Packet Routing Problems on Plane Grids

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Joint work with Omid Amini, Florian Huc and Janez Žerovnik

AEOLUS Workshop on Scheduling

Outline

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  - Statement of the problem
  - Preliminaries
  - Example
- Permutation routing algorithm for triangular grids
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  - Correctness
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- Permutation routing algorithm for hexagonal grids
- (ℓ, k)-routing algorithms
- Conclusions

(ℓ, k)-routing

- The (ℓ, k)**-routing** problem is a **packet routing** problem.
- Each processor is the **origin of at most** ℓ **packets** and the **destination of no more than** k **packets**.
- The goal is to **minimize the number of time steps** required to route all packets to their respective destinations.
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Permutation Routing

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- **Input:**
  - a directed graph $G = (V, E)$ (the *host* graph),
  - a subset $S \subseteq V$ of nodes,
  - and a permutation $\pi : S \rightarrow S$.
    Each node $u \in S$ wants to send a packet to $\pi(u)$.
- **Output:** Find for each pair $(u, \pi(u))$, a path form u to $\pi(u)$ in G.
- **Constraints:**
  - At each step, a packet can either move or stay at a node.
  - No arc can be crossed by two packets at the same step.
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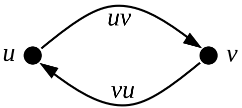
Assumptions

- We consider the **store-and-forward** and **Δ-port** model.
- **Full duplex link**: packets can be sent in the two directions of the link **simultaneously**.



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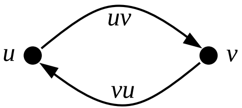

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Previous work

-The permutation routing problem has been studied in:

- Mobile Ad Hoc Networks
- Cube-Connected Cycle Networks
- Wireless and Radio Networks
- All-Optical Networks
- Reconfigurable Meshes...

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- 2-circulant graphs, square grids.
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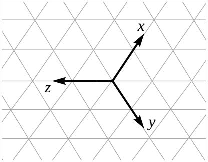
Permutation Routing on Triangular Grids

Notation and preliminary results

Nocetti, Stojmenović and Zhang [IEEE TPDS'02]:

Representation of the relative address of the nodes on a generating system ***i***, ***j***, ***k*** on the directions of the three axis x, y, z.



- This address **is not unique**, but we have that, being (a, b, c) and (a', b', c') the addresses of two $D - S$ pairs,

$$(a, b, c) = (a', b', c') \quad \Leftrightarrow \quad \exists \text{ an integer } d \text{ such that}$$

$$a' = a + d,$$

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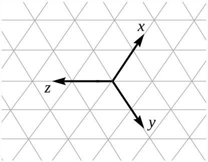

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Notation and preliminary results (2)

- A relative address $D - S = (a, b, c)$ is of the ***shortest path form*** if
  - there is a path C from S to D, $C = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$,
  - and C has the shortest length over all paths going from S to D.

Theorem (*NSZ'02*)

*An address (a, b, c) is of the **shortest path form** if and only if*

i) ***at least one component is zero*** *(that is, $abc = 0$),*

ii) *and **any two components do not have the same sign** (that is, $ab \leq 0,\ ac \leq 0,$ and $bc \leq 0$).*

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Notation and preliminary results (3)

Corollary (*NSZ'02*)

*Any address has a **unique** shortest path form.*

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If $D - S = (a, b, c)$, then the shortest path form is one of those:

$$(0, b-a, c-a),$$
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and thus:

$$|D-S| = \min(|b-a| + |c-a|, |a-b| + |c-b|, |a-c| + |b-c|).$$

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  Any permutation routing algorithm needs at least ℓ_{max} routing steps.

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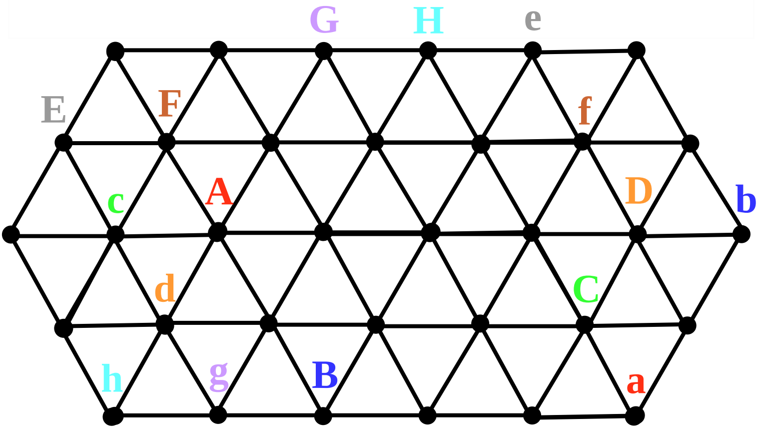
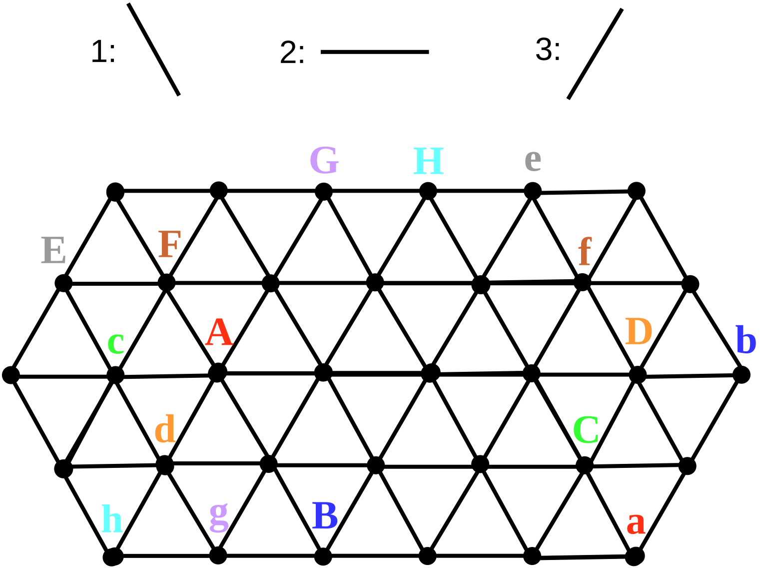
Example of an instance



A non-optimal intuitive algorithm



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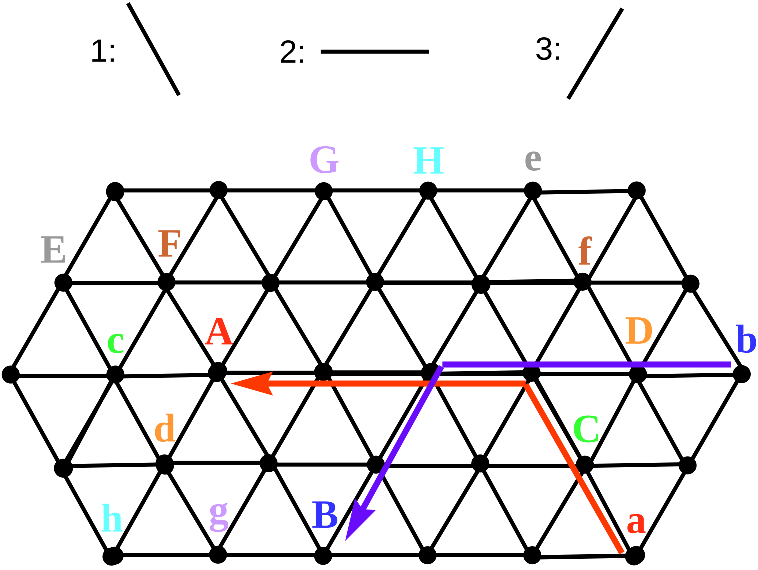

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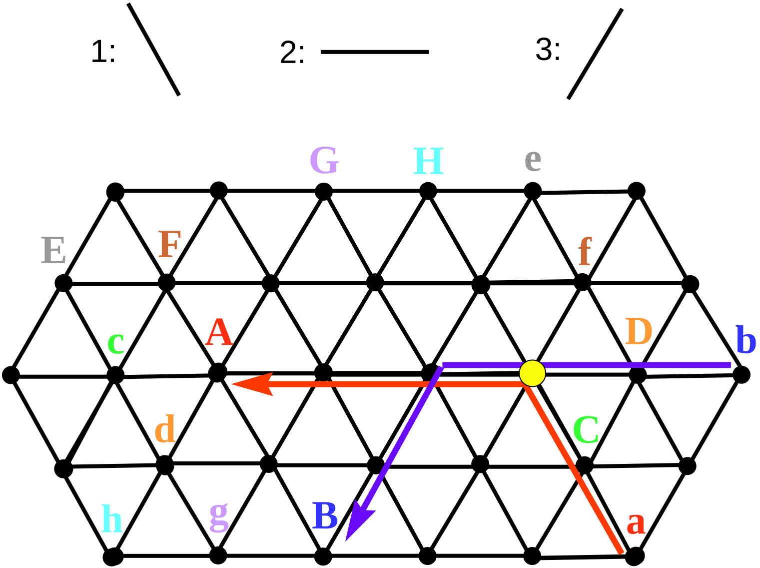

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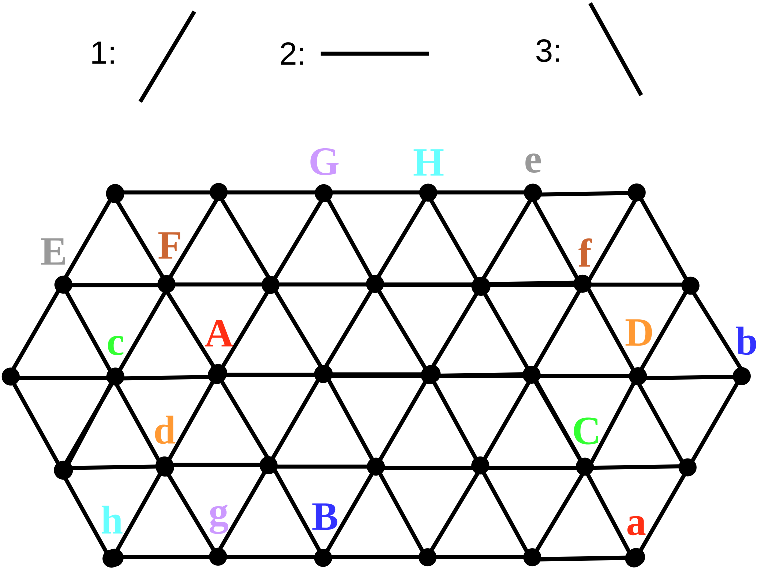

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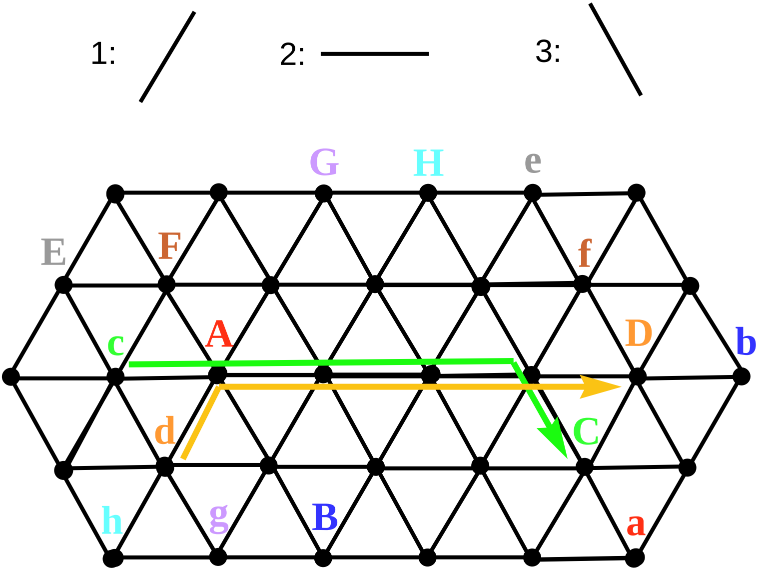

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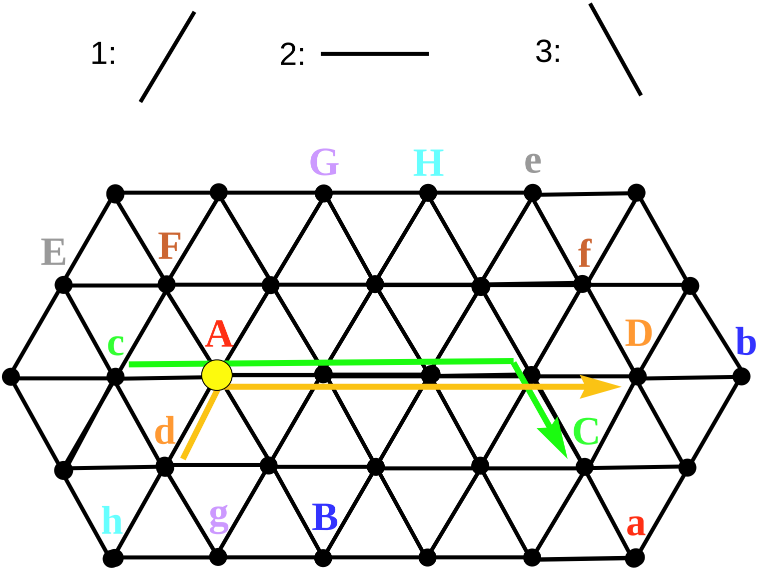

Description of Algorithm $\mathcal{A}$

At each node u of the network:

- ***Preprocessing*:** Initially, if there is a packet at u, compute the relative address $D - S$ of the message in the shortest path form, and add this information to the message.
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  - **a)** If there are packets with **negative components**, send them **immediately** along the direction of this component.
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- Algorithm $\mathcal{A}$ defines for each packet **two directions of movement** (except if a packet has only one non-zero component)
- For instance:
  - if the packet address is of the type $(-, 0, +) \rightarrow$ this packet goes first in the direction $-x$, and after in $+z$

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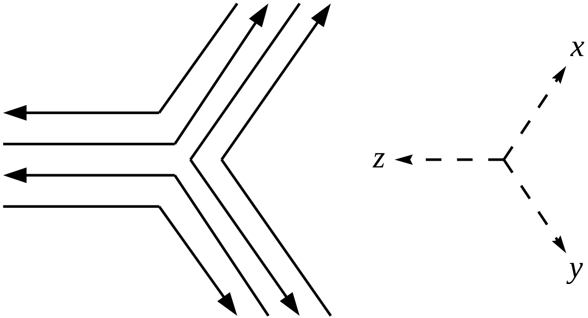
Routing the packets (2)

In this figure all the routing rules are summarized:



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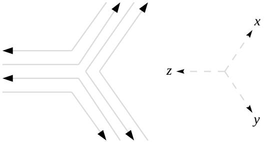

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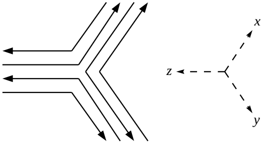

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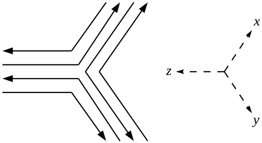

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Optimality

- Using this algorithm, at each step **all the packets with maximum remaining distance move**

  $\longrightarrow$ every step the maximum remaining distance over all packets decreases by one

  $\longrightarrow$ the total running time is at most ℓ_{max}, **meeting the lower bound**.

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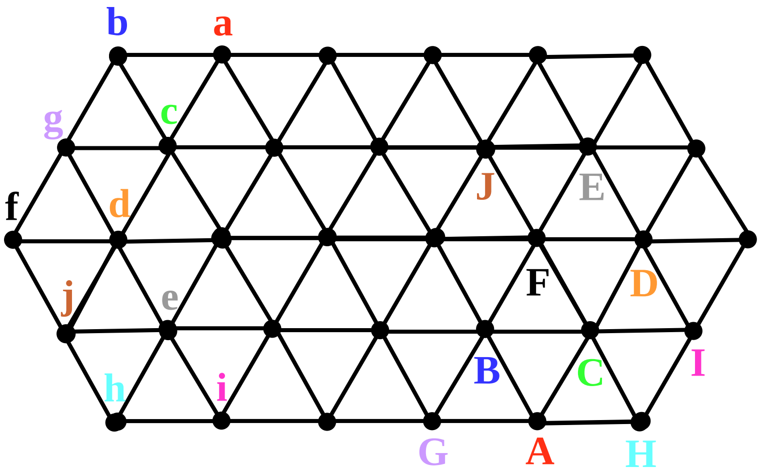

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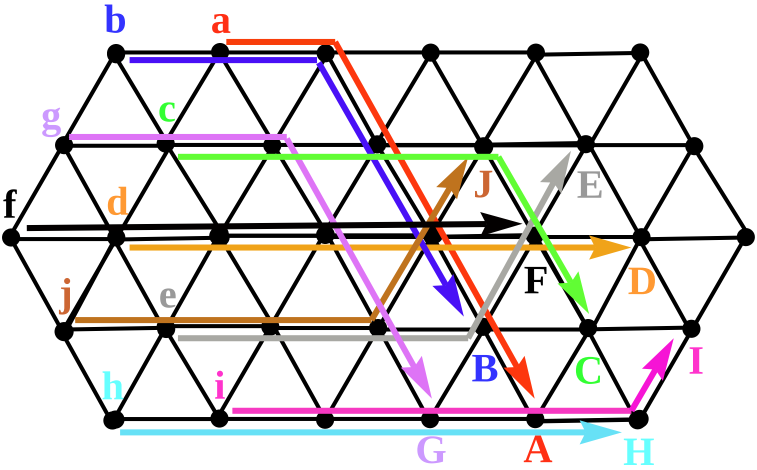
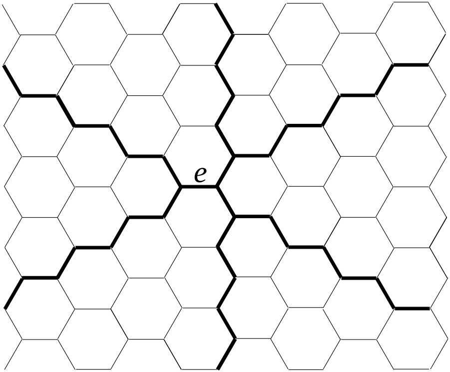


Permutation Routing on Hexagonal Grids

Hexagonal grid

- One can define 3 types of *zigzag chains*:



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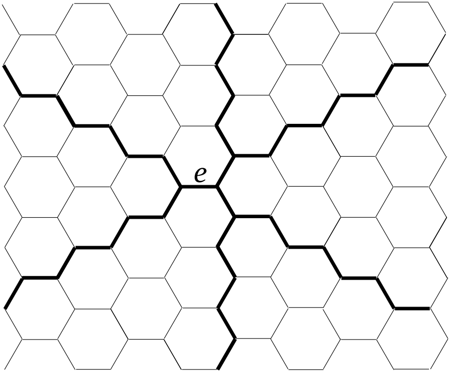

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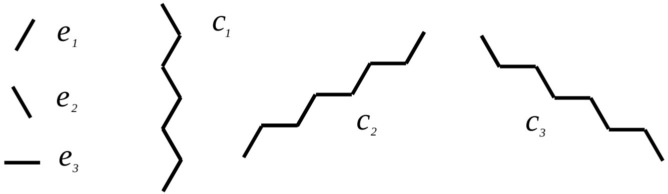
Idea

- There are 3 types of edges and 3 types of chains:



- Each edge belongs to exactly 2 different chains, and conversely each chain is made of 2 types of edges.
- Any 2 chains of different type intersect exactly on one edge.
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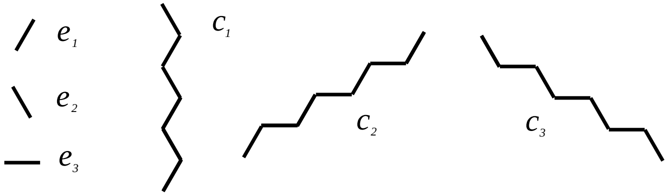

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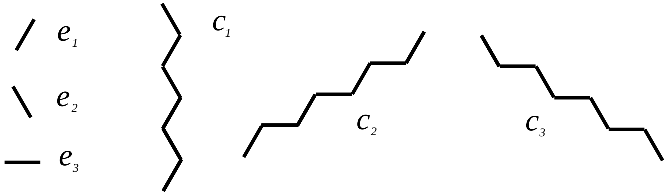

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Optimal algorithm

At each node of the network:

1) During the **first step**, move all packets along the direction of their **negative component**. If a packet's address has only a positive component, move it along this direction.
2) From now on, **change alternatively** between Phase 1 and Phase 2.
3) At each step (the same for both phases):
   a) If there are packets with negative components, send them immediately along the direction of this component.
   b) If not, for each outgoing edge order the packets according to decreasing number of remaining steps, and send the first packet of each queue.

Running time

- Every 2 steps (one of Phase 1 and one of Phase 2) the maximum remaining distance over all packets decreases by one.
- During the first step all packets decrease their remaining distance by one.
- Thus, the total running time is $1 + 2(\ell_{max} - 1) = 2\ell_{max} - 1$.
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(ℓ, k)-Routing

Algorithm (in any grid)

- **Recall**: each node can send at most ℓ packets and receive at most k packets
- **Idea**: represent the request set as a weighted bipartite graph H:
  - split each vertex of the original graph
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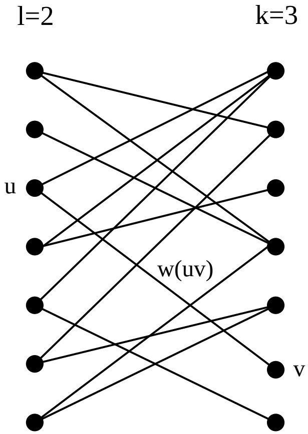
Example



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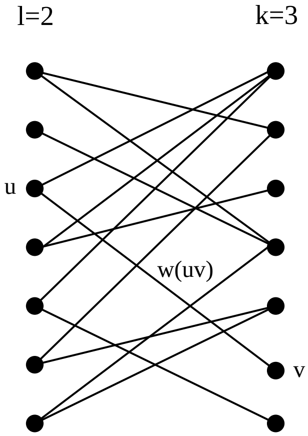

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New problem

- **Problem**: find $m := \max\{\ell, k\}$ matchings in H: $M_1, \ldots, M_m$
- Let $c(M_i) := \max\{w(e) | e \in M_i)\}$, $i = 1, \ldots, m$
- **Objective function**:

$$\min \sum_{i=1}^{m} c(M_i)$$

- **Fact**: $\min \sum_{i=1}^{m} c(M_i)$ is the running time of routing a (ℓ, k)-routing instance using this algorithm
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Thanks!