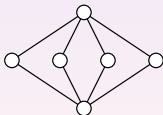


The Structure of $K_{2,4}$ -Minor Free Graphs

Youssou Dieng and Cyril Gavoille

Université de Bordeaux, France

JGA08 6-7 Nov
(Sophia-Antipolis)



Outline

Introduction

Results

Proof

Conclusion

Outline

Introduction

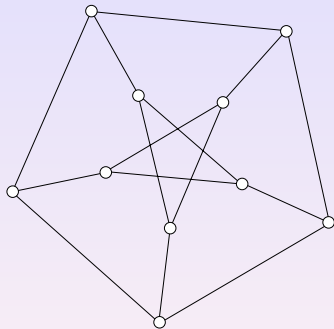
Results

Proof

Conclusion

What is a minor?

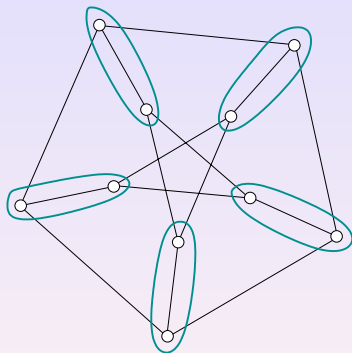
A **minor** of G is a subgraph of a graph obtained from G by edge contraction.



A H -minor free graph is a graph without minor H .

What is a minor?

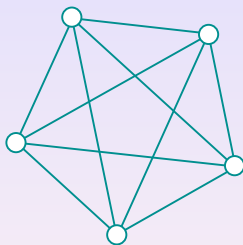
A **minor** of G is a subgraph of a graph obtained from G by edge contraction.



A H -minor free graph is a graph without minor H .

What is a minor?

A **minor** of G is a subgraph of a graph obtained from G by edge contraction.



A H -minor free graph is a graph without minor H .

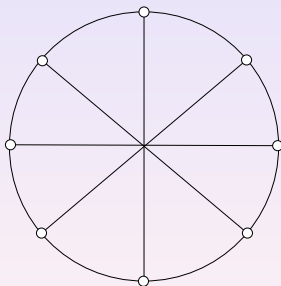
Some H -minor free graph families

- Trees are K_3 -minor free
- Outerplanar graphs are $K_{2,3}$ -minor free
- Planar are K_5 -minor free
- Treewidth- t graphs are K_{t+2} -minor free
- The graphs of any minor closed families \mathcal{F} are H -minor free for some $H = H(\mathcal{F})$.

K_5 -minor free graphs

Theorem (Wagner - 1937)

Every K_5 -minor free graph has a tree-decomposition whose bags intersect in at most 3 vertices, and induced a planar graph or a V_8 .



Corollary: 4-coloring of K_5 -minor free graphs \Leftrightarrow 4CC

H -minor free graphs

Theorem (Robertson & Seymour - Graph Minor 16)

Every H -minor free graph has a tree-decomposition whose bags intersect in $\leq k$ vertices, and induced graphs that either have $\leq k$ vertices, or are k -almost embeddable on a surface Σ on which H has no embedding.

H -minor free graphs

Theorem (Robertson & Seymour - Graph Minor 16)

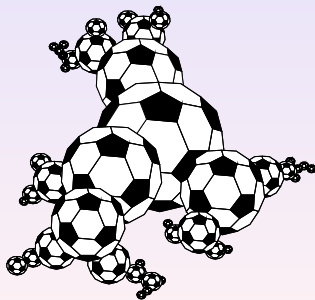
Every H -minor free graph has a tree-decomposition whose bags intersect in $\leq k$ vertices, and induced graphs that either have $\leq k$ vertices, or are k -almost embeddable on a surface Σ on which H has no embedding.



H -minor free graphs

Theorem (Robertson & Seymour - Graph Minor 16)

Every H -minor free graph has a tree-decomposition whose bags intersect in $\leq k$ vertices, and induced graphs that either have $\leq k$ vertices, or are k -almost embeddable on a surface Σ on which H has no embedding.



Wagner's Theorem: $k = 3$ and $\Sigma = \mathbb{S}_0$.

K_6 -minor free: conjectures

Conjecture (Hadwiger - 1943)

Every K_{r+1} -minor free graph has a r -coloring.

Proved for $r \in \{1, \dots, 5\}$.

[Robertson et al. - 1993]

5-coloring of K_6 -minor free graphs \Leftrightarrow 4CC

[Every minimal counter-example is a planar plus one vertex (83 pages)]

However, the structure of K_6 -minor free graph is still unknown. Ken-ichi Kawarabayashi explains in SODA '07 why the problem is important and difficult.

K_6 -minor free: conjectures

Conjecture (Jørgensen - 2001)

Every K_6 -minor free graph has a arboricity at most 3.

K_6 -minor free: conjectures

Conjecture (Jørgensen - 2001)

Every K_6 -minor free graph has a arboricity at most 3.

Conjecture (Jørgensen - 1994)

Every 6-connected K_6 -minor free graph has a vertex u such that $G \setminus \{u\}$ is planar.

DeVos, Hegde, Kawarabayashi, Norine, Thomas, and Wollan have announced that [J94] is true if the graph has many vertices ...

Problem: replace in [J94] “6” by “ r ”.

Outline

Introduction

Results

Proof

Conclusion

Our result

Theorem

Every 2-connected $K_{2,4}$ -minor free graph has two vertices u, v such that $G \setminus \{u, v\}$ is outerplanar.

Our result

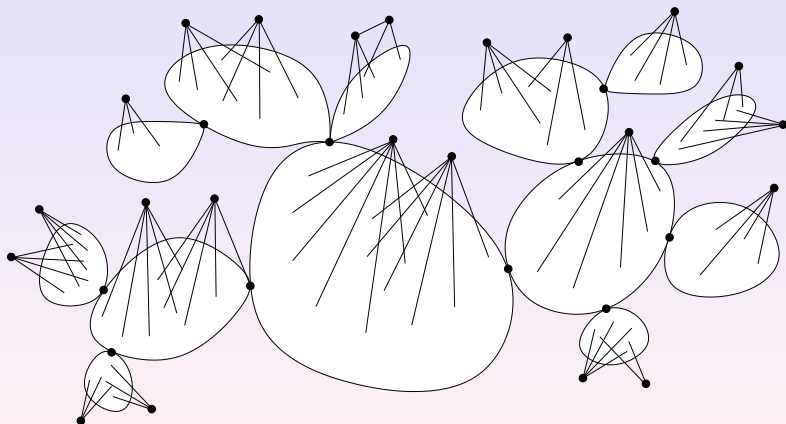
Theorem

Every 2-connected $K_{2,4}$ -minor free graph has two vertices u, v such that $G \setminus \{u, v\}$ is outerplanar.

Actually, in $O(n)$ time (n is the number of vertices of the input graph) we can either extract a $K_{2,4}$ -minor, or find these two vertices.

Applications

The simple geometrical structure of these graphs (almost outerplanar embedding) can be used for “object location problems” (e.g. routing).



Corollaries

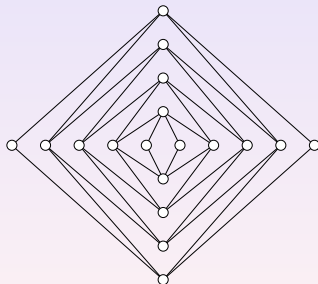
- The treewidth of $K_{2,4}$ -minor free graphs is at most **4**. This is optimal because K_5 . [Previous best bound was 6 by Bodlaender et al. 1997]

Corollaries

- The treewidth of $K_{2,4}$ -minor free graphs is at most **4**. This is optimal because K_5 . [Previous best bound was 6 by Bodlaender et al. 1997]
- Every 2-connected $K_{2,r+2}$ -minor free graph has r vertices whose removal leaves the graph outerplanar, for each $r \in \{0, 1, 2\}$.

Corollaries

- The treewidth of $K_{2,4}$ -minor free graphs is at most **4**. This is optimal because K_5 . [Previous best bound was 6 by Bodlaender et al. 1997]
- Every 2-connected $K_{2,r+2}$ -minor free graph has r vertices whose removal leaves the graph outerplanar, for each $r \in \{0, 1, 2\}$.



[Unfortunately, wrong for $r = 3$. No $K_{2,5}$ -minor!]

Other related result

We have also proved that:

Theorem

Let H be a graph having a $k \times r$ grid straight-line embedding. Then, every H -minor free planar graph has treewidth at most $O(k^{3/2}\sqrt{r})$.

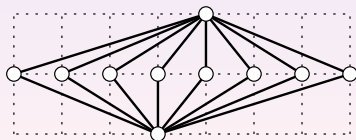
Other related result

We have also proved that:

Theorem

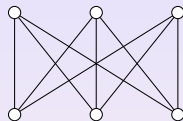
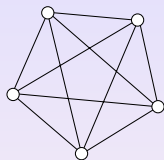
Let H be a graph having a $k \times r$ grid straight-line embedding. Then, every H -minor free planar graph has treewidth at most $O(k^{3/2}\sqrt{r})$.

$K_{2,r}$ has a $3 \times r$ embedding, so $K_{2,r}$ -minor free planar graph has treewidth at most $O(\sqrt{r})$. [Best previous bound was $r + 2$ by Thilikos 1999]



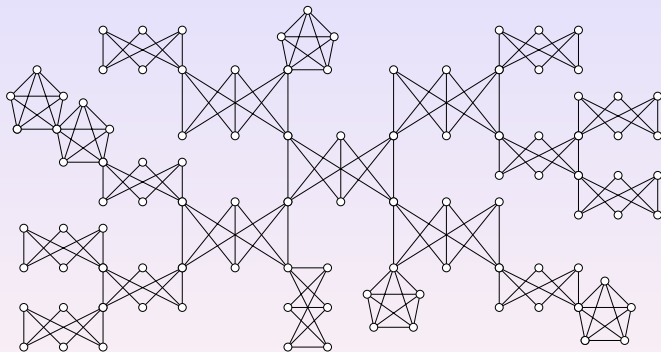
How does a $K_{2,4}$ -minor free graph look?

There are not planar: K_5 and $K_{3,3}$ are $K_{2,4}$ -minor free.



How does a $K_{2,4}$ -minor free graph look?

There are not of bounded genus.



How does a $K_{2,4}$ -minor free graph look?

They have no more than $3n - 3$ edges.

A maximal $K_{2,r}$ -minor free graph has $\lfloor \frac{1}{2}(r+2)(n-1) \rfloor$ edges.
[Chudnovsky-Reed-Seymour 2008]

Outline

Introduction

Results

Proof

Conclusion

Structure of the Proof

Let G be a 2-connected $K_{2,4}$ -minor free graph.

- Part 1. If G is planar, then removing one vertex leaves G outerplanar.
- Part 2. If G is not planar, then removing one vertex leaves G planar.

Structure of the Proof

Let G be a 2-connected $K_{2,4}$ -minor free graph.

Part 1. If G is planar, then removing one vertex leaves G outerplanar.

Part 2. If G is not planar, then removing one vertex leaves G planar.

- Part 1.
- 1 If G is planar, but not outerplanar, G has a special embedding, called LMR-embedding.
 - 2 If G has a LMR-embedding (and $K_{2,4}$ -minor free), then removing one vertex leaves G outerplanar.

Structure of the Proof

Let G be a 2-connected $K_{2,4}$ -minor free graph.

Part 1. If G is planar, then removing one vertex leaves G outerplanar.

Part 2. If G is not planar, then removing one vertex leaves G planar.

Part 2.

- 1 If G is not planar, it is a K_5 or it has a subdivision of $K_{3,3}$ as spanning subgraph.
- 2 ... then removing one vertex leaves G planar.

LMR-embedding

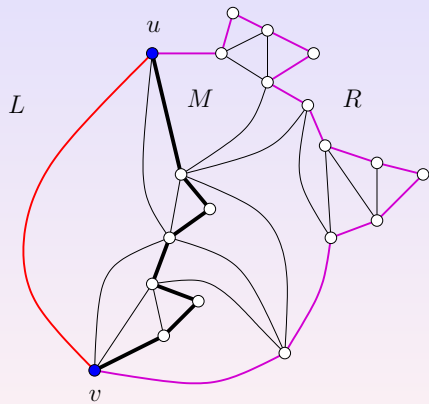
Notation: $\text{IN}(\mathcal{C})$ denotes the bounded region of $\mathbb{R}^2 \setminus \mathcal{C}$.
[\mathcal{C} is a cycle or a curve of \mathbb{R}^2]

Definition

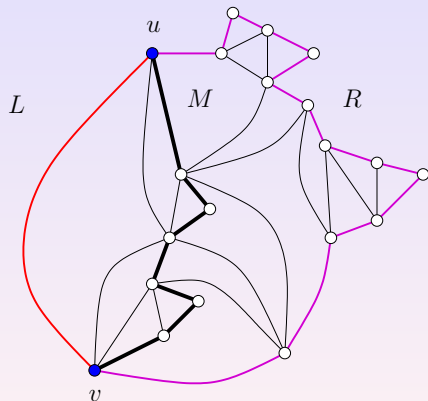
An *LMR-embedding* is a plane embedding such that there exists three paths, L, M, R , sharing only their extremities, and such that:

- $L \cup R$ is the border of the outerface;
- $\text{IN}(L \cup M)$ and $\text{IN}(M \cup R)$ have no vertices;
- $\text{IN}(M \cup R)$ has no edges with both endpoints in M ; and
- M and R has length at least two.

Example



Example



Remarks:

- Paths L, M, R span G
- $G \setminus M$ is outerplanar
- $G \setminus R$ is outerplanar
- $G \setminus L$ maybe not outerplanar

Outline

Introduction

Results

Proof

Conclusion

Conclusion

- $K_{2,4}$ is far from K_6 by an edge ratio of $8/15 \approx 0.53\%$.
Can we give a structure for $K_{2,4} \cup \{e\}$ -minor free graphs?

Conclusion

- $K_{2,4}$ is far from K_6 by an edge ratio of $8/15 \approx 0.53\%$.
Can we give a structure for $K_{2,4} \cup \{e\}$ -minor free graphs?
- Can we Generalize to $K_{2,5}$? and to $K_{2,r}$ with $r \geq 5$???
[need to consider higher connected components]

Conclusion

- $K_{2,4}$ is far from K_6 by an edge ratio of $8/15 \approx 0.53\%$.
Can we give a structure for $K_{2,4} \cup \{e\}$ -minor free graphs?
- Can we Generalize to $K_{2,5}$? and to $K_{2,r}$ with $r \geq 5$???
[need to consider higher connected components]
- Characterize the properties \mathcal{P} satisfying the following meta-theorem:

Given a graph H with property \mathcal{P} , every H -minor free planar graph has treewidth at most $f(\mathcal{P})\sqrt{|V(H)|}$, for some function f .

Example for property \mathcal{P} : planar and not k -connected.