Gamma)\}$$

with

$$\mathcal{Z}_i(\Gamma) = \left\{ z_i \in \mathbb{R}^n : |z_{ij}| \leq 1 \quad \forall j = 1, \ldots, n, \quad \sum_{j=1}^{n} |z_{ij}| \leq \Gamma \right\}$$

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Stated otherwise:

- nominal values $\bar{a}_{ij}$ and deviations $\hat{a}_{ij}$
- $a_{ij} \in [\bar{a}_{ij} - \hat{a}_{ij}, \bar{a}_{ij} + \hat{a}_{ij}]$
- Sum of relative deviations from the nominal values is bounded by Γ

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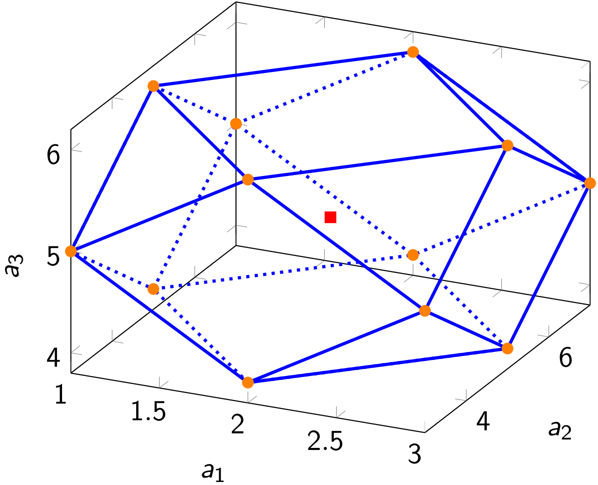
- $a_{ij} \in [\bar{a}_{ij} - \hat{a}_{ij}, \bar{a}_{ij} + \hat{a}_{ij}]$
- Sum of relative deviations from the nominal values is bounded by Γ
  At most Γ many entries might deviate from their nominal value

$$\bar{a} = \begin{pmatrix} 2 \\ 5 \\ 5 \end{pmatrix}, \; \hat{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \; \Gamma = 2$$



Provided by Manuel Kutschka

Robust Counterpart

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} c_i x_i \\
\text{s.t.} \quad & \sum_{j=1}^{n} \bar{a}_{ij} x_j + \max_{z_i \in \mathcal{Z}_i(\Gamma)} \left(\sum_{j=1}^{n} \hat{a}_{ij} z_{ij} x_j \right) \leq b_i \qquad i = 1, \dots, m \\
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Observation

Since $\mathcal{Z}_i$ defines a (bounded) polyhedron, only the extreme points have to be treated.

Robust Counterpart

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} c_i x_i \\ \text{s.t.} \quad & \sum_{j=1}^{n} \bar{a}_{ij} x_j + \max_{z_i \in \mathcal{Z}_i(\Gamma)} \left(\sum_{j=1}^{n} \hat{a}_{ij} z_{ij} x_j \right) \leq b_i \qquad i = 1, \ldots, m \\ & x \geq 0 \end{aligned}$$

Observation

Since $\mathcal{Z}_i$ defines a (bounded) polyhedron, only the extreme points have to be treated.

For $\Gamma \in \mathbb{Z}_+$:

$$\sum_{j=1}^{n} \bar{a}_{ij} x_j + \max_{S \subseteq \{1,\ldots,n\} : |S| \leq \Gamma} \left(\sum_{j \in S} \hat{a}_{ij} x_j \right) \leq b_i \tag{1}$$

Theorem 1 (Bertsimas & Sim [6])

Let $x^\star$ be an optimal solution of the Γ-robust counterpart. If a_{ij}, $j = 1, \ldots, n$, are independent and symmetric distributed random variables in $[\bar{a}_{ij} - \hat{a}_{ij}, \bar{a}_{ij} + \hat{a}_{ij}]$, then

$$\mathcal{P}\left(\sum_{i=1}^{n} a_i x_i^\star > b\right) \leq B(n, \Gamma)$$

with

$$\lim_{n\to\infty} B(n, \Gamma) = 1 - \Phi\left(\frac{\Gamma}{\sqrt{n}}\right)$$

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Instead of the limit: $B(n, \Gamma) \approx 1 - \Phi\left(\frac{\Gamma - 1}{\sqrt{n}}\right)$

Choice of Γ as a function of n so that the probability of constraint violation is less than p%:

	Γ				
n	$p = 5$	$p = 2$	$p = 1$	$p = 0.5$	$p = 0.1$
5	4.7	5.0	5.0	5.0	5.0
10	6.2	7.5	8.4	9.1	10.0
20	8.4	10.2	11.4	12.5	14.8
50	12.6	15.5	17.4	19.2	22.9
100	17.4	21.5	24.3	26.8	31.9
200	24.3	30.0	33.9	37.4	44.7
1,000	53.0	65.9	74.6	82.5	98.7
2,000	74.6	92.8	105.0	116.2	139.2

Note: Result is independent of actual distribution of random variables a_{ij}, only symmetry and independence are required.

Solving the robust counterpart

$$\sum_{j=1}^{n} \bar{a}_{ij} x_j + \max_{S \subseteq \{1,\ldots,n\} : |S| \leq \Gamma} \left(\sum_{j \in S} \hat{a}_{ij} x_j \right) \leq b_i$$

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Observations:

- Inequality (1) can be linearized by

$$\sum_{j \notin S} \bar{a}_{ij} x_j + \sum_{j \in S} (\bar{a}_{ij} + \hat{a}_{ij}) x_j \leq b_i \qquad \forall S \subseteq \{1, \ldots, n\}, |S| \leq \Gamma \quad (2)$$

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- This number of inequalities is exponential if $\Gamma = O(n)$
- Separation can be done in polynomial time
- Alternatively, a compact formulation can be obtained via dualization

Given $x^\star$, find a subset $S \subseteq \{1, \ldots, n\}$ with $|S| \leq \Gamma$ such that

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Given $x^\star$, find a subset $S \subseteq \{1, \ldots, n\}$ with $|S| \leq \Gamma$ such that

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Separation of robust inequalities

Given $x^\star$, find a subset $S \subseteq \{1, \ldots, n\}$ with $|S| \leq \Gamma$ such that

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Separation problem:

$$\begin{aligned} Z_{SEP} = \max \quad & \sum_{j=1}^{n} \hat{a}_{ij} x_j^\star z_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} z_j \leq \Gamma \\ & z_j \in \{0, 1\} \end{aligned}$$

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Optimization = Separation implies polynomial solvability of LP

Compact formulation

Let $\beta_i(x,\Gamma) = \max\limits_{S\subseteq\{1,\dots,n\}:|S|\leq\Gamma} \left(\sum\limits_{j\in S} \hat{a}_{ij}x_j\right)$, and hence (1) reads

$$\sum_{j=1}^{n} \bar{a}_{ij}x_j + \beta_i(x,\Gamma) \leq b_i$$

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or equivalently

$$\begin{aligned} &\sum_{j=1}^{n} \bar{a}_{ij} x_j + \Gamma \pi_i + \sum_{j=1}^{n} \rho_{ij} \leq b_i \\ &\pi_i + \rho_{ij} \geq \hat{a}_{ij} x_j \qquad\qquad \forall j = 1, \dots, n \\ &\pi_i \geq 0, \rho_{ij} \geq 0 \end{aligned}$$

Theorem 2 (Bertsimas & Sim [6])

If only the objective is uncertain and $x \in \{0,1\}^n$, then the robust counterpart can be solved by solving $n+1$ nominal problems of the same type.

Further results

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Corollary 3

The knapsack problem with uncertain objective can be solved in $O(n^2 B)$.

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Theorem 4 (Pferschy et al., 2012)

The knapsack problem with uncertain weights can be solved in $O(n \Gamma B)$.

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- Only marginal complexity increase (compared to deterministic case)

Disadvantages Robust Optimization:

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  $\Rightarrow$ More detailed description of uncertainty

Recoverable robustness [14, 7]

- uncertainty as two-stage process:

  1st stage: a-priori decision

  2nd stage: recovery:
  limited change of first-stage decision
  after realization of uncertainty is known

- optimize worst-case w. r. t. recovery

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Example:

Recoverable Robust Knapsack problem (RRKP) with

- Discrete Scenarios [9]
- Γ Scenarios [8]

Recoverable Robust Network Topology Design (discrete scenarios) [2]

(k, ℓ)-RRKP with Discrete Scenarios

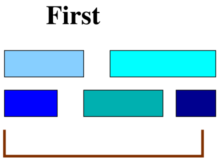
Given
- items $N = \{1, \ldots, n\}$,
- first stage: profits p^0, weight w^0, capacity c^0,

Find
- subset $X \subseteq N$

Such that
- $w^0(X) \leq c^0$,
- total profit

$$p_T(X) = p^0(X)$$

is maximized.



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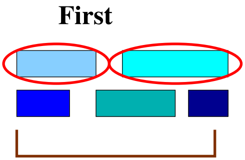

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Given
- items $N = \{1, \ldots, n\}$,
- first stage: profits p^0, weight w^0, capacity c^0,
- scenarios $S \in \mathcal{S}_D$ with profits p^S, weight w^S, capacity c^S,

Find
- subset $X \subseteq N$

Such that
- $w^0(X) \leq c^0$,
- for all $S \in \mathcal{S}_D$ there exists $X^S \in \mathcal{X}(X)$ with $w^S(X^S) \leq c^S$,
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$$p_T(X) = p^0(X) + \min_{S \in \mathcal{S}_D} \max_{X^S} p^S(X^S)$$

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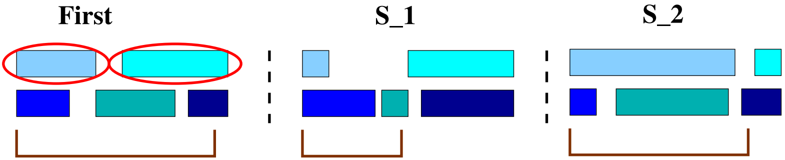

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- recovery set $\mathcal{X}(X)$: delete $\leq k$ items, add $\leq \ell$ items

Find
- subset $X \subseteq N$

Such that
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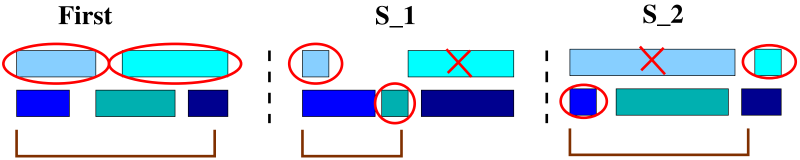

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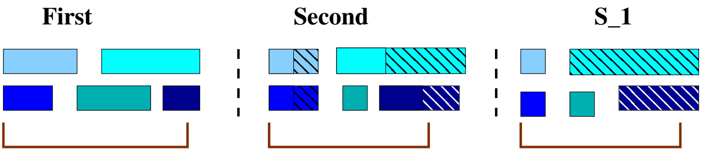

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- first stage: profits p^0, weights w^0, capacity c^0,
- Γ-scenarios: weights $[\bar{w}, \bar{w} + \hat{w}]$, capacity c, $\Gamma \in \mathbb{N}$,
- recovery set $\mathcal{X}(X)$: delete $\leq k$ items from $X \subseteq N$

Find
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Such that
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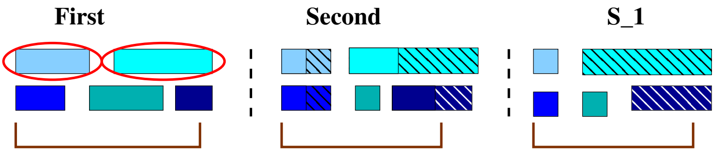

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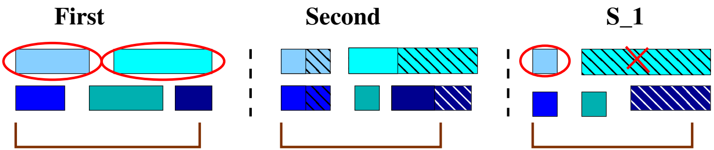

Mathematical Programming formulation:

$$\begin{aligned}
\max \sum_{i \in N} p_i^0 x_i \\
s.t. \sum_{i \in N} w_i^0 x_i & \leq c^0 \\
\sum_{i \in N} \bar{w}_i x_i + \max_{\substack{X \subseteq N \\ |X| \leq \Gamma}} \left(\sum_{i \in X} \hat{w}_i x_i - \max_{\substack{Y \subseteq N \\ |Y| \leq k}} \left(\sum_{i \in Y} \bar{w}_i x_i + \sum_{i \in X \cap Y} \hat{w}_i x_i \right) \right) & \leq c \\
x_i \in \{0, 1\}
\end{aligned}$$

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\max \sum_{i \in N} p_i^0 x_i & \\
s.t. \sum_{i \in N} w_i^0 x_i & \leq c^0 \\
\sum_{i \in N} \bar{w}_i x_i + \max_{\substack{X \subseteq N \\ |X| \leq \Gamma}} \left(\sum_{i \in X} \hat{w}_i x_i - \max_{\substack{Y \subseteq N \\ |Y| \leq k}} \left(\sum_{i \in Y} \bar{w}_i x_i + \sum_{i \in X \cap Y} \hat{w}_i x_i \right) \right) & \leq c \\
x_i \in \{0, 1\} &
\end{aligned}
$$

Question: Compact Linear reformulation?

RRKP with Γ scenarios - MP

Mathematical Programming formulation:

$$
\begin{aligned}
\max \sum_{i\in N} p_i^0 x_i & \\
s.t. \sum_{i\in N} w_i^0 x_i & \le c^0 \\
\sum_{i\in N} \bar{w}_i x_i + \max_{\substack{X\subseteq N\\|X|\le\Gamma}} \left(\sum_{i\in X} \hat{w}_i x_i - \max_{\substack{Y\subseteq N\\|Y|\le k}} \left(\sum_{i\in Y} \bar{w}_i x_i + \sum_{i\in X\cap Y} \hat{w}_i x_i \right)\right) & \le c \\
x_i \in \{0,1\} &
\end{aligned}
$$

Question: Compact Linear reformulation?

Answer: LP duality and enumeration of solution values!

Static vs. Dynamic Routing

Static Routing:

- Capacities have to be installed in integer amounts
- Routing templates fixes percentual distribution of traffic volume along paths

Dynamic Routing:

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Dynamic Routing:

- Capacities have to be installed in integer amounts
- Routing can be adapted to actual traffic volumes (realization from uncertainty set)

$y_{ij}^k(d)$ = fraction of demand $k \in K$ routed along arc $(i,j) \in A$ for realization $d \in \mathcal{D}$.
x_e = number of link capacity modules to be installed on link $e \in E$.

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Integer Linear Programming formulation:

$$\min \sum_{e \in E} \kappa_e x_e$$

$$s.t. \sum_{j \in N(i)} (y_{ij}^k(d) - y_{ji}^k(d)) = \begin{cases} d^k(d) & i = s(k) \\ -d^k(d) & i = t(k) \\ 0 & else \end{cases}, \quad \forall d \in \mathcal{D}, i \in V, \ k \in K$$

$$\sum_{k \in K} y_e^k \leq C x_e, \qquad \forall d \in \mathcal{D}, e \in E$$

$$y(d) \geq 0, x \in \mathbb{Z}_+^{|E|}$$

Theorem (Mattia [15])

The vector $x \in P^x$ if and only if for all length functions $\ell : E \to \mathbb{R}_+$ holds

$$\sum_{e \in E} \ell(e) x_e \geq \max_{d \in \mathcal{D}} \left\{ \sum_{k \in K} d^k(d) \ell(s^k, t^k) \right\}$$

Robust Network Design with Affine Routing:

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where $h_{ij}^{k0}, h_{ij}^{k\bar{k}} \in \mathbb{R}$ for all $ij \in A$, $k, \bar{k} \in K$.

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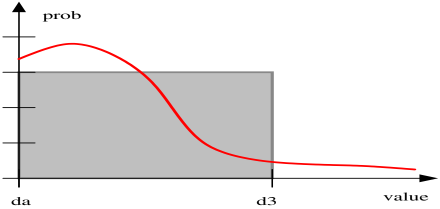
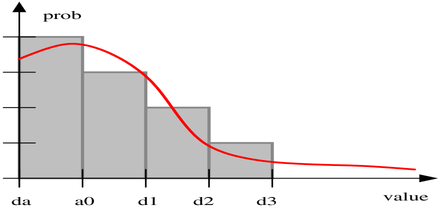
Theorem (Poss & Raack [19])

Let $\mathcal{D}$ be an arbitrary demand uncertainty set. Then

$$OPT_{dyn}(\mathcal{D}) \leq OPT_{aff}(\mathcal{D}) \leq OPT_{stat}(\mathcal{D})$$

Idea: Refinement of Γ-robustness approach





Γ-robustness

- $\bar{d}^k \geq 0$
- $\hat{d}^k \geq 0$
- $[\bar{d}^k, \bar{d}^k + \hat{d}^k]$
- $\Gamma \in \mathbb{N}$

Multi-band robustness

- $\bar{d}^k \geq 0$
- $0 = \hat{d}^k_0 \leq \hat{d}^k_1 \leq \ldots \leq \hat{d}^k_{|B|} = \hat{d}^k$
- $[\bar{d}^k + \hat{d}^k_{b-1}, \bar{d}^k + \hat{d}^k_b]$ forall $b \in B$
- $u_0, u_1, \ldots, u_{|B|} \in \mathbb{N}$

- Γ-robustness $\equiv$ multi-band robustness with $B = \{1\}$, $u_0 = |K|$, $u_1 = \Gamma$
- work by Büsing and D'Andreagiovanni [10]; based on Bienstock

Compact ILP formulation:

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e \\
\text{s.t.} \quad & \sum_{j \in V : ij \in E} f_{ij}^k - \sum_{j \in V : ji \in E} f_{ji}^k = \begin{cases} 1 & , \ i = s^k \\ -1 & , \ i = t^k \\ 0 & , \ \text{otherwise} \end{cases} && \forall i, k \\
& \sum_{k \in K} \bar{d}^k f_e^k + \sum_{b \in B} u_b w_{e,b} + \sum_{k \in K} z_e^k \le C x_e && \forall e \\
& w_{e,b} + z_e^k \ge \hat{d}_b^k f_e^k && \forall b, k \\
& x_e \in \mathbb{Z}_+, \ f_{ij}^k \in [0,1], \ w_{e,b} \ge 0, \ z_e^k \ge 0 && \forall e, ij, b, k
\end{aligned}
$$

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Concluding Remarks

- Robust optimization is an emerging field in mathematical optimization but requires an additional effort to solve robust problems
- Correct modelling of uncertainties in network applications required
- Integration of other (real) networking aspects (survivability, multi-layer)
  $\Rightarrow$ Real applications [3, 12, 11]
  $\Rightarrow$ Robustness & 1+1 Protection [13]

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  $\Rightarrow$ What recovery action is possible?
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- Recoverable Robustness allows a two-stage approach
  $\Rightarrow$ What recovery action is possible?
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- Quality of robust approach has to be evaluated
  $\Rightarrow$ Which value of Γ is enough to obtain robust designs?

A Crash Course in Robust Optimization

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Lehrstuhl II für Mathematik
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