Colouring squares

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Let $G = (V, E)$ be a graph. Then $\chi(G^2) \leq \omega(G^2)^2$. Indeed, G^2 has maximum degree $\Delta(G)^2$, hence $\chi(G^2) \leq \Delta(G)^2 + 1$. On the other hand, $\omega(G^2) \geq \Delta(G)+1$, since every closed neighbourhood is a clique. There exist graphs such that $\chi(G^2) \geq \frac{5}{4}\omega(G^2)$, and several people asked for the smallest function f such that for every graph G,

$$\chi(G^2) \leq f(\omega(G^2)) .$$

In particular, it was asked whether f could be a linear function.

The answer to this question is negative. This follows from a result of Alon and Mohar [1]. They proved the existence of an absolute constant c_1 such that, for any integers $\Delta \geq 2$ and $g \geq 7$, there exists a graph G of girth g and maximum degree Δ such that

$$\chi(G^2) \geq c_1 \cdot \frac{\Delta^2}{\log \Delta} .$$

If G is such a graph, then $\omega(G^2) = \Delta + 1$. Indeed, if C is a maximum clique of G^2, then $V(C)$ is the closed neighbourhood of a vertex in G. Otherwise, $V(C)$ would contain three vertices x_1, x_2, x_3 that are pairwise non-adjacent in G (this holds by the girth condition on G). Since those three vertices are adjacent in G^2, there exist three distinct vertices y_1, y_2, y_3 such that y_i is adjacent in G to x_j precisely when $j \neq i$. This holds because C is not a closed neighbourhood, and G has no cycle of length less than 7. Thus, $x_1, y_2, x_3, y_1, x_2, y_3$ is 6-cycle of G, a contradiction.

It is still interesting to know whether any better upper-bound can be found.

References

[1] N. Alon and B. Mohar: The chromatic number of graph powers. *Combinatorics, Probability and Computing*, 11:1–10, 1993.