Traffic Grooming in Unidirectional WDM Ring Networks using Design Theory

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***Abstract*—We address the problem of traffic grooming in WDM rings with all-to-all uniform unitary traffic. We want to minimize the total number of SONET add-drop multiplexers (ADMs) required. We show that this problem corresponds to a partition of the edges of the complete graph into subgraphs, where each subgraph has at most C edges (where C is the grooming ratio) and where the total number of vertices has to be minimized. Using tools of graph and design theory, we optimally solve the problem for practical values and infinite congruence classes of values for a given C, and thus improve and unify all the preceding results. We disprove a conjecture of [6] saying that the minimum number of ADMs cannot be achieved with the minimum number of wavelengths, and also another conjecture of [16].**
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I. INTRODUCTION

The WDM (Wavelength Division Multiplexing) has significantly increased the available capacity transmission of networks, therefore the bottleneck is now in the nodes (routers) where all the information has to be processed. In order to reduce the cost of the network, it becomes important to reduce the traffic processed at the node. Traffic grooming is the generic term for packing low rate signals into higher speed streams (see the surveys [13, 18, 19]). By using traffic grooming, one can bypass the electronics in the nodes for which there is no traffic sourced or destinated to it. Typically, in a WDM network, instead of having one SONET Add Drop Multiplexer (shortly ADM) on every wavelength at every node, it may be possible to have ADMs only for the wavelength used at that node (the other wavelengths being optically routed without electronic switching).

In the past many papers on WDM networks had for objective to minimize the transmission cost and in particular the number of wavelengths to be used [1, 10, 11], recent research has focused on reducing the total number of ADMs used in the network, trying to minimize it.

It is known that even for the simpler network which is the unidirectional ring, the number of wavelengths and the number of ADMs cannot be simultaneously minimized (see [14], or [6] for uniform traffic). Furthermore, given a traffic matrix expressed in some units of a bandwidth (for example OC-3) where $r_{i,j}$ units have to be transmitted from i to j, the solution will depend on the routing used and how connections are assigned to wavelengths. Hence, the general problem is very difficult.

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Here, we consider the particular case of unidirectional rings (the routing is unique) with static uniform symmetric all-to-all traffic (that is $r_{i,j} = 1$ for all couples (i, j)) and with no possible wavelength conversion.

In that case, for each pair $\{i, j\}$, we associate a circle (or circuit) which contains both the request from i to j and from j to i. If each circle requires only $\frac{1}{C}$ of the bandwidth of a wavelength, we can "groom" C circles on the same wavelength. C is called the *grooming ratio* (or grooming factor). For example, if the request from i to j (and from j to i) is one OC-12 and a wavelength can carry an OC-48, the grooming factor is 4. Given the grooming ratio C and the size N of the ring, the objective is to minimize the total number of (SONET) ADMs used, denoted $A(C, N)$, and so reducing the network cost by eliminating as many ADMs as possible from the "no grooming case". For example, let $N = 4$; we have 6 circles corresponding to the 6 pairs $\{0,1\},\{0,2\},\{0,3\},\{1,2\},\{1,3\},\{2,3\}$. If we don't use grooming, that is if we assign one wavelength per circle, we need 2 ADMs per circle, and thus a total of 12. Suppose now that $C = 4$, that is we can groom 4 circles on one wavelength. One can groom on wavelength 1 the circles associated to $\{0,1\},\{1,2\},\{2,3\},\{3,0\}$ requiring 4 ADMs and on wavelength 2 the circles associated to $\{0,2\}$ and $\{1,3\}$ requiring 4 ADMs and so a total of 8. A better way is to groom the circles associated to $\{0,1\},\{0,2\},\{0,3\}$ using 4 ADMs and those associated with $\{1,2\},\{1,3\},\{2,3\}$ using 3 ADMs and a total of 7 ADMs.

The case we consider has been considered by many authors [6, 12, 14–16, 20, 21, 23–25] and numerical results, heuristics and tables have been given (see for example that in [21]). It presents the advantage of concentrating on the grooming phase (excluding the routing). It can also be applied to groom components of more general connections than two opposite pairs into wavelengths or more general classes. These components are called circles [6, 25] or circuits [21] or primitive rings [8, 9].

Here we show that the problem of minimizing the number of ADMs for the unidirectional ring C_N with a grooming factor C can be expressed as follows: partition the edges of the complete graph on N vertices (K_N) into W subgraphs B_λ, $\lambda = 1, 2, \ldots, W$, having $|E(B_\lambda)|$ edges and $|V(B_\lambda)|$ vertices with $|E(B_\lambda)| \leq C$ and where $\sum_{\lambda=1}^{W} |V(B_\lambda)|$ has to be minimized (the edges of K_N correspond to the circles, the subgraphs B_λ correspond to the wavelengths and a node of B_λ corresponds to an ADM).

We show the importance of choosing graphs B_λ in the par-

TABLE I
$A(C, N)$ FOR $N \leq 16$ AND $C = 3, 4, 12, 16, 48, 64$

	N=3	N=4	N=5	N=6	N=7	N=8	N=9	N=10	N=11	N=12	N=13	N=14	N=15	N=16
C=3	3	7	12	17	21	31	36	48	57	69	78	95	105	124
		6-7	11-12	15-17	21-21	29-31	36-36	45-48	56-57	66-69	78-78	92-95	105-105	120-124
C=4	3	7	10	15	21	28	36	45	55	66	78	91	105	120
C=12	3	4	5	9	12	16	18	24	30	35	39	47	54	60
				9-9	11-12	15-16	18-18	23-24	28-30	33-36	39-39	46-49	53-57	60-64
C=16	3	4	5	6	11	14	18	20	26	32	36	41	45	53-54
					10-11	12-14	16-18	18-20	23-26	28-33	33-37	37-42	42-46	48-57
C=48	3	4	5	6	7	8	9	10	16	19	22	24	30	32
									15-16	17-19	19-22	21-24	26-31	29-34
C=64	3	4	5	6	7	8	9	10	11	15	19	22	25	28
										15-15	18-19	20-22	22-25	24-28

tition with the best ratio $\frac{|E(B_\lambda)|}{|V(B_\lambda)|}$. Indeed, if we denote by $\rho_{\max}(C)$ the maximum ratio among all graphs with at most C edges, we have the following lower bound on the minimum number $A(C, N)$ of ADMs: $A(C, N) \geq \frac{N(N-1)}{2\rho_{\max}(C)}$.

For some values of C the best ratio is obtained for subgraphs with C edges, but it is not the case for other values of C, namely if $\frac{k(k-1)}{2} < C < \frac{(k+1)(k-1)}{2}$, where the best ratio is attained for the complete graph K_k. For these values of C, design theory enables us to give counterexamples to the conjecture of [6] saying that "the minimum number of ADMs, $A(C, N)$, for unidirectional rings with uniform unitary traffic is obtained using the minimum number of wavelengths" is false, the smallest case being $C = 7$. That also give counterexamples for $C = 16$ and to the conjecture of [16].

We show furthermore that the techniques of *design theory* can be used to obtain optimal or quasi-optimal results that improve all the preceding results of the literature and unify them. Thus, it is possible to use the vast effort and the numerical results obtained in the last century in design theory [7], without reinventing them. Note that design theory was also used in [8, 9] for $C = 8$.

Among the results we have obtained on $A(C, N)$ are the following.

- Given C, when N is large enough, $A(C, N) = \frac{N(N-1)}{2\rho_{\max}(C)}$.
- Exact results for $C = 3$, $C = 4$ (already obtained in [16]) and for various congruence classes, for other values of C.
- Table I with the same entries as in [21] but with optimal values. Table I show the values of $A(C, N)$ for $N \leq 16$ and $C = 3, 4, 12, 16, 48, 64$. We have indicated in the line below the best known values from Table III of [21] in the form $l(C, N) - u(C, N)$ where $l(C, N)$ is the lower bound of [21] and $u(C, N)$ the best result obtained by heuristics.

II. NOTATION AND REFORMULATION OF THE PROBLEM

We precise here our notations and show how the problem can be formulated in terms of graph partitioning. Although we restrict ourselves to the case of unidirectional rings with uniform static unit traffic, the ideas can be applied to other situations.

- N will denote the number of nodes of the unidirectional ring $\overrightarrow{C}_N$
- For the unidirectional ring with symmetric traffic, $C_{\{i,j\}}$ will denote a *circle* associated to the pair $\{i, j\}$, that is containing both an unitary request from i to j and from j to i. So $C_{\{i,j\}}$ uses all the arcs of $\overrightarrow{C}_N$.
- R the total number of circles. In the case of unidirectional rings, with uniform unitary traffic, each pair $\{i, j\}$ is associated to a unique circle $C_{\{i,j\}}$ and thus $R = \frac{N(N-1)}{2}$.
- C the grooming ratio (or grooming factor). In the example of [6], C indicates the number of circles a wavelength can contain. Similarly, $\frac{1}{C}$ indicates the part of the bandwidth of a wavelength that can be used by a circle. For example, if a wavelength is running at the line rate of OC-N, it can carry $C = \frac{N}{M}$ low speed OC-M. Typical values of C are $C = 3, 4, 8, 12, 16, 48, 64$.
- Let K_N be the complete graph on N vertices where there is an edge $\{i, j\}$ for each pair of vertices $\{i, j\}$; let C_N be the undirected cycle with N nodes.
- B_λ will denote a subgraph of K_N. $V(B_\lambda)$ (resp $E(B_\lambda)$) denote its vertex (resp edge) set. In the example of the introduction, B_λ corresponds to a wavelength ; an edge $\{i, j\}$ of B_λ correspond to a circle $C_{\{i,j\}}$. So a subgraph can be viewed as the set of circles packed in the wavelength. The grooming factor implies that $|E(B_\lambda)| \leq C$. $V(B_\lambda)$ corresponds to the number of (SONET) ADMs used in the wavelength λ ; indeed we have to use an ADM in all the vertices appearing in a circle $C_{\{i,j\}}$ packed in the wavelength λ.

So, the original problem of minimizing the total number $A(C, N)$ of ADMs in a grooming with grooming ratio C, in the unidirectional ring $\overrightarrow{C}_N$ with unitary static uniform traffic, can be stated as follows.

Problem II.1—ADM:

Inputs :	a number of nodes N and a grooming ratio C
Output :	a partition of the edges of K_N into subgraphs B_λ, $\lambda = 1, \ldots, W$, such that $\lvert E_\lambda \rvert \leq C$
Objective :	minimize $\sum_{1 \leq \lambda \leq W} \lvert V_\lambda \rvert$

Remark: As we said in the introduction, most interest has focused on a different objective function which was to minimize the number W of subgraphs (wavelengths) of the partition. This is in this context an easy problem as $W_{\min} = \lceil \frac{R}{C} \rceil = \left\lceil \frac{N(N-1)}{2C} \right\rceil$.

III. LOWER BOUND

Let $\rho(B_\lambda)$ denote the ratio of a subgraph B_λ, $\rho(B_\lambda) = \frac{|E(B_\lambda)|}{|V(B_\lambda)|}$, and $\rho(m)$ the maximum ratio of a subgraph with m edges. Let $\rho_{\max}(C)$ denote the maximum ratio of subgraphs with $m \leq C$ edges. We have $\rho_{\max}(C) = \max\{\rho(B_\lambda) \mid |E(B_\lambda)| \leq C\} = \max_{m \leq C} \rho(m)$.

Theorem III.1: Any grooming of R circles with a grooming factor C needs at least $\frac{R}{\rho_{\max}(C)}$ ADMs.

Proof: We have $R = \sum_{\lambda=1}^{W} |E(B_\lambda)| \leq \rho_{\max}(C) \sum_{\lambda=1}^{W} |V(B_\lambda)|$. ■

In particular, we get the following lower bound

Theorem III.2—Lower Bound: $A(C,N) \geq \frac{N(N-1)}{2\rho_{\max}(C)}$.

We will now compute $\rho_{\max}(C)$. Let $\varphi(m) = \min\left\{k \mid \frac{k(k-1)}{2} \geq m\right\}$, that is $\varphi(m) = \left\lceil \frac{1+\sqrt{1+8m}}{2} \right\rceil$ and note that any subgraph with m edges has at least $\varphi(m)$ vertices.

Proposition III.3: If $\frac{k(k-1)}{2} \leq C \leq \frac{(k+1)(k-1)}{2}$, then $\rho_{\max}(C) = \frac{k-1}{2}$ and the value is attained for K_k.

If $\frac{(k+1)(k-1)}{2} \leq C \leq \frac{(k+1)k}{2}$, then $\rho_{\max}(C) = \frac{C}{k+1}$ and the value is attained for any graph with C edges and $k+1$ vertices.

Proof:

Case 1: Let $\frac{k(k-1)}{2} \leq C \leq \frac{(k+1)(k-1)}{2}$. If $m \leq \frac{k(k-1)}{2}$, then $\varphi(m) \leq k$ and $\rho(m) \leq \frac{k-1}{2}$. If $\frac{k(k-1)}{2} < m \leq C$, $\varphi(m) = k+1$ and $\rho(m) = \frac{m}{k+1} \leq \frac{C}{k+1}$. As $C \leq \frac{(k+1)(k-1)}{2}$, $\rho(m) \leq \frac{k-1}{2}$. So, we always have $\rho(m) \leq \frac{k-1}{2}$, the equality being attained for K_k (where $m = \frac{k(k-1)}{2}$ and $\rho(m) = \frac{k-1}{2}$).

Case 2: Let $\frac{(k+1)(k-1)}{2} \leq C \leq \frac{(k+1)k}{2}$. If $m \leq \frac{(k+1)(k-1)}{2}$, we have seen that $\rho(m) \leq \frac{k-1}{2} \leq \frac{C}{k+1}$. If $\frac{(k+1)(k-1)}{2} \leq m \leq C \leq \frac{(k+1)k}{2}$, $\varphi(m) = k+1$ and $\rho(m) = \frac{m}{k+1} \leq \frac{C}{k+1}$. The value $\frac{C}{k+1}$ is attained for any graph with C edges and $k+1$ vertices ; such a graph can be obtained by deleting $\frac{(k+1)k}{2} - C$ edges from K_{k+1}. ■

For the sake of illustration, Table II give the values of $\rho_{\max}(C)$ for small values of C.

In view of Theorem III.2, we have interest to chose if possible subgraphs with a ratio equal to $\rho_{\max}(C)$. Note that according to Proposition III.3, these subgraphs do not have necessarily exactly C edges and so the minimum is not necessarily attained for $W = W_{\min}$.

For example, let $C = 7$. If a subgraph has 7 edges, its ratio is at most $\frac{7}{5} = 1.4$. But a subgraph with 6 edges can have a ratio $\frac{6}{4} = 1.5$ (and this is attained for K_4). Any other subgraph has a ratio at most $\frac{5}{4}$. So, in an optimal solution for the number of ADMs, we have to use as subgraphs K_4 and not subgraphs with 7 edges and 5 vertices. But in a solution minimizing the number of wavelengths, we have in contrary to use these last ones. We are now able to give counterexamples to a conjecture of [6].

TABLE II
VALUES OF $\rho_{\max}(C)$ FOR SMALL C

C	1	2	3	4	5	6	7	8	9	10
$\rho_{\max}(C)$	$\frac{1}{2}$	$\frac{2}{3}$	1	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{9}{5}$	2
C	11	12	13	14	15	16	24	32	48	64
$\rho_{\max}(C)$	2	2	$\frac{13}{6}$	$\frac{14}{6}$	$\frac{5}{2}$	$\frac{5}{2}$	3	$\frac{32}{9}$	$\frac{9}{2}$	$\frac{64}{11}$

Proposition III.4: The conjecture of [6] that the minimum number of ADMs, $A(C,N)$, for unidirectional rings $\overrightarrow{C}_N$ with uniform unitary traffic is obtained for $W = W_{\min} = \left\lceil \frac{N(N-1)}{2C} \right\rceil$, is false.

Proof: Let $C = 7$ and $N = 13$. There is a decomposition of K_{13} into 13 subgraphs K_4 (namely the subgraphs $B_i = \{i, i+1, i+4, i+6\}$ for $i = 0, 1, \ldots, 12$, the numbers being taken modulo 13). For this solution, $A = 52$ and it is an optimal one. However, we have $W = 13$ and $W_{\min} = \lceil \frac{78}{7} \rceil = 12$.

Consider now a grooming with 12 subgraphs and let W_7 be the number of subgraphs with 7 edges. The $12 - W_7$ other subgraphs have at most 6 edges each. To groom the 78 circles, we should have $7W_7 + 6(12 - W_7) \geq 78$, which implies $W_7 \geq 6$. But each subgraph with 7 edges needs 5 ADMs. So, there are $5W_7$ vertices to cover $7W_7$ circles. To cover the remaining $78 - 7W_7$ circles, we need by Theorem III.1 at least $\frac{4}{6}(78 - 7W_7)$ vertices (as the ratio of any subgraph is at most $\frac{6}{4}$). So altogether we have a total number of vertices $A \geq 5W_7 + \frac{4}{6}(78 - 7W_7) \geq 52 + \left(5 - \frac{28}{6}\right) W_7$. As $W_7 \geq 6$, we obtain $A \geq 54$. So, conjecture of [6] is false for $C = 7$ and $N = 13$.

For $C = 7$ we can easily build infinite families of counterexamples using known decompositions of K_N into K_4 if $N = 12t + 1$ or $12t + 4$ (see Theorem IV.4). Such a decomposition will use $W = \frac{R}{6}$ subgraphs and $A = \frac{2R}{3}$ ADMs. On the other hand, a solution with $W_{\min} = \lceil \frac{R}{7} \rceil$ subgraphs will need around $\frac{7R}{5}$ ADMs as all the subgraphs have exactly 7 edges and a ratio of $\frac{7}{5}$. So, using $\frac{1}{6}$ more wavelength allows a saving of $\frac{1}{15}$ ADMs (in fact we can use any number of subgraphs between $\frac{R}{7}$ and $\frac{R}{6}$, increasing the number of subgraphs by 1 decreases the number of ADMs of 2). ■

Other counterexamples to the conjecture of [6] are obtained for all values of C such that $\frac{k(k-1)}{2} < C < \frac{(k+1)(k-1)}{2}$, for example $C = 11, 16, 17$. When $C = 16$, $\rho_{\max}(16) = \frac{5}{2}$ and is attained for subgraphs K_6 which have only 15 edges.

That confirms the intuition of [16]. However, it was also conjectured in [16] that $A(16,N) \geq \left\lceil \frac{7N(N-1)}{32} \right\rceil$. But, as we will see in Theorem IV.4, this conjecture is false, since there exists an infinite number of values of N for which $A(16,N) = \frac{N(N-1)}{5}$.

IV. UPPER BOUND AND OPTIMAL RESULTS

Our problem looks similar to design theory. Indeed an $(N, k, 1)$-design is nothing else than a partition of the edges of K_N into subgraphs isomorphic to K_k called blocks in this theory. That corresponds to impose in our partitioning problem that all the subgraphs B_λ are isomorphic to K_k. Note that the classical equivalent definition is : given a set of N elements, find a set of blocks such that each block contains k elements and each pair of elements appears in exactly one block (see the handbook [7]).

More generally, a G-design of order N (see [7] chap. 22 or [4] or [5]) consists on partition of the edges of K_N into subgraphs isomorphic to a given graph G. The interest of the existence of a G-design is shown by the following immediate proposition.

Proposition IV.1: If there exists a G-design of order N, where G is a graph with at most C edges and ratio $\rho_{\max}(C)$, then $A(C, N) = \frac{N(N-1)}{2\rho_{\max}(C)}$.

Necessary conditions IV.2—Existence of a G-design: If there exists a G-design, then

(i) $\frac{N(N-1)}{2}$ should be a multiple of $E(G)$

(ii) $N-1$ should be a multiple of the greatest common divisor of the degrees of the vertices of G.

Wilson [22] has shown that these necessary conditions are also sufficient for large N. From that, we obtain

Theorem IV.3: Given C, for an infinite number of values of N, $A(C, N) = \frac{N(N-1)}{2\rho_{\max}(C)}$.

Unfortunately, the values of N for which Wilson's Theorem applies are very large. However, for small values of C, we can use exact results of design theory. For example, from the existence of a G-design for $G = K_3, K_3 + e, K_4 - e, K_4, K_5 - 3e, K_5 - 2e, K_5 - e, K_5$ and K_6, where $K_p - \alpha e$ (resp. $K_p + \alpha e$) denotes the graph obtained from K_p by deleting (resp. adding) α edges, we obtain

Theorem IV.4:

- $A(3, N) = \frac{N(N-1)}{2}$ when $N \equiv 0$ or $1 \pmod 6$
- $A(4, N) = \frac{N(N-1)}{2}$ when $N \equiv 0$ or $1 \pmod 8$
- $A(5, N) = \frac{2N(N-1)}{5}$ when $N \equiv 0$ or $1 \pmod{10}$
- $A(6, N) = A(7, N) = \frac{N(N-1)}{3}$ when $N \equiv 1$ or $4 \pmod{12}$
- $A(8, N) = \frac{5N(N-1)}{16}$ when $N \equiv 0$ or $1 \pmod{16}$
- $A(9, N) = \frac{5N(N-1)}{18}$ when $N \equiv 0$ or $1 \pmod{18}$
- $A(10, N) = \frac{N(N-1)}{4}$ when $N \equiv 1$ or $5 \pmod{20}$
- $A(16, N) = \frac{N(N-1)}{5}$ when $N \equiv 1 \pmod{30}$

Therefore, as we said before, it disproves the conjecture of [16] for $A(16, N)$.

Note that, if design theory can be used, the problem is slightly different. Indeed, in design theory, one looks for a partition of the edges into isomorphic subgraphs. Such partition exist only for some values of N. For example, for $C = 3$ and $N \not\equiv 1$ or $3 \pmod 6$, one can think that the best solution is obtained by taking as many K_3's as possible, but it does not necessarily lead to an optimal solution.

Consider for example K_6. It can be partitioned into the 4 triangles $\{0, 1, 2\}, \{0, 3, 4\}, \{1, 3, 5\}, \{2, 4, 5\}$ plus the 3 edges $\{0, 5\}, \{1, 4\}, \{2, 3\}$. So, all together we have 5 subgraphs and 18 ADMs. However, we can also partition K_6 into the 3 K_3's $\{0, 1, 2\}, \{1, 3, 4\}, \{2, 4, 5\}$, the star $K_{1,3}$ $\{0|3, 4, 5\}$, and the path P_4 $(1, 5, 3, 2)$. This solution use 5 subgraphs and 17 ADMs.

In [2], we determined exactly the value of $A(3, N)$, proving the following theorem.

Theorem IV.5—[2]:

(i) When N is odd, $A(3, N) = \frac{N(N-1)}{2} + \epsilon$, where $\epsilon = 0$ if $N \equiv 1$ or $3 \pmod 6$, and $\epsilon = 2$ if $N \equiv 5 \pmod 6$;

(ii) When N is even, $A(3, N) = \frac{N(N-1)}{2} + \left\lceil \frac{N}{4} \right\rceil + \epsilon$, where $\epsilon = 1$ if $N \equiv 8 \pmod{12}$, and $\epsilon = 0$ elsewhere.

The proof uses technics inspired of design theory. In the even case, the optimal solutions use a lot of K_3's and some $K_{1,3}$ or P_4. For example, if $n \equiv 0$ or $4 \pmod{12}$, the optimal solution consists of $\frac{N(N-1)}{6} - \frac{N}{4}$ K_3's and $\frac{N}{4}$ $K_{1,3}$. Note that there always exist solutions minimizing both the number of ADMs and the number of subgraphs (wavelengths) so conjecture of [6] is true for $C = 3$.

An other difference with design theory is that we can use in our decomposition a mixture of graphs with ratio $\rho_{\max}(C)$.

For example, if $C = 4$, we can use either a C_4 or $K_3 + e$ (or also a K_3), all these graphs having ratio $\rho_{\max}(4) = 1$. It becomes easy (see [3]) to show that for $N > 4$, K_N can be partitioned into $C_4's$ and $K_3 + e$, giving the result of [16] that $A(4, N) = \frac{N(N-1)}{2}$ with the minimum number of wavelengths.

An other example is given with $C = 12$, where we can use as subgraphs of the partition of K_N either K_5 or any graph with 6 vertices and 12 edges (obtained by deleting 3 edges from K_6). Such a decomposition exists for example if $n \equiv 1 \pmod 4$ in which case $A(12, N) = \frac{N(N-1)}{4}$.

Although the problem is asymptotically solved, for small values of N (corresponding to the practical ones) we have to refine both lower and upper bounds to obtain the exact value of $A(C, N)$.

For upper bounds, we have to construct decompositions. If the graph with $\rho_{\max}(C)$ is K_k, we can use the known results of design theory, and in any case we can use the ideas of this theory in particular all the recursive constructions. This methodology is that used in [8] for $C = 8$. The ideas of partitioning into classes of [6] or others appears as a particular case of this design techniques. For example, in [6] they use mainly subgraphs of the form $K_{\sqrt{C},\sqrt{C}}$. If for $C = 4$, their use is good because $K_{2,2} = C_4$ and $\rho_{\max}(4) = 1$, in all other cases it gives result far from the optimal. For $C = 16$, $K_{4,4}$ has a ratio 2 to be compared with K_6 which has ratio $\frac{5}{2}$. So their solution can have only at most 80% of ADMs compared to the optimal solution. Using a partition into 3 classes, we have shown

Theorem IV.6—[3]: When $C \geq R/3$, $A(C, N) \leq 2N$, except when $N = 4$ and $C = 2$, and when $N = 7$ and $C = 7$.

If exact optimal solutions might be difficult to obtain (in particular to prove the non existence of a solution), solution differing by a small constant can be easily found as there exist a lot of them. That explains why the results obtained by *simulated* *annealing* [21] are relatively good. Furthermore, as we know by the theory the structure of the optimal or quasi-optimal solutions, we can use programming by imposing constraints and looking only to particular solutions. More details will be found in forthcoming papers and the values reported in Table I where obtained in that way.

For lower bounds, if N is small, we have to take into account the fact that subgraphs should have large intersections and so edges are covered many times.

Proposition IV.7: When $C \geq \frac{N(N-1)}{2}$, $A(C,N) = N$.

Proof: There should be at least one ADM in each vertex, so $A(C,N) \geq N$. A solution is obtained by taking a unique subgraph. In that case the traffic is carried on one wavelength (it correspond to the shaded data in [21]). ■

Proposition IV.8: When $\frac{N(N-1)}{4} \leq C \leq \frac{N(N-1)}{2}$, $A(C,N) = N + \varphi\left(\frac{N(N-1)}{2} - C\right)$.

Proof: Recall that $\varphi(m)$ is the smallest integer k such that $\frac{k(k-1)}{2} \geq m$, and let $k_0 = \varphi\left(\frac{N(N-1)}{2} - C\right)$. If each vertices belongs to at least 2 subgraphs then $A(C,N) \geq 2N \geq N + k_0$. So one vertex belongs to exactly one subgraph which should contains the $N-1$ other vertices and at most C edges. To cover the $\frac{N(N-1)}{2} - C$ remaining edges, we need a subgraph with at least k_0 vertices. Therefore, $A(C,N) \geq N + k_0$.

A solution with $N + k_0$ ADMs is obtained by taking two subgraphs, one with k_0 vertices covering $\frac{k_0(k_0-1)}{2}$ edges, where $\frac{k_0(k_0-1)}{2} \geq \frac{N(N-1)}{2} - C$ by definition of k_0. The second subgraphs contains all the vertices and cover the remaining edges in number less than or equal to C. ■

Applying Proposition IV.8, we obtained the results of Table I for $C = 64$ and $12 \leq N \leq 16$, and for $C = 48$ and $11 \leq N \leq 14$. For other values of C and N, we have to use more sophisticated arguments (see [3]).

V. Conclusion

In this article, we have shown how the problem of traffic grooming in unidirectional WDM ring with uniform unitary traffic can be modelized as a problem of partition of the edges of a complete graph. The use of graph theory and design tools has enabled us to solve optimally the problem for practical values and infinite congruence classes of values for a given C. This modelization and the tools can easily extended to uniform but non unitary traffic. Indeed, if the requests satisfy $r_{i,j} = r$, it suffices to consider decomposition of the edges of the complete multipartite graph rK_N (here again, this has been done in design theory for partition into K_k under the name of (N,k,r)-design). We can also extend the ideas to the case of an arbitrary traffic, but it requires to partition general graphs and this is known to be a difficult problem in graph theory. However, our tools provide insight for finding approximate and heuristic solutions for arbitrary traffic. We can also consider networks different from the unidirectional ring, if we are first able to groups the requests into circles (that is the way used in [8,9] for bidirectional rings). Finally, the tools can also be used to groom traffic in a slightly different context, for example , in the RNRT project PORTO our team developed with France Telecom and Alcatel, the traffic was expressed in terms of STM-1 (each one needed one wavelength) and we group them into bands or fibers, typically a fiber containing 8 bands of 4 wavelengths [17].

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