Introduction to the probabilistic method

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The probabilistic method is an efficient technique to prove the existence of combinatorial objects having some specific properties. It is based on the probability theory but it can be used to prove theorems which have nothing to do with probability. The probabilistic method is now widely used and considered as a basic knowledge. It is now exposed in many introductory books to graph theory like [8].

In these notes, we give the most common tools and techniques of this method. The interested reader who would like to know more complicated techniques is referred to the books of Alon and Spencer [5] and Molloy and Reed [21].

We assume that the reader is familiar with basic graph theory. Several books (see e.g. [8, 11, 29]) give nice introduction to graph theory.

1 Basic probabilistic concept

A *(finite) probability space* is a couple $(\Omega, \mathbf{Pr})$ where Ω is a finite set called *sample set*, and $\mathbf{Pr}$ is a function, called *probability function*, from Ω into $[0,1]$ such that $\sum_{\omega\in\Omega}\mathbf{Pr}(\omega)=1$. We will often consider a *uniform distribution* for which $\Pr(\omega)=\frac{1}{|\Omega|}$ for all $\omega\in\Omega$.

The set $\mathcal{G}_n$ of the labelled graphs on n vertices may be seen as the sample set of a probability space $(\mathcal{G}_n, \mathbf{Pr})$. The result of the selection of an element G of this sample set according to the probability function $\mathbf{Pr}$ is called a *random graph*.

The simplest example of such a probability space is the one with uniform distribution that is for which all the graphs $G\in\mathcal{G}_n$ have the same probability. Since $|\mathcal{G}_n| = 2^{\binom{n}{2}}$, the probability function $\mathbf{Pr}$ is $\mathbf{Pr}(G) = 2^{-\binom{n}{2}}$ for all $G\in\mathcal{G}_n$. A natural way of seeing this probability space is to consider the edges of the complete graph K_n one after another and to choose each of them for $E(G)$ with probability $\frac{1}{2}$, all these choices being made independently from each other. The result of such a procedure is a spanning subgraph of K_n with all the $G\in\mathcal{G}_n$ occurring equally likely. A more elaborate probability space over $\mathcal{G}_n$ can be obtained by fixing a real number p between

0 and 1 and choosing each edge with probability p, all the choices being made independently. Then $1-p$ is the probability of a given edge not to be chosen. Hence the probability function is $\mathbf{Pr}(G) = p^m(1-p)^{\binom{n}{2}-m}$ for all $G \in \mathcal{G}_n$ with m edges. This probability space is denoted $\mathcal{G}_{n,p}$. For example, $\mathcal{G}_{3,p}$ has for sample set the $2^{\binom{3}{2}} = 8$ spanning subgraphs of K_3 depicted in Figure 1 with the indicated probability function.

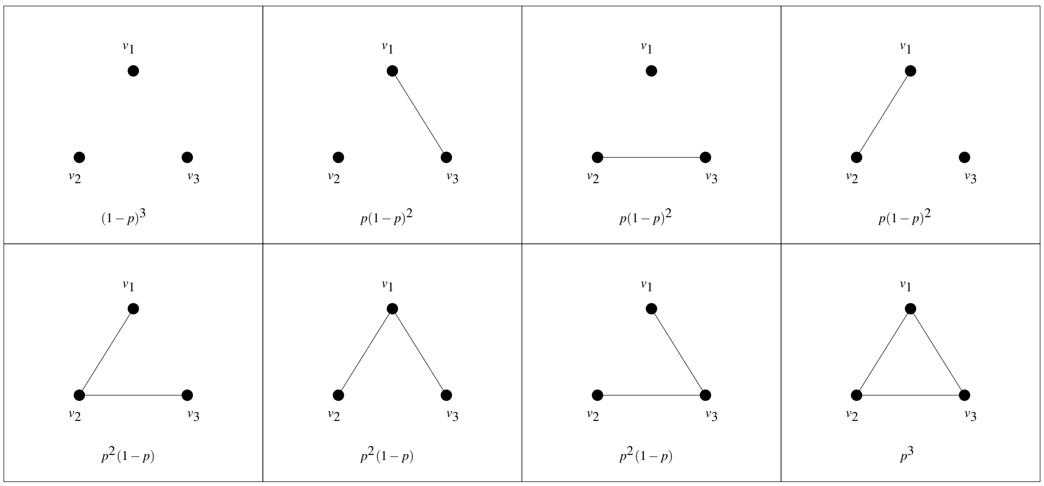

Figure 1: The probability space $\mathcal{G}_{3,p}$

Observe that the smaller p is, the larger the probability to have a graph with few edges.

To each graph property, like connectivity for example, corresponds the set of graphs of $\mathcal{G}_n$ verifying this property. The probability of a random graph to satisfy this particular property is the sum of the probability of the graphs of this subset of $\mathcal{G}_n$. For example, the probability that a random graph of $\mathcal{G}_{3,p}$ is connected is $3p^2(1-p)+p^3 = p^2(3-2p)$, the probability that it is bipartite is $(1-p)^3+3p(1-p)^2+3(1-p)p^2 = (1-p)(1+p+p^2)$, and the probability that it is both connected and bipartite is $3p^2(1-p)$.

This is captured by the notion of event. In a probability space $(\Omega, \mathbf{Pr})$, an *event* is a subset A of Ω. The *probability* of an event A is defined by

$$\mathbf{Pr}(A) = \sum_{\omega \in A} \mathbf{Pr}(\omega).$$

By definition, we have

$$\mathbf{Pr}(A \cup B) = \mathbf{Pr}(A) + \mathbf{Pr}(B) - \mathbf{Pr}(A \cap B) \tag{1}$$

In particular, $\mathbf{Pr}(A \cup B) \leqslant \mathbf{Pr}(A) + \mathbf{Pr}(B)$ and more generally,

Proposition 1. SUBADDIDIVITY OF PROBABILITIES

$$\mathbf{Pr}(A_1 \cup \cdots \cup A_n) \leqslant \mathbf{Pr}(A_1) + \cdots + \mathbf{Pr}(A_n).$$

The *complement* of an event A is the event $\overline{A} = \Omega \setminus A$. Trivially $\mathbf{Pr}(\overline{A}) = 1 - \mathbf{Pr}(A)$.

1.1 Conditional probability and independence

Let A and B be two events. The *(conditional) probability of A, given B*, denoted $\mathbf{Pr}(A|B)$, is the probability of A assuming that B has occurred. Formally, $\mathbf{Pr}(A|B) = \frac{\mathbf{Pr}(A\cap B)}{\mathbf{Pr}(B)}$ (if $\mathbf{Pr}(B) = 0$, then $\mathbf{Pr}(A|B) = \mathbf{Pr}(A)$).

Intuitively, A is independent of B if knowing if B occurs or not does not affect the probability that A occurs. Formally, A is *independent* of B if $\mathbf{Pr}(A|B) = \mathbf{Pr}(A)$, that is if $\mathbf{Pr}(A\cap B) = \mathbf{Pr}(A)\,\mathbf{Pr}(B)$. Observe that it implies that $\mathbf{Pr}(B|A) = \mathbf{Pr}(B)$, i. e. B is independent of A. Hence, one can speak about *independent events*. Two non-independent events are said *dependent*.

For example, if A is the event that G is connected and B the event that G is bipartite in the probability space $\mathcal{G}_{3,p}$, then (unless $p = 0$ or $p = 1$)

$$\mathbf{Pr}(A)\,\mathbf{Pr}(B) = p^2(3-2p)(1-p)(1+p+p^2) \neq 3p^2(1-p) = \mathbf{Pr}(A\cap B).$$

So these two events are dependent. In other words, knowing that G is connected has an influence on the probability of being bipartite, and vice-versa.

An event A is *independent* of a set of events $\{B_j \mid j\in J\}$ if, for all subset J' of J, $\mathbf{Pr}\left(A \mid \cap_{j\in J'} B_j\right) = \mathbf{Pr}(A)$. This condition is in fact equivalent to the following stronger condition: for all $J_1, J_2 \subset J$ such that $J_1\cap J_2 = \emptyset$ then $\mathbf{Pr}\left(A \mid \bigcap_{j\in J_1} B_j \cap \bigcap_{j\in J_2} \overline{B_j}\right) = \mathbf{Pr}(A)$.

Let A_i, $i\in I$ be a (finite) set of events. They are *pairwise independent* if for all $i\neq j$ A_i and A_j are independent. Events are *mutually independent* if each of them is independent from the set of the others. It is important to note that events may be pairwise independent but not mutually independent. (See Exercise 3.)

1.2 Random variables and expected value

A large part of graph theory concerns the study of basic parameters such as connectivity, clique number or chromatic number. The values of these parameters give some information on the graph and its properties. In the context of random graphs, such functions are called random variables, because they depend on the graph which is selected. More generally, a *random variable* on a probability space $(\Omega, \mathbf{Pr})$ is a function from the sample space into $\mathbb{R}$.

In the combinatorial context, random variables are frequently integer-valued. A typical example is the one of indicator variables. Each event A in a probability space (Ω, P) has an associated *indicator variable* X_A, defined by

$$X_A(\omega) = \begin{cases} 1, & \text{if } \omega\in A, \\ 0, & \text{otherwise.} \end{cases}$$

The *expected value* of a random variable X is

$$\mathbf{E}(X) = \sum_{\omega\in\Omega} \mathbf{Pr}(\omega) X(\omega).$$

Intuitively, $\mathbf{E}(X)$ is the value that is expected if we make a large number of random trials and take the average of the outcomes for X. For example, if X is the random variable which is equal to 1 with probability p and 0 with probability $1-p$, then the expected value of X is $p \times 1 + (1-p) \times 0 = p$.

If X denotes the number of components $G \in \mathcal{G}_{3,p}$, then

$$\mathbf{E}(X) = 3 \times (1-p)^3 + 2 \times 3p(1-p)^2 + 1 \times (3p^2(1-p) + p^3) = 3 - 3p + p^3.$$

If X_A is the indicator variable associated to an event A, then

$$\mathbf{E}(X_A) = \mathbf{Pr}(X_A = 1) = \mathbf{Pr}(A).$$

A very useful tool for calculating the expected value is its linearity.

Proposition 2. LINEARITY OF THE EXPECTED VALUE
If $X = \sum_{i=1}^{n} \lambda_i \cdot X_i$, then $\mathbf{E}(X) = \sum_{i=1}^{n} \lambda_i \cdot \mathbf{E}(X_i)$.

Proof.

$$\begin{aligned} \mathbf{E}(X) = \mathbf{E}\left(\sum_{i=1}^{n} \lambda_i \cdot X_i\right) &= \sum_{\omega \in \Omega} \mathbf{Pr}(\omega) \sum_{i=1}^{n} \lambda_i \cdot X_i(\omega) \\ &= \sum_{i=1}^{n} \lambda_i \left(\sum_{\omega} \mathbf{Pr}(\omega) \times X_i(\omega)\right) \\ &= \sum_{i=1}^{n} \lambda_i \cdot \mathbf{E}(X_i) \end{aligned}$$

□

For example, in a graph of $\mathcal{G}_{n,p}$, there are $\binom{n}{2}$ edges, each of them being present with probability p. Thus the expected value of the number of edges is $p\binom{n}{2}$.

This is a particular case of a more general paradigm that appears frequently. Many variables are the sum of 0-1 variables and so their expected value can be calculated as the sum of the expected values of these 0-1 variables. The random variable which is the sum of n 0-1 variables equals to 1 with probability p and 0 with probability $(1-p)$ is denoted $BIN(n,p)$. The Linearity of the Expected Value yields $\mathbf{E}(BIN(n,p)) = np$.

It is important to emphasize that the Linearity of the Expected Value is valid whether or not the random variables are independent.

2 Basic method

The general principle of the probabilistic method is the following. We want to show the existence of a combinatorial object satisfying a given property P. To do this, we consider a random object in a well chosen probability space and we compute the probability that such an objet satisfies the property P. If we show that this probability is greater than 0, then we deduce that an object with property P exists: indeed, if no object were satisfying P, then the probability would be 0.

2.1 2-colouring hypergraph

Let us give a first simple example regarding proper 2-colouring of hypergraphs. A 2-*colouring* of a hypergraph is a mapping from its vertex set into a set of two colours $\{red, blue\}$. A hyperedge is *monochromatic* if all its vertices are coloured the same. A 2-colouring is *proper* if no hyperedge is monochromatic. A hypergraph is 2-*colourable* if it admits a proper 2-colouring.

Theorem 3. *If H is a k-uniform hypergraph with less than 2^{k-1} hyperedges then H is 2-colourable.*

Proof. Let us colour randomly and independently the vertices *red* with probability $1/2$ and *blue* with probability $1/2$. In other words, we consider a uniform random 2-colouring. For every hyperedge e, let A_e be the event that e is monochromatic. Then $\mathbf{Pr}(A_e) = 2 \times 2^{-k} = 2^{1-k}$ because the probability of being monochromatic *red* (or *blue*) is 2^{-k}. By the Subadditivity of Probabilities, the probability that at least one of these events occurs is at most $\mathbf{Pr}\left(\bigcup_{e\in E(H)} A_e\right) \leqslant \sum_{e\in E(H)} \mathbf{Pr}(A_e) = |E(H)| \times 2^{1-k} < 1$. Hence the probability that none of these events occurs is $\mathbf{Pr}\left(\bigcap_{e\in E(H)} \overline{A_e}\right) = 1 - \mathbf{Pr}\left(\bigcup_{e\in E(H)} A_e\right) > 0$. □

2.2 The Crossing Lemma

In order to convince the reader of the power of the probabilistic method, we present a remarkably simple application of this proof technique to crossing numbers of graphs. We obtain a lower bound for the crossing number of a graph in terms of its order and size, and then use this bound to derive a theorem in combinatorial geometry.

The *crossing number* $cr(G)$ of a graph G is the least number of crossings in a plane embedding of G. By Euler's Formula, this parameter satisfies the trivial lower bound $cr(G) \geqslant m - 3n$ (in fact $cr(G) \geqslant m - 3n + 6$ for $n \geqslant 3$. The following much stronger lower bound was given by Ajtai et al. [1] and, independently, by Leighton [17]. Its very short probabilistic proof is due to Alon; see [5].

Lemma 4. CROSSING LEMMA
Let G be a simple graph with $m \geqslant 4n$. Then

$$cr(G) \geqslant \frac{1}{64}\frac{m^3}{n^2}.$$

Proof. Consider a planar embedding $\widetilde{G}$ of G with $cr(G)$ crossings. Let S be a random subset of V obtained by choosing each vertex of G independently with probability $p := 4n/m$, and set $H := G\langle S\rangle$ and $\widetilde{H} := \widetilde{G}\langle S\rangle$.

Define random variables X, Y, Z on Ω as follows: X is the number of vertices, Y the number of edges, and Z the number of crossings of $\widetilde{H}$. The trivial bound noted above, when applied to H, yields the inequality $Z \geqslant cr(H) \geqslant Y - 3X$. By Linearity of the Expected Value, $E(Z) \geqslant E(Y) - 3E(X)$. Now $E(X) = pn$, $E(Y) = p^2 m$ (each edge having two ends) and $E(Z) = p^4 cr(G)$ (each crossing being defined by four vertices). Hence

$$p^4 cr(G) \geqslant p^2 m - 3pn$$

Dividing both sides by p^4, we have:

$$cr(G) \geqslant \frac{pm - 3n}{p^3} = \frac{n}{(4n/m)^3} = \frac{1}{64}\frac{m^3}{n^2}.$$

□

Székely [24] realized that the Crossing Lemma could be used to derive very easily a number of theorems in combinatorial geometry, some of which hitherto had been regarded as extremely challenging. We now give the proof of one of them.

Consider a set of n points in the plane. Any two of these points determine a line, but it might happen that some of these lines pass through more than two of the points. Specifically, given a positive integer k, one may ask how many lines there can be which pass through at least k points. For instance, if n is a perfect square and the points are in the form of a square grid, there are $2\sqrt{n}+2$ lines which pass through $\sqrt{n}$ points. Is there a configuration of points which contains more lines through this number of points? The following theorem of [**?**] gives a general bound on the number of lines which pass through more than k points.

Theorem 5. *Let P be a set of n points in the plane, and let ℓ be the number of lines in the plane passing through at least $k+1$ of these points, where $1 \leqslant k \leqslant 2\sqrt{2n}$. Then $\ell < 32n^2/k^3$.*

Proof. Form a graph G with vertex set P whose edges are the segments between consecutive points on the lines which pass through at least $k+1$ points of P. This graph has at least $k\ell$ edges and crossing number at most $\binom{\ell}{2}$. Thus either $k\ell < 4n$, in which case $\ell < 4n/k \leqslant 32n^2/k^3$, or $\ell^2/2 > \binom{\ell}{2} \geqslant cr(G) \geqslant (k\ell)^3/64n^2$ by the Crossing Lemma, and again $\ell < 32n^2/k^3$. □

3 The First Moment Method

The First Moment Method is the most fundamental tool of the probabilistic method. It is based on two simple statements yet surprisingly powerful.

Theorem 6. FIRST MOMENT PRINCIPLE
If $\mathbf{E}(X) \leqslant t$, then $\mathbf{Pr}(X \leqslant t) > 0$.

Proof. Intuitively, the expected value is the (weighted) average of X over all possible outcomes. If all the outcomes are greater than t then the average is necessarily greater than t.

Formally, since the sample space is finite, X can only take a finite set I of values. Thus, $\mathbf{E}(X) = \sum_{i\in I} i \times \mathbf{Pr}(X = i)$. If $\mathbf{Pr}(X \leqslant t) = 0$, then we have $\mathbf{E}(X) = \sum_{i>t} i \times \mathbf{Pr}(X = i) > t \times \sum_{i>t} \mathbf{Pr}(X = i) = t$. □

Similarly, one can show the following three statements also known as the First Moment Principle:

- *If $\mathbf{E}(X) \geqslant t$, then $\mathbf{Pr}(X \geqslant t) > 0$.*
- *If $\mathbf{E}(X) < t$, then $\mathbf{Pr}(X < t) > 0$.*

- *If* $\mathbf{E}(X) > t$, *then* $\mathbf{Pr}(X > t) > 0$.

Theorem 7. MARKOV'S INEQUALITY
For every non-negative random variable X, $\mathbf{Pr}(X \geqslant t) \leqslant \dfrac{\mathbf{E}(X)}{t}$.

Proof. Since $\mathbf{E}(X) = \sum_i i \times \mathbf{Pr}(X = i)$ and because X is never negative, $\mathbf{E}(X) \geqslant \sum_{i \geqslant t} i \times \mathbf{Pr}(X = i) \geqslant t \times \mathbf{Pr}(X \geqslant t)$. □

To apply the First Moment Method, we need to make a judicious choice of the random variable X and to compute its expected value. Often X is a non-negative integer and we show that $\mathbf{E}(X)$ is less than one. Hence either the First Moment Principle or Markov Inequality imply that $\mathbf{Pr}(X = 0) = \mathbf{Pr}(X < 1) > 0$.

Since $\mathbf{E}(X) = \sum_i i \times \mathbf{Pr}(X = i)$, one could think at first glance that to calculate $\mathbf{E}(X)$, one has to calculate $\mathbf{Pr}(X = i)$ for all i, which is at least as difficult as calculating directly $\mathbf{Pr}(X \leqslant t)$. However, the Linearity of the Expected Value often makes possible to calculate $\mathbf{E}(X)$ without calculating all the $\mathbf{Pr}(X = i)$.

Theorem 3 may be proved using the First Moment Principle (or Markov's Inequality) instead of the Subadditivity of Probabilities.

Alternative proof of Theorem 3. Consider a uniform random 2-colouring of H. For all hyperedge e, let X_e be the indicator random variable that is equal to 1 if e is monochromatic and equal to 0 otherwise. Then $X = \sum_{e \in E(H)} X_e$ is the number of monochromatic hyperedges of H. Since every hyperedge is monochromatic with probability 2^{1-k}, then for all $e \in E(H)$, $\mathbf{E}(X_e) = 2^{1-k}$. Thus, by the Linearity of the Expected Value, $\mathbf{E}(X) = \sum_{e \in E(H)} \mathbf{E}(X_e) = |E(H)| \times 2^{1-k} < 1$. Hence, the First Moment Principle (or Markov's Inequality) implies that the probability that $X = 0$ is positive. In other words, the probability that H has no monochromatic hyperedge is positive. □

In fact, the Subadditivity of Probabilities is a particular case of the First Moment Principle. (See Exercise 4). The later one allows however sometimes to prove more general results. For example, Theorem 3 can be generalised in the following way:

Theorem 8. *A k-uniform hypergraph H with m hyperedges can be* 2*-coloured in such a way that at most* $\frac{m}{2^{k-1}}$ *hyperedges are monochromatic.*

Proof. Let X be the random variable that is the number of monochromatic hyperedges in a uniform random 2-colouring of H. The preceding calculation yields $\mathbf{E}(X) = m \times 2^{1-k}$. Hence by the First Moment Principle $\mathbf{Pr}(X \leqslant m \times 2^{1-k}) > 0$. □

In fact, this result can be slightly improved. Indeed, there exist 2-colourings that have strictly more than $\mathbf{E}(X) = m \times 2^{1-k}$ monochromatic hyperedges. For example, in the two 2-colourings for which all the vertices receive the same colour, all the hyperedges are monochromatic. Hence we need some 2-colouring having fewer monochromatic than $\mathbf{E}(X) = m \times 2^{1-k}$ hyperedges to "compensate" them.

Theorem 9. *A k-uniform hypergraph H with m hyperedges can be* 2*-coloured in such a way that strictly less than* $\frac{m}{2^{k-1}}$ *hyperedges are monochromatic.*

This is a particular case of a variant of the First Moment Principle:

Proposition 10. *If* $\mathbf{E}(X) \leqslant t$ *and* $\mathbf{Pr}(X > t) > 0$*, then* $\mathbf{Pr}(X < t) > 0$.

Proof. By the contrapositive. Assume that $\mathbf{Pr}(X < t) = 0$. Then $\mathbf{E}(X) = t \times \mathbf{Pr}(X = t) + \sum_{i>t} i \times \mathbf{Pr}(X = i)$. If $\mathbf{Pr}(X > t) > 0$, then $\mathbf{E}(X) > t \times \mathbf{Pr}(X = t) + t \times \mathbf{Pr}(X > t) = t$. □

4 Alteration

The basic probabilistic method described above works as follows: Trying to prove that a structure with certain desired properties exists, one defines an appropriate probability space of structures and then shows that the desired properties hold in this space with positive probability. However, there are situations where the "random" structure does not have all the desired properties but may have a few "blemishes". With a small alteration we remove the blemishes, giving the desired structure.

A *stable set* in a graph is a set of pairwise non-adjacent vertices. The *stability number* of a graph G, denoted $\alpha(G)$, is $\max\{|S| \mid S \text{ is a stable set}\}$.

Theorem 11. *Let* $d \geqslant 1$ *be a real number and let* G *be a graph with* n *vertices and* $nd/2$ *edges. Then* $\alpha(G) \geqslant n/2d$.

Proof. In order to prove this theorem, we will consider a random subset S. But instead of proving that with positive probability S is a stable set (i.e. $G\langle S\rangle$ has no edges), we will show that the number of edges in $G\langle S\rangle$ is small compared to the number of vertices. We then remove one endvertex of each edge in order to obtain a stable set.

Let $S \subset V(G)$ be a random subset defined by $\mathbf{Pr}(v \in S) = p$, p to be determined, the events $v \in S$ being mutually independent. Let X and Y be the number of vertices and edges, respectively, in the graph $G\langle S\rangle$.

Clearly, $E(X) = np$. Moreover, for each edge $e = uv$, let Y_e be the indicator random variable for the event $e \in E(G\langle S\rangle)$ (i.e. $Y_e = 1$ if $e \in E(G\langle S\rangle)$ and 0 otherwise). For any such e, $\mathbf{E}(Y_e) = \mathbf{Pr}(\{u,v\} \subset S) = p^2$. So, by Linearity of the Expected Value, $\mathbf{E}(Y) = \sum_{e\in E(G)} \mathbf{E}(Y_e) = \frac{nd}{2}p^2$. Again by Linearity of the Expected Value, $\mathbf{E}(X - Y) = np - \frac{nd}{2}p^2$. We set $p = 1/d$ (here using $d \geqslant 1$) to maximize this quantity, giving $\mathbf{E}(X - Y) = \frac{n}{2d}$.

By the First Moment Principle, there exists a specific S for which the number of vertices in $G\langle S\rangle$ minus the number of edges in $G\langle S\rangle$ is at least $n/2d$. Select one vertex from each edge of $G\langle S\rangle$ and delete it. This leaves a set S' with at least $n/2d$ vertices, which is stable since all edges have been destroyed. □

A well-known example of the alteration method is the celebrated result of Erdős stating the existence of graphs with arbitrarily large girth and chromatic number.

A *colouring* of a graph $G = (V,E)$ is a mapping $c : V \to S$. The elements of S are called *colours*. If $|S| = k$, then we say that c is a k-colouring. A colouring is *proper* if $c(u) \neq c(v)$ for any edge $uv \in E(G)$. A graph is k*-colourable* if it has a proper k-colouring. The *chromatic number* $\chi(G)$ is the least k such that G is k-colourable. A proper k-colouring may also be seen as a partition of the vertex set of G into k disjoint stable sets $S_i = \{v \mid c(v) = i\}$ for $1 \leqslant i \leqslant k$. Hence $\chi(G) \geqslant \frac{|V(G)|}{\alpha(G)}$.

The *girth* of a graph is the length of a smallest cycle or $+\infty$ if the graph has no cycles.

Theorem 12 (Erdős [12]). *For any two positive integers g and k , there exists a graph G with girth larger than g and chromatic number larger than k.*

Proof. Set $\theta < \frac{1}{g}$. Let n be sufficiently large and G be a random graph in $\mathcal{G}_{n,p}$ with $p = n^{\theta-1}$. Let X be the number of cycles in G of length at most g. There are $\frac{n!}{2i(n-i)!}$ potential cycles of length i because there are $\frac{n!}{(n-i)!}$ sequences of i distinct vertices and each cycle of length i corresponds to $2i$ of these sequences. Furthermore, each potential cycle is in G with probability p^i. Hence according to the Linearity of the Expected Value

$$\mathbf{E}(X) = \sum_{i=3}^{g} \frac{n!}{2i(n-i)!} p^i \leqslant \sum_{i=3}^{g} \frac{n^{\theta i}}{2i} = o(n),$$

because $\theta g < 1$. So by Markov's Inequality

$$\mathbf{Pr}\left(X \geqslant \frac{n}{2}\right) \leqslant \frac{2\,\mathbf{E}(X)}{n} = o(1).$$

Set $x = \left\lceil \frac{3}{p} \ln(n) + 1 \right\rceil$. We have

$$\mathbf{Pr}(\alpha(G) \geqslant x) \leqslant \binom{n}{x} (1-p)^{\binom{x}{2}}.$$

But $\binom{n}{x} < n^x$ and $(1-p)^t < \exp(-pt)$. Hence,

$$\mathbf{Pr}(\alpha(G) \geqslant x) < (n \cdot \exp(-p(x-1)/2))^x = \left(n \times n^{-3/2}\right)^x = o(1).$$

Let n be large enough for the two above events to have probability less than $\frac{1}{2}$. Then there exists a graph H with less than $n/2$ cycles of length at most g such that $\alpha(H) < x < 3n^{1-\theta} \ln(n)$. Let us remove from H one vertex per cycle of length at most g. This produces a graph G on at least $n/2$ vertices with girth greater than g. Moreover $\alpha(G) \leqslant \alpha(H)$. Hence

$$\chi(G) \geqslant \frac{|V(G)|}{\alpha(G)} \geqslant \frac{n/2}{3n^{1-\theta} \ln(n) + 1} > \frac{1}{2} \cdot \frac{n^{\theta}}{6 \ln(n)}.$$

For n sufficiently large $\chi(G)$ is greater than k. □

5 Local Lemma

In typical proofs of combinatorial results, we generally have to show that the probability of some event is positive. In lots of cases, we have a set of "bad events" $A_i, i \in I$ and we want to show that with positive probability none of them occurs, that is $\mathbf{Pr}\left(\bigcap_{i\in I}\overline{A_i}\right) > 0$. For example, assume we want to know if a graph G has a proper k-colouring. A naive approach would be to randomly colour G in k colours and then examine whether this random k-colouring is a proper colouring. This will be the case if the ends of each edge of G receive distinct colours. Therefore, if we denote by A_e the event that the ends of e are assigned the same colour, we are interested in the probability $\mathbf{Pr}\left(\bigcap_{e\in E(G)}\overline{A}_e\right)$ that none of these bad events occurs. If we can show that this probability is positive, we have a proof that G is k-colourable.

One possibility is to use the Subadditivity of Probabilities which implies the result if $\sum_{i\in I}\mathbf{Pr}(A_i) < 1$. We can also conclude if the events are mutually independent (and with probability less than 1). Indeed, in this case,

$$\mathbf{Pr}\left(\bigcap_{i\in I}\overline{A_i}\right) = \prod_{i\in I}\mathbf{Pr}(\overline{A_i}) = \prod_{i\in I}(1-\mathbf{Pr}(A_i)) > 0.$$

Unfortunately, in general, the considered events are not mutually independent. The Local Lemma, established by Erdős and Lovász in 1975, show that the probability $\mathbf{Pr}\left(\bigcap_{i\in I}\overline{A_i}\right)$ remains strictly positive if the A_i are of small probability and, to some extent, sufficiently independent to each other. There are many different variants of the Local Lemma. We present here the two most common versions: the General Local Lemma and one of its corollary the Symmetric Local Lemma. The other variants of the Local Lemma will not be discussed here. The Symmetric Local Lemma is easy to handle and is enough for lots of proofs. Therefore it is the most commonly used. It is presented with several applications in Subsection 5.1. However, sometimes the Symmetric Local Lemma do not apply and we need to use the General Local Lemma. An example is given in Subsection 5.2.

Lemma 13 (Erdős and Lovász [13]). GENERAL LOCAL LEMMA
Let A_i, $i \in I$ be a set of events in a probability space $(\Omega,\mathbf{Pr})$, and let I_i, $i \in I$, be subsets of I. Suppose that for all $i \in I$,

(i) A_i is independent of the set of events $\{A_j \mid j \notin I_i\}$,

(ii) there exists a real number p_i such that $0 < p_i < 1$ and $\mathbf{Pr}(A_i) \leqslant p_i\prod_{j\in I_i}(1-p_j)$.

Then $\mathbf{Pr}\left(\bigcap_{i\in I}\overline{A_i}\right) \geqslant \prod_{i\in I}(1-p_i) > 0$.

Proof. For any $J \subset I$ we denote $\bigcap_{j\in J}\overline{A_j}$ by $\overline{A_J}$. We prove by induction with respect to the lexicographic order of the pair $(|J_1|,|J_2|)$ that for any two disjoint subsets J_1 and J_2 of I,

$$\mathbf{Pr}\left(\overline{A_{J_1}}\cap\overline{A_{J_2}}\right) \geqslant \mathbf{Pr}(\overline{A_{J_1}})\prod_{j\in J_2}(1-p_j).$$

For $J_1 = \emptyset$ and $J_2 = I$, this is the desired result.

If $J_2 = \emptyset$, then $\overline{A_{J_2}} = \Omega$ and $\prod_{j \in J_2}(1-p_j) = 1$, so

$$\mathbf{Pr}\left(\overline{A_{J_1}} \cap \overline{A_{J_2}}\right) = \mathbf{Pr}(\overline{A_{J_1}}) \geqslant \mathbf{Pr}(\overline{A_{J_1}}) \prod_{j \in J_2}(1-p_j).$$

If $J_2 = \{i\}$, then $\overline{A_{J_2}} = \overline{A_i}$ and $\prod_{j \in J_2}(1-p_j) = 1-p_i$. Setting $J'_1 = J_1 \setminus I_i$ and $J'_2 = J_1 \cap I_i$, we have:

$$\mathbf{Pr}(A_i \cap \overline{A_{J_1}}) \leqslant \mathbf{Pr}(A_i \cap \overline{A_{J'_1}}) = \mathbf{Pr}(A_i)\,\mathbf{Pr}(\overline{A_{J'_1}}).$$

By assumption, and the fact that $J'_2 \subset I_i$,

$$\mathbf{Pr}(A_i) \leqslant p_i \prod_{j \in I_i}(1-p_j) \leqslant p_i \prod_{j \in J'_2}(1-p_j).$$

Because $|J'_1| + |J'_2| = |J_1| < |J_1| + |J_2|$, we have by induction,

$$\mathbf{Pr}(\overline{A_{J'_1}}) \prod_{j \in J'_2}(1-p_j) \leqslant \mathbf{Pr}\left(\overline{A_{J'_1}} \cap \overline{A_{J'_2}}\right).$$

Therefore,

$$\mathbf{Pr}(A_i \cap \overline{A_{J_1}}) \leqslant p_i\,\mathbf{Pr}\left(\overline{A_{J'_1}} \cap \overline{A_{J'_2}}\right)$$

and so

$$\begin{aligned}
\mathbf{Pr}(\overline{A_{J_1}} \cap \overline{A_{J_2}}) = \mathbf{Pr}(\overline{A_{J_1}} \cap \overline{A_i}) &= \mathbf{Pr}(\overline{A_{J_1}}) - \mathbf{Pr}(A_i \cap \overline{A_{J_1}}) \\
&\geqslant \mathbf{Pr}(\overline{A_{J_1}}) - p_i\,\mathbf{Pr}(\overline{A_{J_1}}) = \mathbf{Pr}(\overline{A_{J_1}})(1-p_i).
\end{aligned}$$

If $|J_2| \geqslant 2$, we consider a partition of J_2 in two non-empty sets J'_1 and J'_2. Then

$$\mathbf{Pr}(\overline{A_{J_1}} \cap \overline{A_{J_2}}) = \mathbf{Pr}(\overline{A_{J_1}} \cap \overline{A_{J'_1 \cup J'_2}}) = \mathbf{Pr}(\overline{A_{J_1}} \cap \overline{A_{J_1}} \cap \overline{A_{J_2}} = \mathbf{Pr}(\overline{A_{J_1 \cup J'_1}} \cap \overline{A_{J'_2}}).$$

We now apply induction twice. Because $|J'_2| < |J_2|$,

$$\mathbf{Pr}\left(\overline{A_{J_1 \cup J'_1}} \cap \overline{A_{J'_2}}\right) \geqslant \mathbf{Pr}(\overline{A_{J_1 \cup J'_1}}) \prod_{j \in J'_2}(1-p_j) = \mathbf{Pr}(\overline{A_{J_1}} \cap \overline{A_{J'_1}}) \prod_{j \in J'_2}(1-p_j)$$

and since $|J_1 \cup J'_1| < |J_1 \cup J_2|$,

$$\mathbf{Pr}(\overline{A_{J_1}} \cap \overline{A_{J'_1}}) \geqslant \mathbf{Pr}(\overline{A_{J_1}}) \prod_{j \in J'_1}(1-p_j).$$

Hence

$$\mathbf{Pr}(\overline{A_{J_1}} \cap \overline{A_{J_2}}) \geqslant \mathbf{Pr}(\overline{A_{J_1}}) \prod_{j \in J'_1}(1-p_j) \prod_{j \in J'_2}(1-p_j) = \mathbf{Pr}(\overline{A_{J_1}}) \prod_{j \in J_2}(1-p_j).$$

$\square$

5.1 Symmetric Local Lemma

We will denote by e the constant equal to $exp(1)$.

Lemma 14. SYMMETRIC LOCAL LEMMA
Let A_i, $i \in I$ be a set of events in a probability space $(\Omega, \mathbf{Pr})$ such that, for each $i \in I$,

(i) $\mathbf{Pr}(A_i) \leqslant p$ *and*

(ii) A_i *is independent of all other events but at most* d.

If $\mathrm{e}\, p(d+1) \leqslant 1$, *then* $\mathbf{Pr}\left(\bigcap_{i \in I} \overline{A_i}\right) > 0$.

Proof. Set $p_i = p$ for all $i \in I$, in the General Local Lemma. Now set $p = \frac{1}{d+1}$ in order to maximize $p(1-p)^d$ and apply the inequality $\left(\frac{d}{d+1}\right)^d = \left(1 - \frac{1}{d+1}\right)^d > \frac{1}{\mathrm{e}}$. □

Theorem 15. *Let H be a k-uniform hypergraph in which each hyperedge intersects at most d other hyperedges. If $\mathrm{e}(d+1) \leqslant 2^{k-1}$, then H is 2-colourable.*

Proof. Let us consider a uniform random 2-colouring of H in which every vertex is coloured independently *red* with probability $1/2$ and *blue* with probability $1/2$. For every hyperedge e, let A_e be the event that e is monochromatic. Then $\mathbf{Pr}(A_e) = 2^{1-k} = p$.

Claim: *Every A_e is independent of all the events A_f such that $e \cap f = \emptyset$.*

This claim implies that every A_e is independent of all the events but at most d. Hence, if $\mathrm{e}(d+1) \leqslant 2^{k-1}$, then $\mathrm{e}\, p(d+1) \leqslant 1$ and the Symmetric Local Lemma yields the result.

It remains to prove the claim. Intuitively it seems clear but let us check it.
Set $e = \{v_1, \dots, v_k\}$. Let $f_1, \dots, f_r$ be edges that do not intersect e and Γ the set of 2-colourings of H for which the event $B = A_{f_1} \cap \dots \cap A_{f_r}$ occurs. For each 2-colouring c of $G - \{v_1, \dots, v_k\}$, let T_c be the set of the 2^k different 2-colourings of G that extend c. It is easy to verify that for all c, Γ contains either all the colourings T_c or none of them. Hence, there exist l and 2-colourings $c_1, \dots, c_l$ such that Γ is the disjoint union $T_{c_1} \cup \dots \cup T_{c_l}$. Thus $\mathbf{Pr}(B) = \frac{2^k l}{2^n}$.
Now in each T_c, there are exactly two 2-colourings for which e is monochromatic, and so $\mathbf{Pr}(A_e \cap B) = \frac{2l}{2^n}$. Thus $\mathbf{Pr}(A_e | B) = (\frac{2l}{2^n}) / \mathbf{Pr}(B) = 2^{-(k-1)} = \mathbf{Pr}(A_e)$ as claimed. □

The claim in the preceding proof is a particular case of a more general principle allowing to show independence. In a very large majority of the proofs, the independence is proved with this principle.

Theorem 16. INDEPENDENCE PRINCIPLE
Suppose that $X = X_1, \dots, X_m$ is a sequence of independent random trials. Suppose moreover that $A_1, \dots, A_N$ is a set of events such that each A_i is determined by $F_i \subset X$. If $F_i \cap \left(\bigcup_{j \in J} F_j\right) = \emptyset$ then A_i is independent of $\{A_j \mid j \in J\}$.

Proof. The proof of this principle is similar to the of the above claim. It is left in Exercise 17. □

A *list-assignment* of a graph G is an application L which assigns to each vertex $v \in V(G)$ a prescribed list of colours $L(v)$. A list-assignment is a *k-list-assignment* if each list is of size at least k. An *L-colouring* of G is a colouring c such that $c(v) \in L(v)$ for all $v \in V(G)$. A graph G is *L-colourable* if there exists a proper L-colouring of G.

Theorem 17. *Let L be an l-list assignment of a graph G. If for every vertex v every colour $i \in L(v)$ appears in the list of at most $\frac{l}{2\mathrm{e}}$ neighbours of v, then G is L-colourable.*

Proof. Let us consider a random colouring of G for which every vertex v is assigned a colour of its list uniformly (with probability $1/l$). For each edge $e = xy$ and each colour $i \in L(x) \cap L(y)$, let $A_{e,i}$ be the event that x and y are both coloured i. Let us denote by $\mathcal{A}$ the set of such events. We will use the Symmetric Local Lemma to show that with positive probability no event of $\mathcal{A}$ occurs and so the colouring is proper.

Firstly, for every e and i, $\mathbf{Pr}(A_{e,i}) = \frac{1}{l^2} = p$. Secondly, let us consider the dependence of the events. If $e = xy$, then $A_{e,i}$ is determined by the colours assigned to x and y uniquely. Hence, setting $\mathcal{A}_x = \{A_{f,j} \mid x \text{ is an endvertex of } f, j \in L(x)\}$ and $\mathcal{A}_y = \{A_{f,j} \mid y \text{ is an endvertex of } f, j \in L(y)\}$, by the Independence Principle, $A_{e,i}$ is independent of all events of $\mathcal{A} \setminus (\mathcal{A}_x \cup \mathcal{A}_y)$. Now as $L(x)$ has l elements and for all $i \in L(x)$, x has at most $l/2\mathrm{e}$ neighbours coloured i, we have $|\mathcal{A}_x| \leqslant \frac{l^2}{2\mathrm{e}}$. Similarly, $|\mathcal{A}_y| \leqslant \frac{l^2}{2\mathrm{e}}$ and so $|\mathcal{A}_x \cup \mathcal{A}_y| \leqslant \frac{l^2}{\mathrm{e}}$. Since $A_{e,i}$ is in $\mathcal{A}_x \cup \mathcal{A}_y$, it is independent of all other events but at most $d = \frac{l^2}{\mathrm{e}} - 1$.

Now $\mathrm{e}\,p(d+1) \leqslant 1$, so the Symmetric Local Lemma gives the result. □

Molloy and Reed [21] conjectured that the $\frac{l}{2\mathrm{e}}$ in the above theorem can be replaced by $l-1$.

Conjecture 18 (Molloy and Reed [21])**.** Let L be an l-list assignment of a graph G. If for every vertex v every colours $i \in L(v)$ appears in the list of at most $l-1$ neighbours of v, then G is L-colourable.

Using different techniques, Haxell [15] has shown that the result holds if the value is $l/2$ and by iteratively applying the Local Lemma, Reed and Sudakov [22] have shown that $l - o(l)$ is sufficient.

As it is often the case with the Local Lemma, once the bad events are chosen, the proof is direct. However, choosing appropriate bad events is sometimes astute. For example, in Exercise 16, we give definitions of two natural bad events for the proof of Theorem 17 which yield no proof.

5.2 An application of the General Local Lemma

A cycle in a graph is *hamiltonian* if it contains all the vertices.

Conjecture 19 (Sheehan [23])**.** Every hamiltonian k-regular graph, $k \geqslant 3$, has at least two hamiltonian cycles.

Sheehan's Conjecture has been proved for odd k. (See Chapter 18 of [8].) Hence it can be restricted without loss of generality to 4-regular graphs. For if C is a hamiltonian cycle of G, then the spanning subgraph $G \setminus E(C)$ is regular of positive even degree, and hence has a 2-factor F, (i.e. a 2-regular spanning subgraph). The graph $H := F \cup C$ is a 4-regular spanning subgraph with a hamiltonian cycle C. If we could prove that H had a second hamiltonian cycle, then G would also have this second hamiltonian cycle.

Theorem 20 (Thomassen [26])**.** *For $k \leqslant 73$, every hamiltonian k-regular graph has at least two hamiltonian cycles.*

The bound $k \geqslant 73$ in this theorem was reduced to $k \geqslant 23$ by Haxell et al. [16]. However, Sheehan's Conjecture remains open.

The proof of Theorem 20 uses a general sufficient condition for the existence of at least two hamiltonian cycles in a graph having one. The argument is based on the following concepts. Consider a (not necessarily proper) 2-edge-colouring of a graph G in red and blue. A set S of vertices of G is called *red-stable* if no two vertices of S are joined by a red edge, and *blue-dominating* if every vertex of $V \setminus S$ is adjacent by a blue edge to at least one vertex of S.

Lemma 21 (Thomassen [26])**.** *Let G be a graph and let C be a hamiltonian cycle of G. Colour the edges of C red and the remaining edges of G blue. If there is a red-stable blue-dominating set S in G, then G has a second hamiltonian cycle.*

Proof of Theorem 20. Let G be a hamiltonian k-regular graph, and let C be a hamiltonian cycle of G. As in Lemma 21, we colour the edges of C red and the remaining edges of G blue. We now select each vertex of G independently, each with probability p, so as to obtain a random subset S of V. We show that, for an appropriate choice of p, this set S is, with positive probability, a red-stable blue-dominating set. The theorem then follows on applying Lemma 21.

For each element of $E(C) \cup V(G)$, we define a "bad" event, as follows.

- A_e: both ends of edge e of C belong to S.
- B_v: neither vertex v of G nor any vertex joined to v by a blue edge belongs to S.

We have $\mathbf{Pr}(A_e) = p^2$ and $\mathbf{Pr}(B_v) = (1-p)^{k-1}$, because each vertex v has blue degree $k-2$.

The set A_e is determined by the two trials for its endvertices, and the set B_v is determined by the trial of v and the neighbours to which it is linked by a blue edge. By the Independence Principle, for every edge $e = uv$ of C, the event A_e is independent of all events but the two A_f for f an edge having u or v has endvertex, and the $2k-2$ events B_w for w in $\{u,v\}$ or joined by a blue edge to either u or v. The $k-1$ vertices determining B_v are each involved in two events A_e, and are together involved in a total of at most $(k-2)^2$ other events B_w. Thus, by the Independence Principle, B_v is independent to all events except at most $(2k-2)$ events A_f and $(k-2)^2$ events B_w. In order to apply the Local Lemma, we must therefore select a value for p and numbers x (associated with each event A_e) and y (associated with each event B_v) such that

$$p^2 \leqslant x(1-x)^2(1-y)^{2k-2} \quad \text{and} \quad (1-p)^{k-1} \leqslant y(1-x)^{2k-2}(1-y)^{(k-2)^2}.$$

We may simplify these expressions by setting $x = a^2$ and $y = b^{k-1}$:

$$p \leqslant a(1-a^2)\left(1-b^{k-1}\right)^{k-1} \quad \text{and} \quad 1-p \leqslant b\left(1-a^2\right)^2\left(1-b^{k-1}\right)^{k-3}.$$

Thus

$$1 \leqslant a\left(1-a^2\right)\left(1-b^{k-1}\right)^{k-1} + b\left(1-a^2\right)^2\left(1-b^{k-1}\right)^{k-3}.$$

For $k \geqslant 73$, a solution to this inequality is obtained by setting $a = .25$ and $b = .89$, resulting in a value of .2305 for p. □

6 Chernoff's Bound

The *binomial random variable* $\text{BIN}(n,p)$ is the sum of n independent zero-one random variables where each is equal to 1 with probability p.

Chernoff's Bound is an important result stating that $\text{BIN}(n,p)$ is concentrated around its expected value. See [18, 5].

Theorem 22. CHERNOFF'S BOUND
For every $t \in [0, np]$,

$$\mathbf{Pr}\left(|\text{BIN}(n,p) - np| > t\right) < 2\exp\left(-\frac{t^2}{3np}\right).$$

There are other versions of the Chernoff's Bound: for every $t > 0$,

$$\mathbf{Pr}\left(|\text{BIN}(n,p) - np| > t\right) < 2\exp\left(t - \ln\left(1+\frac{t}{np}\right)(np+t)\right).$$

Chernoff's Bound is a tool to bound the probability of some events. It can be an element of proofs following the First Moment Method as shown in the two following Subsections, but it is also very convenient for the use of the Local Lemma as examplified in Section 6.3.

6.1 Hypergraph colouring

Let H be a hypergraph and let c be a 2-colouring of its vertices. For all hyperedge e, the *discrepancy* of e is the absolute value of the difference between the number of vertices of e in each colour class. The *discrepancy of H with respect to c* is the maximum discrepancy of a hyperedge of H. The *discrepancy of H*, $disc(H)$, is the minimum over all 2-colourings of the vertex set of H, of the discrepancy of H with respect to the 2-colouring.

For example, if H is k-uniform, then $disc(H) < k$ if and only if H is 2-colourable. In a certain sense, $disc(H)$ measure how "balanced" a 2-colouring we can obtain for H.

Theorem 23. *Let H be a k-uniform hypergraph with k hyperedges. Then $disc(H) \leqslant \sqrt{8k\ln(k)}$.*

Proof. We may assume $k \geqslant 9$ as if $k \leqslant 8$, $disc(H) \leqslant k < \sqrt{8k\ln(k)}$.

Consider a random 2-colouring of H with *red* and *blue* obtained by assigning to each vertex a random colour with probability $1/2$ for each colour, and where the choices corresponding to different vertices are mutually independent. For any hyperedge e, the number of vertices of e which receive colour *red* is distributed like $BIN(k, \frac{1}{2})$. Applying Chernoff's Bound with $t = \sqrt{2k\ln(k)}$, we get that the probability that the discrepancy of e is greater than $2t$ is at most:

$$\mathbf{Pr}\left(\left|BIN\left(k,\frac{1}{2}\right)-\frac{k}{2}\right| > t\right) < 2\exp\left(-\frac{t^2}{3\times\frac{k}{2}}\right) = 2k^{-\frac{4}{3}} < \frac{1}{k}.$$

Hence by Linearity of the Expected Value, the expected number of hyperedges with discrepancy greater than $2t$ is less than 1. Thus with positive probability, there are no such hyperedges, and the desired 2-colouring exists. □

We refer the reader to [5] and [9] for a more thorough discussion of discrepancy.

6.2 Choosablity of complete multipartite graphs with equal colour classes

A graph G is *k-choosable* if it is L-colourable for every k-list-assignment L. The *choice number*, *choosability* or *list chromatic number* of G, denoted $ch(G)$, is the least k such that G is k-choosable.

For two positive integers m and r, we denote by K_{m*r} the complete r-partite graph with m vertices in each vertex class. The graph K_{m*1} is a graph with m vertices and no edges and so $ch(K_{m*1}) = 1$. The graph K_{1*r} is the complete graph on r vertices K_r, so $ch(K_{1*r}) = r$. Erdős, Rubin and Taylor [14] showed that $ch(K_{2*r}) = r$ for all r. Alon determined, up to a constant factor, the choice number of all the remaining cases.

Theorem 24 (Alon [3])**.** *There exist two positive constants c_1 and c_2 such that for every $m \geqslant 2$ and for every $r \geqslant 2$*

$$c_1 r\log m \leqslant ch(K_{m*r}) \leqslant c_2 r\log m.$$

Proof. Since $ch(K_{m*r})$ is a non-decreasing function of r, we will assume that r is a power of 2. Since $ch(K_{m*r}) \leqslant rm$ and $rm \leqslant c_2 r\log m$ for all $m \leqslant c_2$, we may assume that $m > c_2$. Let us denote by $V_1, \dots V_r$ the colour classes of K_{m*r} and V its vertex set. Let L be a $c_2 r\log m$-list assignment of K_{m*r} and $S = \bigcup_{v\in V} L(v)$ be the set of all colours.

We consider two possible cases.

Case 1: $r \leqslant m$**.** Let $f : S \to \{1, 2, \dots, r\}$ be a random function, obtained by choosing, for each colour $c \in S$, randomly and independently, the value of $f(c)$ according to a uniform distribution. The colours c for which $f(c) = i$ will be the ones to be used for colouring the vertices in V_i . To complete the proof for this case, let us show that with positive probability for every i, $1 \leqslant i \leqslant r$, and for every vertex $v \in V_i$ there is at least one colour $c \in L(v)$ such that $f(c) = i$.

Fix an i and a vertex $v \in V_i$. The probability that there is no colour $c \in L(v)$ such that $f(c) = i$ is

$$\left(1-\frac{1}{r}\right)^{c_2 r\log m} \leqslant \exp(-c_2\log m) \leqslant \frac{1}{m^{c_2}} < \frac{1}{rm},$$

where the last inequality follows from the fact that $r \leqslant m$ and $c_2 > 2$. There are rm possible choices of i, $1 \leqslant i \leqslant r$ and $v \in V_i$, and hence, the probability that for some i and some $v \in V_i$ there is no $c \in L(v)$ so that $f(c) = i$ is smaller than 1. This completes the proof of Case 1.

Case 2: $r > m$**.** To prove this case we will use a (more or less standard) splitting trick. It was first used in [2].

Set $R_1 = \{1, \ldots, r/2\}$ and $R_2 = \{r/2+1, \ldots, r\}$. Let $f : S \to \{1,2\}$ be a random function obtained by choosing, for each $c \in S$ randomly and independently, $f(c) \in \{1,2\}$ according to a uniform distribution. The colours c for which $f(c) = 1$ will be used for colouring the vertices in $\bigcup_{i \in R_1} V_i$, whereas the colours c for which $f(c) = 2$ will be used for colouring the vertices in $\bigcup_{i \in R_2} V_i$.

For every vertex v, set $L_0(v) = L(v)$ and define $L_1(v) = L_0(v) \cap f^{-1}(1)$ if v belongs to $\bigcup_{i \in R_1} V_i$ and $L_1(v) = L_0(v) \cap f^{-1}(2)$ if v belongs to $\bigcup_{i \in R_2} V_i$. Then the problem of finding a proper L-colouring of K_{m*r} is decomposed into two independent problems: the ones of finding proper colourings of the two complete $r/2$-partite graphs on the vertex classes $\bigcup_{i \in R_1} V_i$ and $\bigcup_{i \in R_2} V_i$, by assigning to each vertex v a colour from $L_1(v)$. Set $l_0 = c_2 r \log m$. Then $|L_1(v)|$ is a binomial random variable $BIN(l_0, 1/2)$. Hence by Chernoff's Bound,

$$\mathbf{Pr}\left(|L_1(v)| < \frac{1}{2}l_0 - \frac{1}{2}l_0^{2/3}\right) \leqslant \exp\left(-\frac{1}{2}c_2^{1/3} r^{1/3} (\log m)^{1/3}\right).$$

The total number of vertices is $rm < r^2$. Since $r > m > c_2$ and c_2 can be chosen to be a sufficiently large constant (independent of r and m), one can easily check that for all $r > m > c_2$: $r^2 \cdot \exp\left(-\frac{1}{2}c_2^{1/3} r^{1/3} (\log m)^{1/3}\right) << 1$. Hence for all sufficiently large c_2, with high probability,

$$|L_1(v)| \geqslant \frac{1}{2}l_0 - \frac{1}{2}l_0^{2/3}$$

for all vertex v. Setting $l_1 = \min\{L_1(v) \mid v \in V\}$, we can make sure that $l_1 \geqslant \frac{1}{2}l_0 - \frac{1}{2}l_0^{2/3}$. Hence we have reduced the problem of showing that the choice number of K_{m*r} is at most l_0 to that of showing that the choice number of $K_{m*(r/2)}$ is at most l_1.

Repeating the above decomposition technique (which we can repeat as long as $r/2^i > m$) we obtain a sequence l_i , where $l_0 = c_2 r \log m$ and

$$l_{i+1} \geqslant \frac{1}{2}l_i - \frac{1}{2}l_i^{2/3}.$$

In order to show that the choice number of K_{m*r} is at most l_0, it suffices to show that for some i, $ch(K_{m*(r/2^i)}) \leqslant l_i$. Let the number of iterations j be chosen so that j is the minimum integer satisfying $r/2^j \leqslant m$. Clearly, in this case, $r/2^j > m/2 \geqslant c/2$.

Claim 24.1. $l_j \geqslant \frac{l_0}{2^{j+1}}$.

Proof. Let us define $z_i = l_i^{1/3}$. Then for all $1 \leqslant i \leqslant j$,

$$z_{i+1}^3 \geqslant \frac{z_i^3 - z_i^2}{2} \geqslant \frac{(z_i - 1)^3}{2},$$

and so

$$z_{i+1} \geqslant \frac{z_i - 1}{2^{1/3}}.$$

Let $x = \frac{1}{2^{1/3}-1} < 4$. Then $x = \frac{1+x}{2^{1/3}}$ and set $t_i = z_i + x$. Then $t_{i+1} - x \geqslant \frac{t_i - x - 1}{2^{1/3}}$, so $t_{i+1} \geqslant \frac{t_i}{2^{1/3}}$. Hence

$$t_j \geqslant \frac{t_0}{2^{j/3}} = \frac{z_0 + x}{2^{j/3}} \geqslant \frac{z_0}{2^{j/3}} = \left(\frac{l_0}{2^j}\right)^{1/3}.$$

Therefore

$$l_j = z_j^3 = (t_j - x)^3 \geqslant \left(\left(\frac{l_0}{2^j}\right)^{1/3} - x\right)^3 = \frac{l_0}{2^j} - O\left(\left(\frac{l_0}{2^j}\right)^{2/3}\right). \quad (2)$$

Since $r/2^j \geqslant m/2 \geqslant c_2/2$ and $c_2 \geqslant 4$, it follows that $\frac{l_0}{2^j} = \frac{c_2 r \log m}{2^j} \geqslant \frac{c_2^2 \log m}{2} > c_2$. Thus if c_2 is sufficiently large then the right hand side of (2) is at least $l_0/2^{j+1}$. □

Let us now show that the choice number of $K_{m*(r/2^j)}$ is at most l_j. We have $r/2^j \leqslant m$ and $l_j \geqslant l_0/2^{j+1} \geqslant \frac{c_2}{2}\frac{r}{2^j}\log m$. Hence for a succiently large c_2 the result follows from Case 1. This completes the proof. □

6.3 Total colouring

A *total colouring* of G is a mapping f from $V(G) \cup E(G)$ into a set S of *colours* such that:

- adjacent vertices have different colours;
- incident edges have different colours;
- each edge and its endvertices have different colours.

If $|S| = k$ then f is a *k-total colouring*. A graph is *k-total colourable* if it has a k-total colouring. The *total chromatic number* $\chi^T(G)$ of a graph G is the least k such that G is k-total colourable. The colour classes in a total colouring are called *total stable sets*.

Since a vertex and the edges incident to it need different colours, we have $\chi^T(G) \geqslant \Delta(G) + 1$.

Using distinct colours for vertices and edges, we get $\chi^T(G) \leqslant \chi(G) + \chi'(G)$. Hence applying Brooks' and Vizing's Theorems and analyzing the case of odd cycles and complete graphs, we obtain:

$$\chi^T(G) \leqslant 2\Delta(G) + 1.$$

This upper bound is clearly not the best possible and the following is conjectured.

Total Colouring Conjecture $\chi^T(G) \leqslant \Delta(G) + 2$.

Total colouring was introduced by Vizing [27, 28] and independently by Behzad [7]. They both formulated the Total Colouring Conjecture.

Theorem 25. *For any graph G with maximum degree Δ sufficiently large, $\chi^T(G) \leqslant \Delta + 2\Delta^{\frac{3}{4}}$.*

Proof. Since every graph with maximum degree Δ is the subgraph of a Δ-regular graph, we may assume that G is Δ-regular.

Set $k = \left\lfloor \Delta^{\frac{1}{3}} \right\rfloor$ and $l = \left\lfloor \frac{\Delta + \Delta^{\frac{3}{4}}}{k} \right\rfloor$. Let us fix an edge-colouring of G with colours $1, \dots, \Delta+1$. Such an edge-colouring exists by Vizing's Theorem.

We then find a colouring of G with colours in $\{1, \dots, kl\}$ as follows. We first partition $V(G)$ into k subsets $V_1, \dots, V_k$ such that

(i) for each vertex v and part i, $|N(v) \cap V_i| \leqslant l-1$,

(ii) for each vertex v, there are at most $\Delta^{\frac{3}{4}} - 2$ edges $e = uv$ such that $u \in V_i$ and the colour of e belongs to $C_i = \{(i-1)l+1, \dots, il\}$.

Next we refine this partition into a proper colouring, assigning to the vertices of V_i a colour in C_i. This is possible using the greedy algorithm because by (i), the subgraph induced by V_i has maximum degree at most $l-1$.

We now have a proper vertex colouring and a proper edge-colouring. However, the union of these two colourings may not yet be a total colouring because some edges may have the same colour as one of its endvertices. Let R be the graph induced by such so called *conflictuous* edges. By (ii), we have $\Delta(R) \leqslant \Delta^{\frac{3}{4}} - 1$. Indeed, at each vertex v, at most $\Delta^{\frac{3}{4}} - 2$ edges uv are conflictuous because of u and at most one is conflictuous because of v. Hence one can recolour the edges of R using at most $\Delta^{\frac{3}{4}}$ new colours. We then obtain a total colouring of G with at most $kl + \Delta^{\frac{3}{4}} \leqslant \Delta + 2\Delta^{\frac{3}{4}}$ colours.

It remains to prove that the partition satisfying (i) and (ii) actually exists. To do so, we assign each vertex to a part uniformly at random, where these choices are made independently. For every $v \in V(G)$ and every $1 \leqslant i \leqslant k$, let $A_{v,i}$ be the event that (i) fails to hold for (v,i) and let B_v be the event that (ii) fails to hold for v. We shall use the Symmetric Local Lemma to show that with positive probability none of these events occurs.

B_v and $A_{v,i}$ are determined by the colours of the vertices adjacent to v. Hence by the Independence Principle, they are independent of all events but those concerning vertices at distance at most two of v. Thus every event is independent of all events except at most $(k+1)\Delta^2 \leqslant \Delta^3 - 1$. We shall prove that the probability of each event is at most $\frac{1}{\mathrm{e}\Delta^3}$. Thus, by the Symmetric Local Lemma (Lemma 14), there exists a partition satisfying (i) and (ii).

Consider first the event B_v. Let R_v be the set of edges uv such that $u \in V_i$ and e has a colour in C_i. Since there are k parts, the probability that this occurs for a given edge e is $\frac{1}{k}$. Moreover, the choices are made independently, so the size of R_v is the sum of Δ independent 0-1variables, each of which is 1 with probability $\frac{1}{k}$. Hence Chernoff's Bound applied to $BIN\left(\Delta, \frac{1}{k}\right)$ yields:

$$\mathbf{Pr}\left(\left||R_v| - \frac{\Delta}{k}\right| > \frac{\Delta}{k}\right) \leqslant 2\exp\left(-\frac{\Delta}{3k}\right).$$

Since $k = \left\lfloor \Delta^{\frac{1}{3}} \right\rfloor$ and $\frac{\Delta^{\frac{3}{4}}}{2} > \frac{\Delta}{k}$, for Δ sufficiently large, $\mathbf{Pr}(B_v) \leqslant 2\exp\left(-\Delta^{1/2}\right) < \frac{1}{\mathrm{e}\Delta^3}$.

Consider now $A_{v,i}$. The size of $N(v) \cap V_i$ s the sum of Δ independent 0-1variables, each of which is 1 with probability $\frac{1}{k}$. Hence applying Chernoff's Bound as above, we obtain that for Δ sufficiently large

$$\mathbf{Pr}(A_{v,i}) \leqslant \mathbf{Pr}\left(\left| |N(v) \cap V_i| - \frac{\Delta}{k} \right| > \frac{\Delta^{\frac{3}{4}}}{k}\right) \leqslant 2\exp\left(\frac{-\Delta^{1/6}}{3}\right) < \frac{1}{\mathrm{e}\Delta^3}.$$

□

7 Other concentrations inequalities

Unfortunately, all the random variables are not binomial and in general we cannot use Chernoff's Bound to prove that a random variable is concentrated around its expected value. However, there are many other concentration bounds that may be applied to random variable satisfying some given properties. This bound are used in the same way as Chernoff's Bound b ut for more general type of random variables. We now list some of them.

The following is a simple corollary of Azuma's Inequality [6, 21].

Theorem 26. SIMPLE CONCENTRATION BOUND
Let X be a non-negative random variable determined by the independent trials $T_1, \dots, T_n$. Suppose that for every set of possible outcomes of the trials

(i) changing the outcome of any one trial can affect X by at most c.

Then

$$\mathbf{Pr}(|X - \mathbf{E}(X)| > t) \leqslant 2\exp\left(-\frac{t^2}{c^2 n}\right).$$

Talagrand's Inequality requires another condition, but often provides a stronger bound when $\mathbf{E}(X)$ is much smaller than n. Rather than providing Talagrand's original statement [25], we present the following useful corollary [21].

Theorem 27. TALAGRAND'S INEQUALITY
Let X be a non-negative random variable determined by the independent trials $T_1, \dots, T_n$. Suppose that for every set of possible outcomes of the trials

(i) changing the outcome of any one trial can affect X by at most c; and

(ii) for each $s > 0$, if $X \geqslant s$ then there is a set of at most rs trials whose outcomes certify that $X \geqslant s$.

Then for every $t \in [0, \mathbf{E}(X)]$,

$$\mathbf{Pr}\left(|X - \mathbf{E}(X)| > t + 60c\sqrt{r\mathbf{E}(X)}\right) \leqslant 4\exp\left(-\frac{t^2}{8c^2 r\mathbf{E}(X)}\right).$$

McDiarmid extended Talagrand's Inequality to the setting where X depends on independent trials and permutations, a setting that arises in this paper. Again, we present a useful corollary [21] rather than the original inequality [19].

Theorem 28. McDiarmid's Inequality
Let X be a non-negative random variable determined by the independent trials $T_1,\dots,T_n$ and m independent permutations $\Pi_1,\dots,\Pi_m$. Suppose that for every set of possible outcomes of the trials

(i) changing the outcome of any one trial can affect X by at most c;

(ii) interchanging two elements in any one permutation can affect x by at most c; and

(iii) for each $s>0$, if $X \geqslant s$ then there is a set of at most rs trials whose outcomes certify that $X \geqslant s$.

Then for every $t \in [0,\mathbf{E}(X)]$,

$$\mathbf{Pr}\left(|X-\mathbf{E}(X)| > t + 60c\sqrt{r\mathbf{E}(X)}\right) \leqslant 4\exp\left(-\frac{t^2}{8c^2r\mathbf{E}(X)}\right).$$

In both Talagrand's Inequality and McDiarmid's Inequality, if $60c\sqrt{r\mathbf{E}(X)} \leqslant t \leqslant \mathbf{E}(X)$ then by substituting $t/2$ for t in the above bounds, we obtain the more concise

$$\mathbf{Pr}\left(|X-\mathbf{E}(X)| > t\right) \leqslant 4\exp\left(-\frac{t^2}{32c^2r\mathbf{E}(X)}\right).$$

8 Exercises

Exercise 1. Calculate the expected value of the number of isolated vertices (i.e. incidents to no edge) in a random graph in $\mathcal{G}_{n,p}$.

Exercise 2. Let $G \in \mathcal{G}_{n,\frac{1}{2}}$. For all $S \subseteq V(G)$, let A_S be the event that S is stable in G. Show that if S and T are two distinct subsets of k vertices then A_S and A_T are independent if and only if $|S \cap T| \leqslant 1$.

Exercise 3. A *random k-colouring* fo a graph G is an element of the probability space $(\Omega,\mathbf{Pr})$ where Ω is the set of all k-colourings (i.e. partition of V into k sets $(V_1,V_2,\dots,V_k)$), all this colourings being equally likely (so happening with probability k^{-n}). For every edge e of G, let A_e be the event that the two endvertices of e receive the same colour. Show that:

a) for any two edges e and f of G, the events A_e and A_f are independent.

b) if e, f and g are three edges of a triangle of G, the events A_e, A_f and A_g are dependents.

Exercise 4. Show the Subadditivity of Probabilities from the First Moment Principle.

Exercise 5. Suppose $k \geqslant 2$ and let H be a k-uniform hypergraph with 4^{k-1} edges. Show that there is a 4-colouring of $V(H)$ such that no edge is monochromatic.

Exercise 6. Let H be a graph, and let $n > |V(H)|$ be an integer. Assume there is a graph on n vertices and t edges containing no copy of H, and assume that $tk > n^2 \ln n$. Show that there is a colouring of the edges of K_n, the complete graph on n vertices by k colours with no monochromatic copy of H.

Exercise 7. Show that there exists a 2-edge-colouring of K_n with at most $\binom{n}{p} 2^{1-\binom{p}{2}}$ monochromatic K_p.

Exercise 8. A *tournament* T is a digraph such that for all pair of distinct vertices $x \neq y$ exactly one of the arcs xy and yx is in $E(T)$. In other words, a tournament is the orientation of a complete graph.

A tournament satisfies *Property* P_k if for every set of k vertices, there exists a vertex dominating all vertices of S.

a) Show that if $\binom{n}{k}(1-2^{-k})^{n-k} < 1$, then there exists on tournament on n vertices which satisfies *Property* P_k.

b) Deduce that there exists a tournament on $\lceil 4k^2 2^k \rceil$ vertices which satisfies *Property* P_k.

Exercise 9. A random tournament on n vertices is the orientation of K_n such that every edge xy is oriented from x to y with probability $1/2$ and from y to x with probability $1/2$, all these choices being made independently.

a) Show that the expected number of hamiltonian paths in a random tournament of order n is $n! \cdot 2^{-(n-1)}$.

b) Deduce that, for all $n \geqslant 1$, there is a tournament on n vertices with at least $n! \cdot 2^{-(n-1)}$ hamiltonian paths.

Exercise 10. Let $G = (V, E)$ be a graph and $(v_1, v_2, \ldots, v_n)$ an ordering of the vertices.

1) Prove that $S = \{v_i \mid N(v_i) \subset \{v_1, \ldots, v_{i-1}\}\}$ is stable.

2) Deduce that $\alpha(G) \geqslant \sum_{v \in V} \frac{1}{d(v)+1}$.

3) Show that $\alpha(G) \geqslant \frac{|V(G)|^2}{2|E(G)| + |V(G)|}$.

Exercise 11. Let G be a bipartite graph on n vertices and L a $\lceil \log_2 n \rceil$-list assignment of G. Prove G is L-colourable.

Exercise 12. Show that for n sufficiently large, there exists a graph with n vertices such that $\chi(G) \geqslant \frac{n}{2}$ and $\omega(G) \leqslant n^{3/4}$. (*Hint:* What can we say about the chromatic number of the complement of a triangle-free graph?)

Exercise 13. Show by the First Moment Principle that every graph having a matching of size m has a bipartite subgraph with at least $\frac{1}{2}(|E(G)|+m)$ edges.
(*Hint:* how can we choose a random bipartition such that the edges of the matching have their endvertices in opposite parts?)

Exercise 14. Let α and c be fixed with $\alpha > 5/6$. Let $p = n^{-\alpha}$. The aim of the exercise is to show that almost always every $c\sqrt{n}$ vertices of $G \in \mathcal{G}_{n,p}$ is 3-colourable.

Let T be a minimal set such that $G\langle T\rangle$ is not 3-colourable.

1) Show that if T has t vertices then T has at least $3t/2$ edges.

2) Conclude.
Hint: One could use the inequalities $\binom{n}{t} \leqslant \left(\frac{n\,\mathrm{e}}{t}\right)^t$ and $\binom{\binom{t}{2}}{\frac{3t}{2}} \leqslant \left(\frac{t\,\mathrm{e}}{3}\right)^{3t/2}$.

Exercise 15. Let G be a graph on n vertices which is not complete. The goal of this exercise is to show the following result proved independently by Chetwynd and Häggkvist [10] and McDiarmid and Reed [20]. *If $k! \geqslant n$, then there is a total colouring of G with at most $\chi'(G)+k+1$ colours.*

Set $q = \chi'(G)$.

1) Show that there exists a proper (vertex) colouring c with q colours.

2) Let $\mathcal{M} = \{M_1, \dots, M_q\}$ the set of q matchings which are the colour classes of a proper q-edge-colouring of G. To every bijection $\Pi : \mathcal{M} \to \{1, \dots, q\}$, we associate the *conflict graph* R_Π which is the subgraph of G whose edges are the xy such that $c(x) = \Pi(xy)$ or $c(y) \in \Pi(xy)$.
Show that there exists Π such that $\Delta(R_\Pi) \leqslant k$.

3) Deduce $\chi_T(G) \leqslant q+k+1$.

Exercise 16. Show what is wrong when we try to prove Theorem 17 by colouring each vertex independently and uniformly with a random colour in its list and applying the Symmetric Local Lemma to the following bad events:

1) For all vertex v, A_v is the event that v receives the same colour as one of its neighbours.

2) For every edge e, A_e is the event that the two endvertices of e receive the same colour.

Exercise 17. Prove the Independence Principle.

Exercise 18. Let G be a graph and let $(V_1, V_2, \dots, V_k)$ be a partition of $V(G)$ into k sets, each of cardinality at least $2\,\mathrm{e}\,\Delta(G)$. Show that there is a stable set S in G such that $|S \cap V_i| = 1$, $1 \leqslant i \leqslant k$.

Exercise 19. Let D be a digraph with minimum outdegree δ^+ and maximum indegree Δ^- and let k be an integer. Show that if $\mathrm{e}(\Delta^-\delta^+ + 1)(1-\frac{1}{k})^{\delta^+} \leqslant 1$ then D contains a cycle of length divisible by k. (Alon and Linial [4])

Exercise 20. Let H be a k-uniform hypergraph in which every hyperedge intersects at most d others. Show that if $d+1 \leqslant 2\exp(l^2/6k)$, then $disc(H) \leqslant l$.

Exercise 21. Using Chernoff's Bound show that the probability that $G \in \mathcal{G}_{n,\frac{1}{2}}$ has a bipartite subgraph with more than $\frac{1}{8}n^2 + n^{3/2}$ edges is $o(1)$.

Exercise 22. Show that there exists a positive constant c so that for every n there is a graph G on n vertices such that $ch(G) + ch(\overline{G}) \leqslant c\sqrt{n \log n}$. (*Hint:* Use Theorem 24) (Alon [3])

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