Trees with three leaves are $(n+1)$-unavoidable.

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Abstract

We prove that every tree of order $n \geq 5$ with three leaves is $(n+1)$-unavoidable. More precisely, we prove that every tree A with three leaves of order n is contained in every tournament T of order $n+1$ except if $(T;A)$ is $(R_5;S_3^+)$ or its dual, where R_5 is the regular tournament on five vertices and S_3^+ is the *outstar* of degree three, that is, the tree consisting of a root dominating three leaves.

Key words: tournament, tree, unavoidable

1 Introduction.

A *tournament* is an orientation of a complete graph. An *oriented tree* is an orientation of a tree; in particular, an *oriented path* is an orientation of a path. Throughout this paper, since we only consider oriented trees and oriented paths, we abbreviate 'oriented tree' to 'tree' and 'oriented path' to 'path'. A digraph is said to be *n-unavoidable* if every tournament of order n contains it as a subgraph. Let $f(n)$ be the smallest integer such that every (oriented) tree of order n is $f(n)$-unavoidable. Sumner (see [6]) noted that $f(n) \geq 2n-2$. Indeed, consider the *outstar* (resp. the *instar*) of degree $n-1$, S_{n-1}^+ (resp. S_{n-1}^-), that is the oriented tree consisting of a root dominating (resp. dominated by) $n-1$ leaves. The regular tournaments of order $2n-3$ have no vertex with outdegree or indegree greater than $n-2$ and thus do not contain S_{n-1}^+ and S_{n-1}^- . Sumner also conjectured that equality holds:

Conjecture 1 (Sumner) Every tree of order n is $(2n-2)$-unavoidable.

Häggkvist and Thomason [1] proved that $f(n) \leq 12n$ and also obtained the asymptotic bound $f(n) \leq (4+o(1))n$. Havet and Thomassé [5] improved both of these bounds by showing that $f(n) \leq 3.5n$. They also proved Sumner's

conjecture for arborescences.
Denote $g(k)$ the smallest integer such that every tree of order n with k leaves is $(n+g(k))$-unavoidable. Häggkvist and Thomason [1] proved that $g(k) \leq 2^{512k^3}$. The outstar S_k^+ (and instar S_k^-) in regular tournaments of order $2k-3$ show that $g(k) \geq k-1$. Havet and Thomassé (see [2]) conjectured that equality holds :

Conjecture 2 (Havet and Thomassé) *Every tree of order n with k leaves is $(n+k-1)$-unavoidable, i.e. $g(k) = k-1$.*

This conjecture implies Sumner's because a tree of order n has at most $n-1$ leaves. Since trees with two leaves are paths, a result of Thomason [7] confirms this conjecture for $k = 2$. Havet [2] proved that $g(3) \leq 5$ and also verified the conjecture [3] for a large class of trees.

If true, the bound $k-1$ of Conjecture 2 is best possible because of the outstar S_k^+ (and instar S_k^-) in regular tournaments of order $2k-1$. However, we conjecture that for n sufficiently large compared to k, every tree of order n with k leaves is $(n+k-2)$-unavoidable. It has been established for paths (trees with two leaves) by Havet and Thomasse [4], who proved that every path of order $n \geq 8$ is n-unavoidable. For $k \geq 3$, we have the following conjecture:

Conjecture 3 *Let T be a tournament of order $n+k-2$ and A a tree with $k \geq 3$ leaves of order n. If T is not a regular tournament or if A is neither an outstar nor an instar then T contains C.*

The bound $k-2$ of Conjecture 3 is tight even when n goes to infinity. Indeed, consider a tournament T consisting of $n-2k+4$ vertices dominating a regular tournament on $2k-3$ vertices, and a tree A of order n that is the merging of one directed inpath of length $n-k$ and $k-1$ outpaths of length one. Clearly, T does not contain A.

In this paper, we prove that Conjecture 3 holds for $k = 3$.

Theorem 4 *Let T be a tournament of order $n+1$ and A a tree of order n with three leaves. If $(T;A) \notin \{(R_5;S_3^+),(R_5;S_3^-)\}$ then T contains C.*

Therefore, Conjecture 2 is also true for $k = 3$, that is $g(3) = 2$.

2 Reduction to a merging of outpaths

Let $P = (x_1, \cdots, x_n)$ be a path. We say that x_1 is the *origin* of P. If $x_1 \rightarrow x_2$, P is an *outpath*, otherwise P is an *inpath*. The *directed outpath* of order n is the path $P = (x_1, \cdots, x_n)$ in which $x_i \rightarrow x_{i+1}$ for all i, $1 \leq i < n$.

Let $P_1, P_2, \dots P_k$ be paths. The *merging* of $P_1, P_2, \dots, P_k$, denoted $\bigvee_{i=1}^{k} P_i$, is the digraph $\bigcup_{i=1}^{k} P'_i$ where P'_i is isomorphic to P_i with origin x and $V(P'_i) \cap V(P'_j) = \{x\}$, $1 \leq i < j \leq k$. A tree with three leaves is thus the merging of three paths.

In [2], Havet proved the following :

Theorem 5 (Havet [2]) *The merging of order n of two outpaths and one inpath is $(n+1)$-unavoidable.*

Thus, by duality, in order to prove Theorem 4, it suffices to prove that a merging of order n of three outpaths is $(n+1)$-unavoidable unless it is S_3^+. Henceforth, we study mergings of three outpaths.

3 Mergings of outpaths

A tournament is *strong* if, for any two vertices x and y, there exists a directed outpath with origin x and terminus y. A tournament T is *k-strong*, if $T - Y$ is strong for any set Y of $k-1$ vertices.

3.1 Mergings of outpaths in 2-strong tournaments

The minimum indegree of the vertices of a tournament T is denoted by $\delta^-(T)$. We first show that a tournament T of order $n+1$ contains a merging of order n of three outpaths, provided that two of the paths are long compared to $\delta^-(T)$.

Lemma 6 *Let T be a 2-strong tournament of order $n+1$ and $C = P_1 \vee P_2 \vee P_3$ a merging of order n of three outpaths. If $|P_1| + |P_2| - 5 \geq \delta^-(T)$, then T contains C.*

We may assume that P_1 and P_2 are at least as long as P_3, so that $|P_1| + |P_2| \geq \left\lceil \frac{2n-2}{3} \right\rceil + 2$. On the other hand, $\delta^-(T) \leq \left\lfloor \frac{n}{2} \right\rfloor$. Thus every tree of order $n \geq 17$ or $n = 15$ with three leaves is contained in every 2-strong tournament of order $n+1$. By studying small tournaments, we obtain the following result :

Theorem 7 *Let T be a 2-strong tournament of order $n+1$ and C the merging of three paths. If $(T; C) \notin \{(5A; S_3^+), (5A; S_3^-)\}$ then T contains C.*

To complete the proof of Theorem 4, we now study the mergings of outpaths in non 2-strong tournaments.

The *blocks* of a path P are the maximal subdipaths of P. We enumerate the blocks of P from its origin to its terminus. The first block of P is denoted $B_1(P)$ and its length $b_1(P)$.

Theorem 8 *Let T be a non 2-strong tournament of order $n+1$ and $C = P_1 \vee P_2 \vee P_3$ a merging of order n of three outpaths such that $b_1(P_3) = 1$. Then P contains C.*

In Remark 3.1 of [2], Havet proved the following:

Lemma 9 *Let $C = P_1 \vee P_2 \vee P_3$ be a merging of order n of three outpaths with first block of length at least two. If one of the P_i is not a directed path then C is $(n+1)$-unavoidable.*

Lemma 10 *A merging of order n of three directed outpaths is contained in every non 2-strong tournament of order $n+1$.*

Theorem 8 and Lemmas 9 and 10 yield that every merging of order n of three outpaths is $(n+1)$-unavoidable. By Theorem 7 and 5, we now obtain Theorem 4.

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