# The Structure of $K_{2,4}$-Minor Free Graphs

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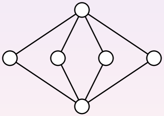

# Outline

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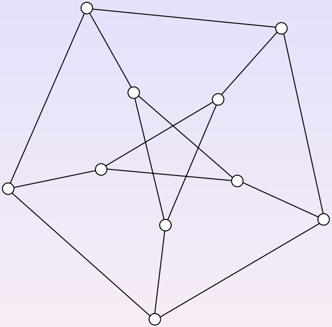
# What is a minor?

A **minor** of $G$ is a subgraph of a graph obtained from $G$ by edge contraction.



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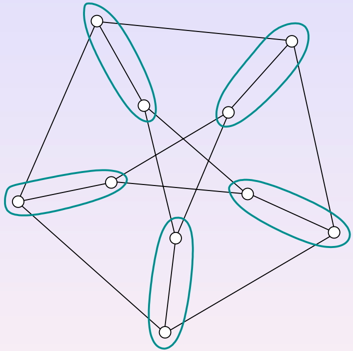

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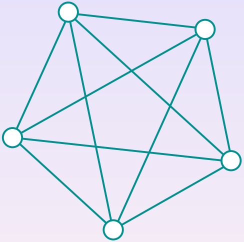

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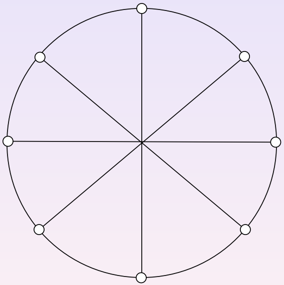
# Some $H$-minor free graph families

- Trees are $K_3$-minor free
- Outerplanar graphs are $K_{2,3}$-minor free
- Planar are $K_5$-minor free
- Treewidth-$t$ graphs are $K_{t+2}$-minor free
- The graphs of any minor closed families $\mathcal{F}$ are $H$-minor free for some $H = H(\mathcal{F})$.

# $K_5$-minor free graphs

**Theorem (Wagner - 1937)**

*Every $K_5$-minor free graph has a tree-decomposition whose bags intersect in at most 3 vertices, and induced a planar graph or a $V_8$.*



**Corollary:** 4-coloring of $K_5$-minor free graphs $\Leftrightarrow$ 4CC

# $H$-minor free graphs

**Theorem (Robertson & Seymour - Graph Minor 16)**

*Every $H$-minor free graph has a tree-decomposition whose bags intersect in $\leqslant k$ vertices, and induced graphs that either have $\leqslant k$ vertices, or are $k$-almost embeddable on a surface $\Sigma$ on which $H$ has no embedding.*

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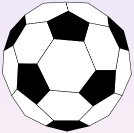

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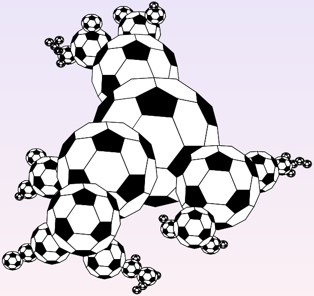

Wagner's Theorem: $k = 3$ and $\Sigma = \mathbb{S}_0$.

# $K_6$-minor free: conjectures

**Conjecture (Hadwiger - 1943)**

*Every $K_{r+1}$-minor free graph has a $r$-coloring.*

Proved for $r \in \{1, \ldots, 5\}$.

[Robertson et al. - 1993]

5-coloring of $K_6$-minor free graphs $\Leftrightarrow$ 4CC

[Every minimal counter-example is a planar plus one vertex (83 pages)]

However, the structure of $K_6$-minor free graph is still unknown. Ken-ichi Kawarabayashi explains in SODA '07 why the problem is important and difficult.

# $K_6$-minor free: conjectures

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**Conjecture (Jørgensen - 1994)**

*Every 6-connected $K_6$-minor free graph has a vertex $u$ such that $G \setminus \{u\}$ is planar.*

DeVos, Hegde, Kawarabayashi, Norine, Thomas, and Wollan have announced that [J94] is true if the graph has many vertices ...

**Problem:** replace in [J94] "6" by "$r$".

# Outline

# Our result

**Theorem**

*Every 2-connected $K_{2,4}$-minor free graph has two vertices $u, v$ such that $G \setminus \{u, v\}$ is outerplanar.*

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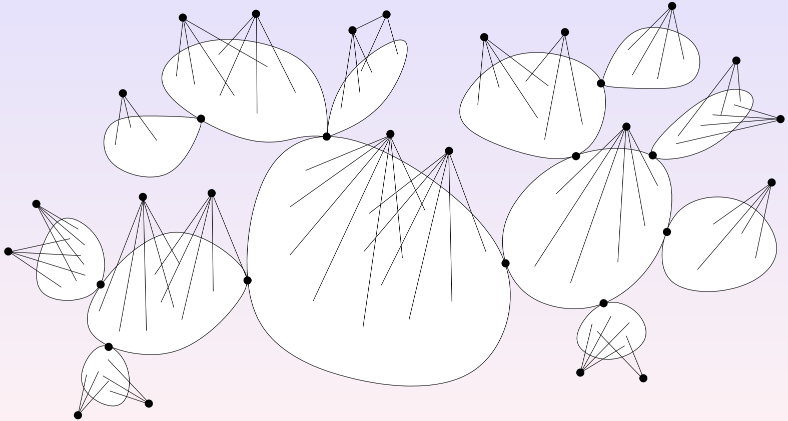
Actually, in $O(n)$ time ($n$ is the number of vertices of the input graph) we can either extract a $K_{2,4}$-minor, or find these two vertices.

# Applications

The simple geometrical structure of these graphs (almost outerplanar embedding) can be used for "object location problems" (e.g. routing).



# Corollaries

- The treewidth of $K_{2,4}$-minor free graphs is at most **4**. This is optimal because $K_5$. [Previous best bound was 6 by Bodlaender et al. 1997]

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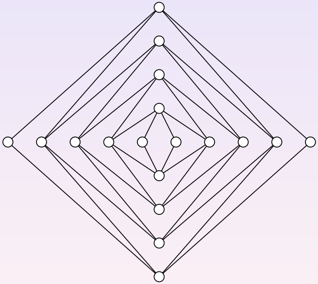

[Unfortunately, wrong for $r = 3$. No $K_{2,5}$-minor!]

# Other related result

We have also proved that:

**Theorem**

*Let $H$ be a graph having a $k \times r$ grid straight-line embedding. Then, every $H$-minor free planar graph has treewidth at most $O(k^{3/2}\sqrt{r})$.*

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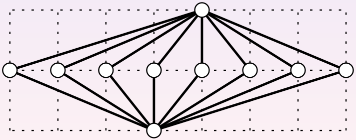
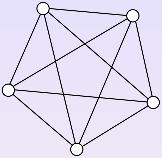
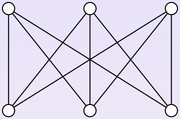
$K_{2,r}$ has a $3 \times r$ embedding, so $K_{2,r}$-minor free planar graph has treewidth at most $O(\sqrt{r})$. [Best previous bound was $r + 2$ by Thilikos 1999]



# How does a $K_{2,4}$-minor free graph look?

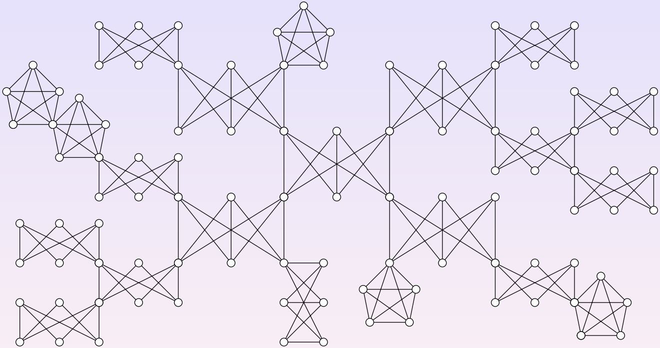
There are not planar: $K_5$ and $K_{3,3}$ are $K_{2,4}$-minor free.





# How does a $K_{2,4}$-minor free graph look?

There are not of bounded genus.



# How does a $K_{2,4}$-minor free graph look?

They have no more than $3n - 3$ edges.

A maximal $K_{2,r}$-minor free graph has $\left\lfloor \frac{1}{2}(r+2)(n-1) \right\rfloor$ edges.
[Chudnovsky-Reed-Seymour 2008]

# Outline

# Structure of the Proof

Let $G$ be a 2-connected $K_{2,4}$-minor free graph.

Part 1. If $G$ is planar, then removing one vertex leaves $G$ outerplanar.

Part 2. If $G$ is not planar, then removing one vertex leaves $G$ planar.

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Part 1.

1. If $G$ is planar, but not outerplanar, $G$ has a special embedding, called LMR-embedding.
2. If $G$ has a LMR-embedding (and $K_{2,4}$-minor free), then removing one vertex leaves $G$ outerplanar.

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Part 2.

1. If $G$ is not planar, it is a $K_5$ or it has a subdivision of $K_{3,3}$ as spanning subgraph.
2. ... then removing one vertex leaves $G$ planar.

# LMR-embedding

Notation: $\textsc{In}(\mathcal{C})$ denotes the bounded region of $\mathbb{R}^2 \setminus \mathcal{C}$.

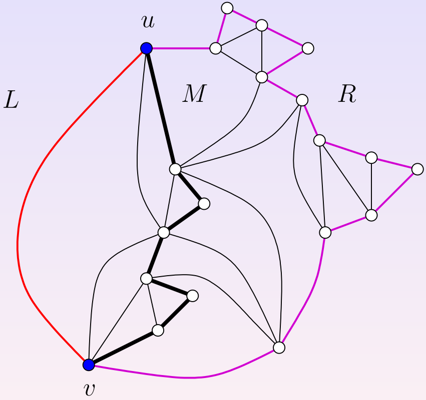
[$\mathcal{C}$ is a cycle or a curve of $\mathbb{R}^2$]

## Definition

An *LMR-embedding* is a plane embedding such that there exists three paths, $L, M, R$, sharing only their extremities, and such that:

- $L \cup R$ is the border of the outerface;
- $\textsc{In}(L \cup M)$ and $\textsc{In}(M \cup R)$ have no vertices;
- $\textsc{In}(M \cup R)$ has no edges with both endpoints in $M$; and
- $M$ and $R$ has length at least two.

# Example



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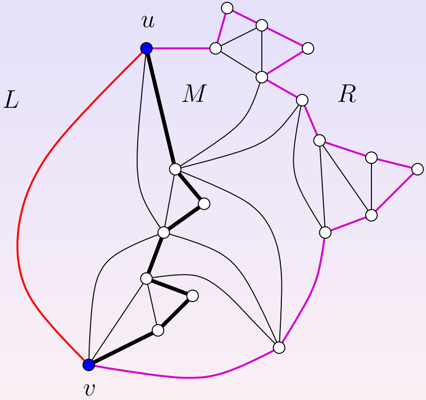


**Remarks:**

- Paths $L, M, R$ span $G$
- $G \setminus M$ is outerplanar
- $G \setminus R$ is outerplanar
- $G \setminus L$ maybe not outerplanar

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  Can we give a structure for $K_{2,4} \cup \{e\}$-minor free graphs?
- Can we Generalize to $K_{2,5}$? and to $K_{2,r}$ with $r \geqslant 5$???
  [need to consider higher connected components]

# Conclusion

- $K_{2,4}$ is far from $K_6$ by an edge ratio of $8/15 \approx 0.53\%$. Can we give a structure for $K_{2,4} \cup \{e\}$-minor free graphs?
- Can we Generalize to $K_{2,5}$? and to $K_{2,r}$ with $r \geqslant 5$??? [need to consider higher connected components]
- Characterize the properties $\mathcal{P}$ satifying the following meta-theorem:

  Given a graph $H$ with property $\mathcal{P}$, every $H$-minor free planar graph has treewidth at most $f(\mathcal{P})\sqrt{|V(H)|}$, for some function $f$.

  Example for perperty $\mathcal{P}$: planar and not $k$-connected.