Diameters, Centers, and Approximating Trees of δ-hyperbolic Geodesic Spaces and Graphs

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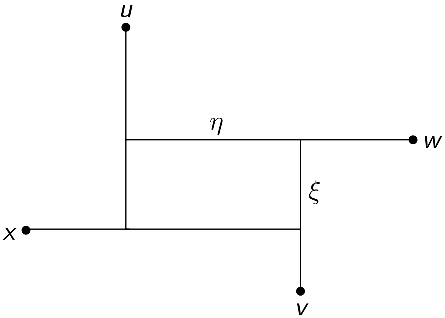
δ-Hyperbolicity (M. Gromov, 1987)

for any four points u, v, w, x of a metric space (X, d), the two larger of the distance sums $d(u,v) + d(w,x)$, $d(u,w) + d(v,x)$, $d(u,x) + d(v,w)$ differ by at most 2δ.



$$\min\{\eta, \xi\} \leq \delta$$

δ-Hyperbolicity measures the local deviation of a metric from a tree metric: *a metric is a tree metric iff it is 0-hyperbolic.*

Goal and results

Our goal

To establish local-to-global results about the "ressemblance" of geodesic δ-hyperbolic metric spaces and δ-hyperbolic graphs to trees.

Our results

(i) We show that approximating the diameter $diam(S)$, the radius $rad(S)$, and the center $C(S)$ of a subset S in a δ-hyperbolic geodesic space or graph with an $O(\delta)$-additive error can be done in the same way as for trees. This leads to very simple algorithms for fast approximating (and in some cases, for computing in linear time) of $diam(S), rad(S)$, and $C(S)$.

(ii) We present a simple linear-time construction of distance approximating trees of δ-hyperbolic graphs with n vertices having the same additive distortion $O(\delta \log n)$ as Gromov's construction.

(iii) We establish that several classes of geometrically defined graphs have bounded hyperbolicity.

Diameter, radius, and center

Diameter

Let S be a finite set of points of a metric space (X, d).
Diameter: $diam(S) = \max\{d(u, v) : u, v \in S\}$.
Diametral pair: any pair of points $x, y \in S$ such that $d(x, y) = diam(S)$.

Furthest neighbors

The set $F(x)$ of furthest neighbors of a point $x \in X$ in S consists of all points of S at the maximum distance from x. The eccentricity $ecc(x)$ of $x \in X$ is the distance from x to any point of $F(x)$.

Center and radius

The center $C(S)$ of S is the set of points of X with minimum eccentricity. The radius $rad(S)$ of S is the eccentricity of central points, i.e., $rad(S)$ is the smallest radius of a ball of (X, d) enclosing all points of S (a ball $B(c, r) = \{x \in X : d(c, x) \leq r\}$ consists of all points $x \in X$ at distance at most r to c).

Known results

Fast computation of diameter, radius, and center

is a basic algorithmic problem in computational geometry and graph theory with applications in operation research, data clustering, location theory, and analysis of complex networks.

Spaces admitting fast algorithms

Linear $O(n)$, $O(n \log n)$, and subquadratic algorithms are known for trees, n-point sets in $\mathbb{R}^2$ and $\mathbb{R}^3$, simple polygons and simple rectilinear polygons endowed with intrinsic geodesic or link metric, and some classes of graphs (chordal graphs, cactus networks, some plane triangulations and quadrangulations). Most of these algorithms are not simple.

Known algorithmic results about δ-hyperbolicity

The internet topology embeds with better accuracy into low-dimensional hyperbolic space than into Euclidian space of comparable dimension. PTAS for the Traveling Salesman Problem, efficient nearest neighbor search, distance labeling schemes and routing schemes, and approximation algorithms for covering and packing by balls.

Tree-folklore

C. Jordan (1869)

C. Jordan established that the center of a tree is a single vertex or an edge.

Diameter

The diameter $diam(S)$ of a set S in a tree T can be found in linear time by running the following folklore algorithm:

Algorithm 2FP

1. Pick an arbitrary point u of T
2. Find a furthest neighbor u of v in S
3. Find a furthest neighbor w of v in S
4. Return $d(v, w)$ as $diam(S)$ and v, w as a diametral pair of S

Center

To find the center of S it suffices to add the following step:

5. Return the midpoint c of the unique (v, w)-path of T

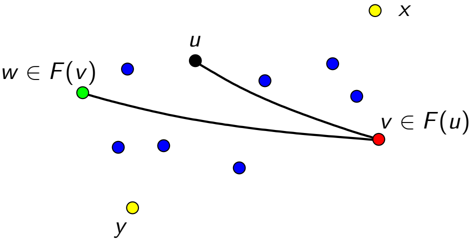
Diameter

Proposition 1

For a finite subset S of a δ-hyperbolic space (X, d) and any $u \in X$, if $v \in F(u)$ and $w \in F(v)$, then $d(v, w) \geq diam(S) - 2\delta$.
The pair $\{v, w\}$ can be computed using $O(|S|)$ distance calculations.

Lemma 1

Let u, v, x, y be four points in a δ-hyperbolic space (X, d). If we have $d(u, v) \geq \max\{d(u, x), d(u, y)\}$ then $\max\{d(v, x), d(v, y)\} \geq d(x, y) - 2\delta$.



Radius

Proposition 2

For a finite set S of a δ-hyperbolic geodesic space, $diam(S)/2 \leq rad(S) \leq diam(S)/2 + 2\delta$.

Lemma 2

(Helly property for balls) If $B(s, r_s), i \in S$, is a family of pairwise intersecting balls of a δ-hyperbolic geodesic space (or graph) then the intersection $\bigcap\{B(s, r_s) + 2\delta) : s \in S\}$ is nonempty.

Corollary 1

For a finite set S of a δ-hyperbolic geodesic space, $d(v, w)/2 \leq rad(S) \leq d(v, w)/2 + 3\delta$.

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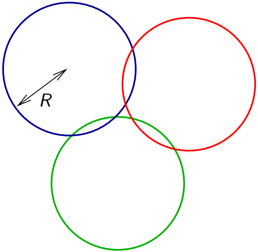

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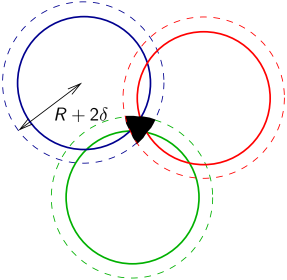
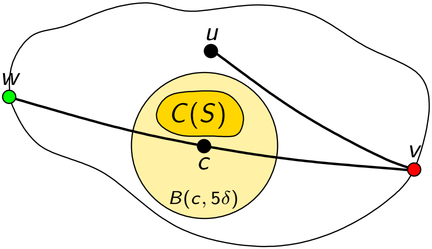


Center I



Proposition 3

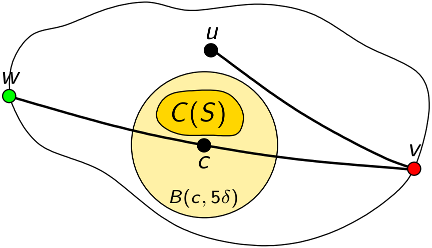
For a finite set S of a δ-hyperbolic geodesic space, $diam(C(S)) \leq 4\delta$.

Let c be the middle of a geodesic $[v, w]$ between v and w.

Proposition 4

The inequality $ecc(c) \leq rad(S) + 5\delta$ holds for all δ-hyperbolic geodesic spaces and graphs. Moreover $C(S) \subseteq B(c, 5\delta)$ $(C(G) \subseteq B(c, 5\delta + 1)$ for δ-hyperbolic graphs).

Center II



Corollary 2

For a finite subset $S \subseteq V$ of a δ-hyperbolic graph $G = (V, E)$ with maximum degree $\Delta(G)$ and δ bounded by a constant, a vertex c with $ecc(c) \leq rad(S) + 2\delta$ can be computed in $O(|E|)$ time and the center $C(S)$ can be computed in $O(|\Delta(G)|^{5\delta+1}|E|)$ time. If the degrees of vertices of G are uniformly bounded, then $C(S)$ can be computed in linear $O(|E|)$ time.

Distance approximating trees I

Theorem (Gromov, 1987)

For any δ-hyperbolic metric space (X, d) on n points and any fixed basepoint $s \in X$, there a tree T and a map $\varphi : X \to T$ such that

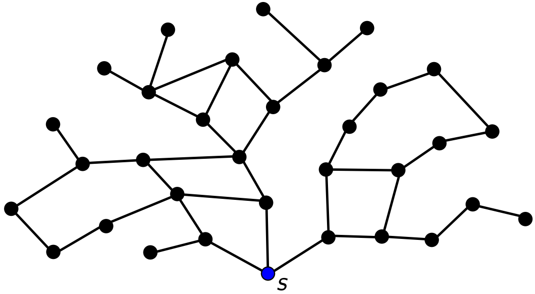
- $d_T(\varphi(s), \varphi(x)) = d(s, x)$ pour tout $x \in X$,
- $d(x, y) - 2\delta \log_2 n \leq d_T(\varphi(x), \varphi(y)) \leq d(x, y)$ for all $x, y \in X$.

The tree T can be constructed using $O(n^2)$ distance computations.

Proposition 5

For a δ-hyperbolic graph $G = (V, E)$ it is possible to construct in $O(|E|)$ time a tree $T = (V, F)$ which $(16 + 12\delta + 8\delta \log_2 n)$-approximate the distances of G.

Distance approximating trees II

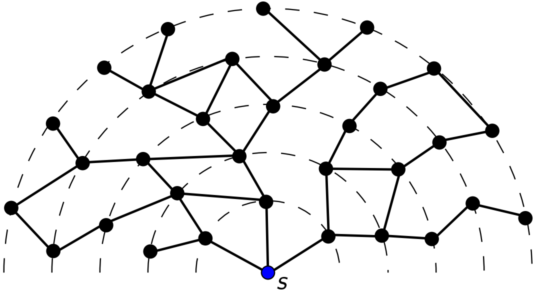


A layering of G is the partition of V into the concentric spheres

$$L^i = \{u \in V : d(s,u) = i\}, i = 0, 1, 2, \ldots .$$

A layering partition of G is a partition of each L^i into clusters $L^i_1, \ldots, L^i_{p_i}$: $u, v \in L^i$ belong to the same cluster L^i_j iff they can be connected by a path outside the ball $B_{i-1}(s)$ of radius $i-1$ centered at s.

Distance approximating trees II



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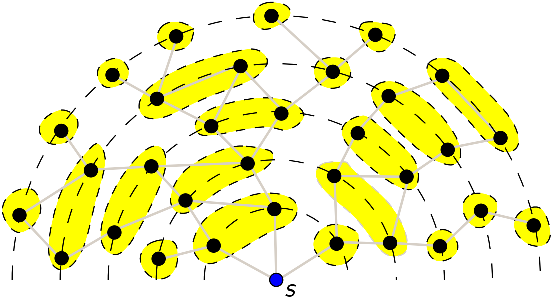

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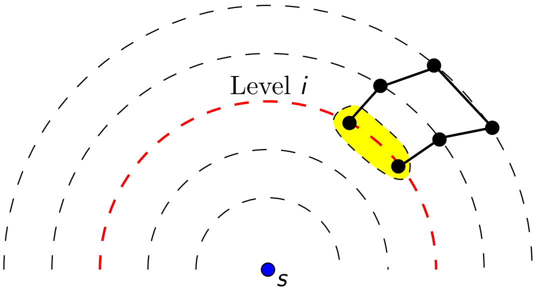

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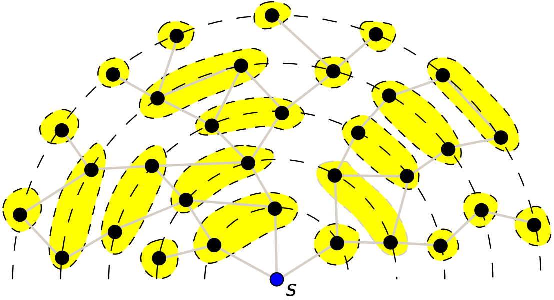

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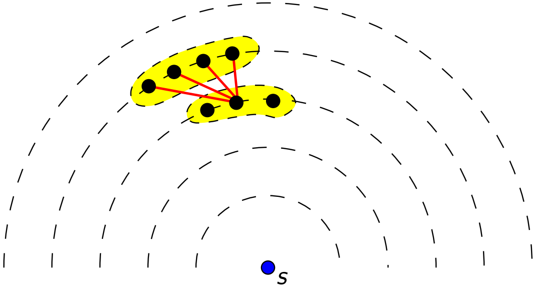

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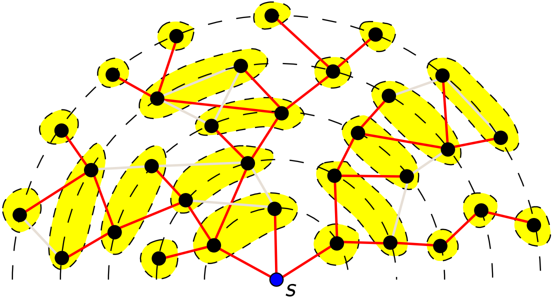

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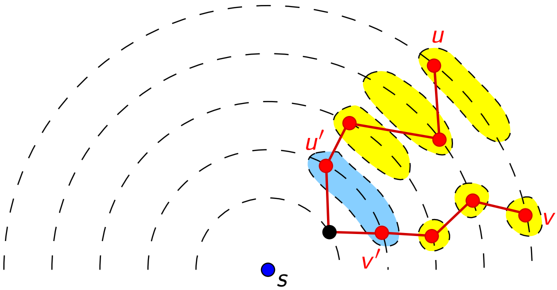

$$d_T(u, v) \le d_G(u, v) \le d_T(u, v) + d_G(u', v')$$

Claim. The diameter of each cluster L^i_j of a δ-hyperbolic graph G with n vertices is at most $\Lambda_n := 16 + 12\delta + 8\delta \log_2 n$.

Proposition 5

For a δ-hyperbolic graph $G = (V, E)$, this construction gives a tree $T = (V, F)$ which Λ_n-approximate the distances of G.