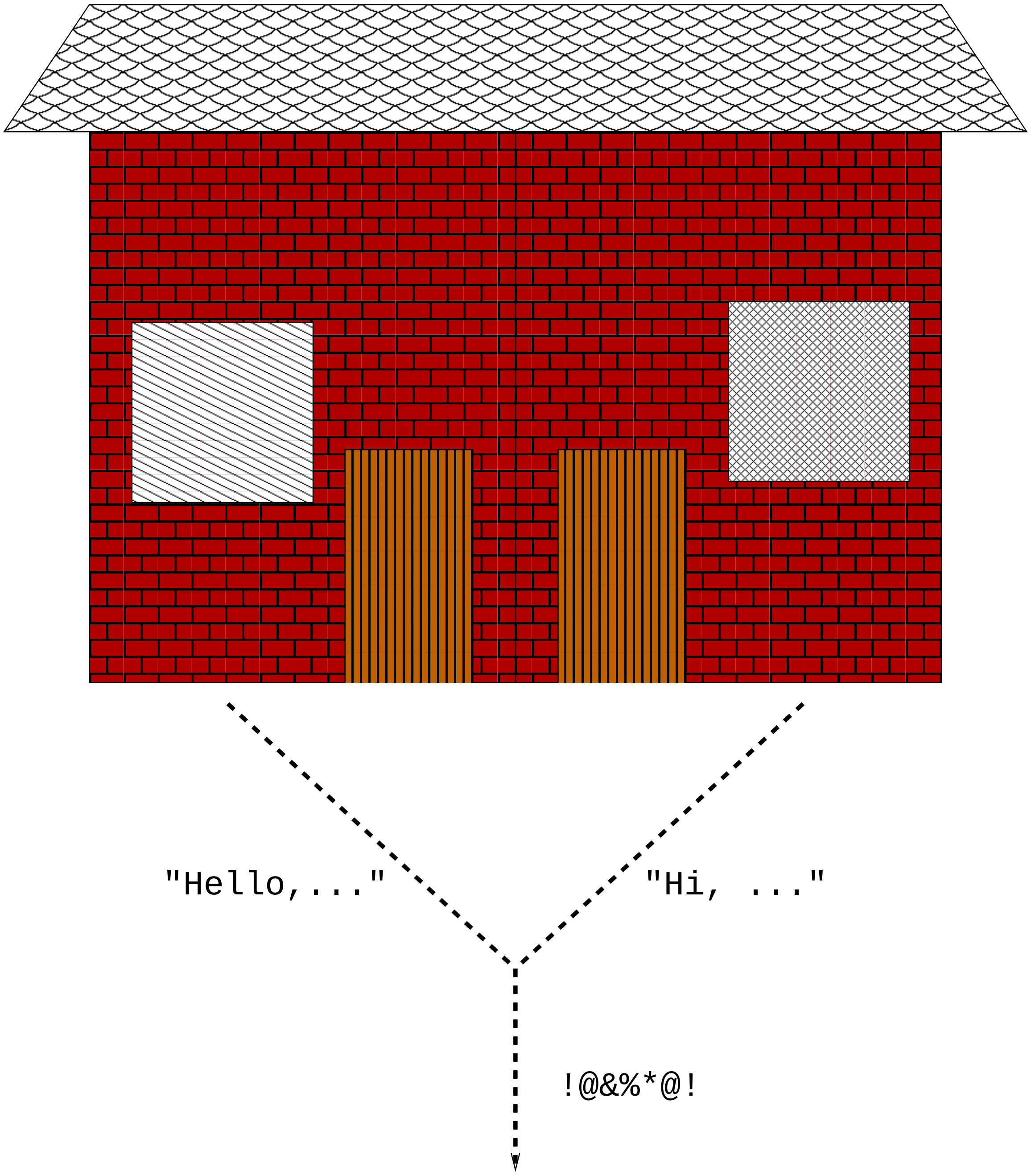
Contention Resolution in Multiple-Access Channels

Leslie Ann Goldberg, Warwick

Credits: Paterson, Al-Ammal, Jerrum, Kannan, MacKenzie, Srinivasan

Warm-up: A toy problem



Strategy 1. Flip coin during each time slot. If heads then try

$\frac{1}{4}$: no tries $\frac{1}{4}$: two tries $\frac{1}{2}$: one try

Expected time to first success:

$$\frac{1}{1/2} = 2.$$

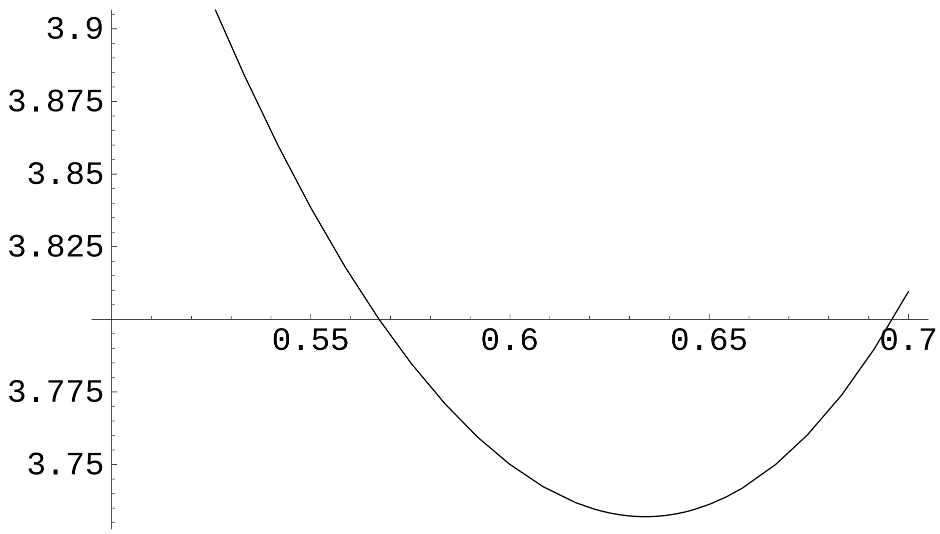
Expected total time: 4

Strategy 2. Use biased coin. $\Pr[\text{heads}] = p$

Expected total time:

$$\frac{1}{2p(1-p)} + \frac{1}{p}.$$

For $p \approx 0.6$, expected total time is ≈ 3.7



Strategy 3. After collision, choose between first slot and second slot u.a.r.

$$E = \frac{1}{2}2 + \frac{1}{4}(1 + E) + \frac{1}{4}(2 + E).$$

$$E = 3.5.$$

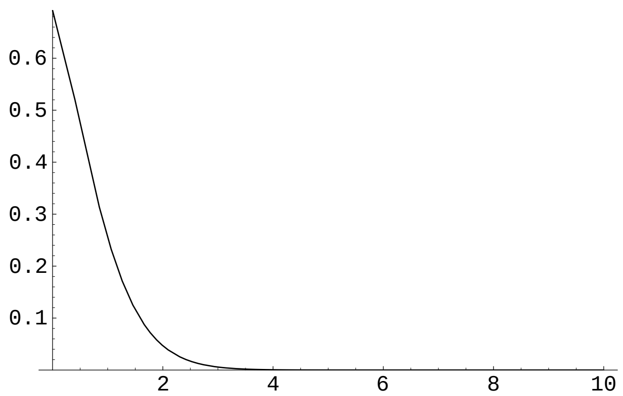
Optimal strategy After collision, choose between first, second and third slots with probabilities $\approx 0.53, 0.37, 0.1$.

$$E \approx 3.4.$$

The Model: The two players only listen when they send. If they listen at all slots, then strategy 1 is optimal. Expected time to first success: 2. Second success in the next slot.

Multiple Access Channel Communication Model (Queue-Free)

- Slotted time $0, 1, 2, \ldots$
- only 1 packet per slot succeeds, if more than 1 packet tries, *all* are blocked collision
- blocked packets (backlog) try again later according to the protocol
- Number of arrivals is Poisson with mean λ. Probability of k arrivals is $\lambda^k e^{-\lambda}/k!$.



ALOHA protocol, 1970
Abramson, University of Hawaii

Each backlogged packet retries repeatedly with probability p at each step (time-homogeneous) Markov chain

State Space Ω

Transition Matrix P
For each $(i, j) \in \Omega^2$, $P(i, j)$ gives the probability of transitioning from state i to state j

A sequence $\{X_0, X_1, \ldots\}$ of random variables with values in Ω

$\Pr(X_t = x_t \mid X_1 = x_1, \ldots, X_{t-1} = x_{t-1}) = P(x_{t-1}, x_t)$.

ALOHA Protocol

$\Omega = \{0, 1, 2, \ldots\}$
state = number of backlogged packets

transition probabilities

From state i

$z(i)$: probability of zero tries. $(1-p)^i$.

$u(i)$: probability of a unique try. $i\,p\,(1-p)^{i-1}$.

$P(i,j)$: probability of going from i to j.

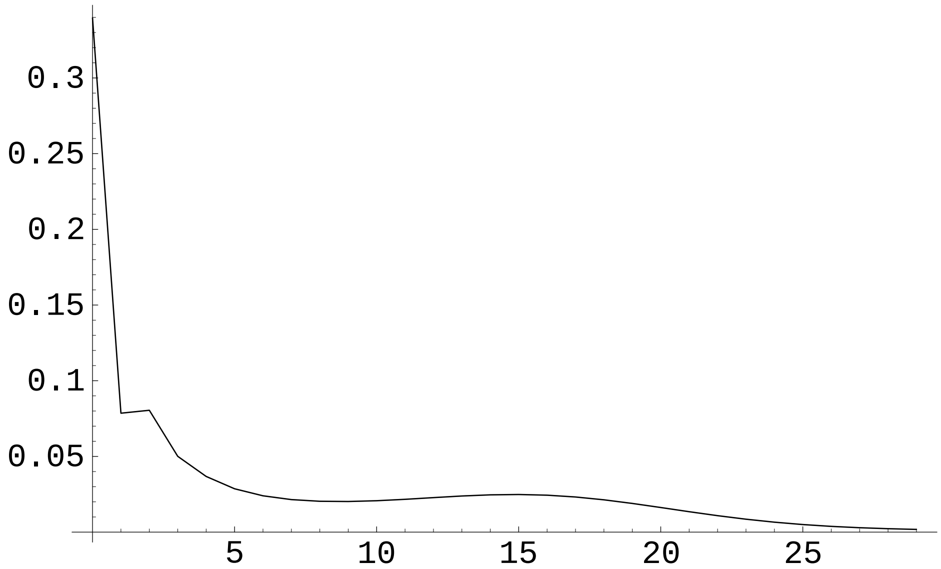
$P(i,i-1) = e^{-\lambda} u(i)$.

$P(i,i+1) = \lambda e^{-\lambda}(1 - z(i))$.

ALOHA protocol with packet-dropping

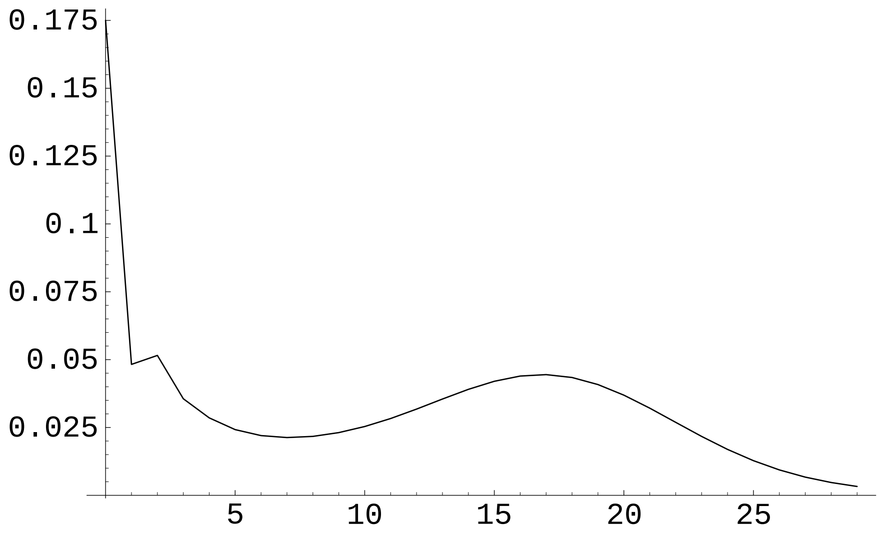
Each backlogged packet retries repeatedly with probability $p = 0.25$ at each step and disappears with probability $r = 0.02$ at each step.

Stationary distribution, $\lambda = 0.34$

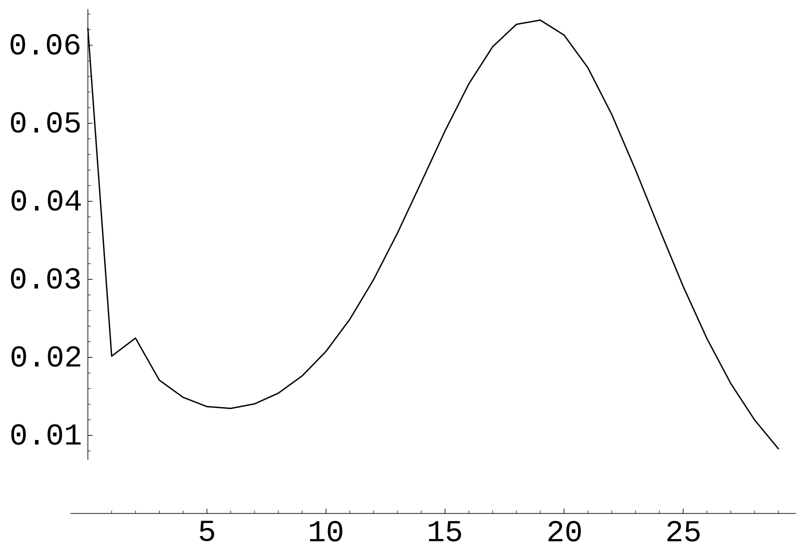


Bistability

Stationary distribution, $\lambda = 0.37$



Stationary distribution, $\lambda = 0.4$



Backoff Protocol

- backoff counter for each packet: Initially 0, incremented by 1 after each collision

- packet with backoff count b sends with probability p_b

Binary Exponential Backoff $p_b = 2^{-b}$

Ethernet protocol, 1973

IEEE 802.3 CSMA/CD Ethernet protocol:

$$\begin{array}{lcll} p_b & = & 2^{-b} & b \leq 10 \\ p_b & = & 2^{-10} & 10 < b \leq 16 \\ p_b & = & 0 & b > 16 \quad \text{Drop Packet!} \end{array}$$

How bad is binary exponential backoff?

Kelly and MacPhee 1987 If $\lambda \geq \ln 2 \approx 0.7$, expected number of successful packets is finite even when protocol is run forever

Kelly and MacPhee's Method

Externally Jammed Channel

$\pi_i =$ Pr(packet tries on its ith step)

$q_j = \pi_0 + \cdots + \pi_j$

$\Phi = \{t \mid \text{at most one attempt at } t\}$

attempts at t = Poisson$(\lambda\, q_t)$

$$E(|\Phi|) = \sum_{t=1}^{\infty} (1 + \lambda\, q_t) e^{-\lambda\, q_t}$$

For $\lambda \geq \lambda^*$, $E(|\Phi|)$ finite

Removing the external jamming

Poisson$(\lambda + \epsilon)$

How bad is binary exponential backoff?

Aldous 1987 For any $\lambda > 0$,

$$\frac{\#\text{ successes up to time } t}{t} \to 0 \text{ as } t \to \infty$$

Also, the corresponding Markov chain is transient

Definitions

Markov chain $X = \{X_0, X_1, \ldots\}$

State i is recurrent if
$\Pr(X_t = i \text{ for some } t \geq 1 \mid X_0 = i) = 1$.

Otherwise, i is transient.

irreducible Markov chain (can get from every state to every other one): Either every state is recurrent (in which case the chain is recurrent) or every state is transient (in which case the chain is transient)

transient Markov chain $\Rightarrow$ very bad protocol

$f_{i,j}(n) = \Pr(X_1 \neq j, \ldots, X_{n-1} \neq j, X_n = j \mid X_0 = i)$.

$f_{i,j} = \sum_{n=1}^{\infty} f_{i,j}(n)$.

transient: For all i, $f_{i,i} < 1$.

$\Pr(\text{revisit } j \text{ at least } N \text{ times} \mid X_0 = i) = f_{i,j} f_{j,j}^{N-1}$

Let C be any finite set of "good" states (that are good for the protocol)

$\Pr(\text{visit } C \text{ at least } N \text{ times} \mid X_0 = i) \leq$

$\sum_{j \in C} f_{i,j}\, f_{j,j}^{\left(\frac{N}{|C|}-1\right)}$ (exponentially small in N)

The expected number of times that C gets visited is finite!

Modelling a backoff protocol as a Markov chain

Backoff Protocol

• backoff counter for each packet: Initially 0, incremented by 1 after each collision

• packet with backoff count b sends with probability p_b

Protocol: Sequence $p_0, p_1, \ldots$

state $x = (x_1, x_2, \ldots)$, where
x_b = number of messages with backoff count b.

transition:

1. $x_0 \leftarrow \text{Poisson}(\lambda)$.

2. Each of the x_b messages with backoff count b sends (independently) with probability p_b.

3. new state.

Binary Exponential Backoff $p_b = 2^{-b}$

How bad is binary exponential backoff?

Aldous 1987 For any $\lambda > 0$,

$$\frac{\#\text{ successes up to time } t}{t} \to 0 \text{ as } t \to \infty$$

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Aldous's method

$*$: unlikely but possible (bad) event

t: expected number of attempts is at least $c\lambda \log t$.

$*$: for all small b, at least $\lambda 2^b$ messages have backoff count b

How bad is binary exponential backoff?

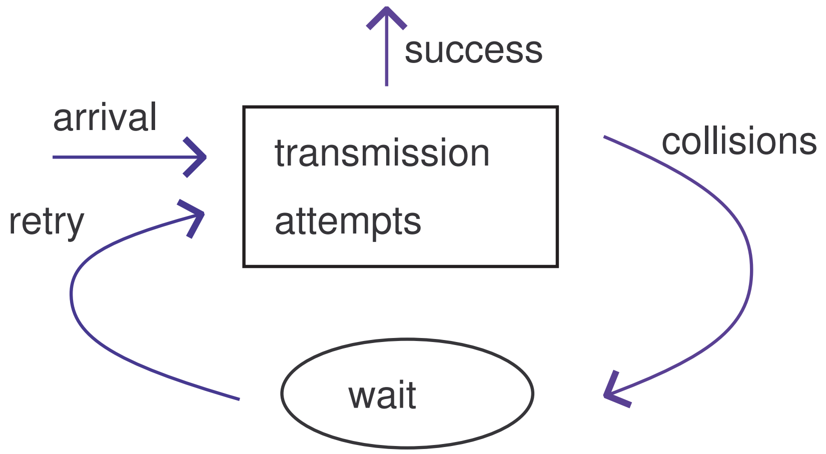
Aldous 1987 For any $\lambda > 0$,

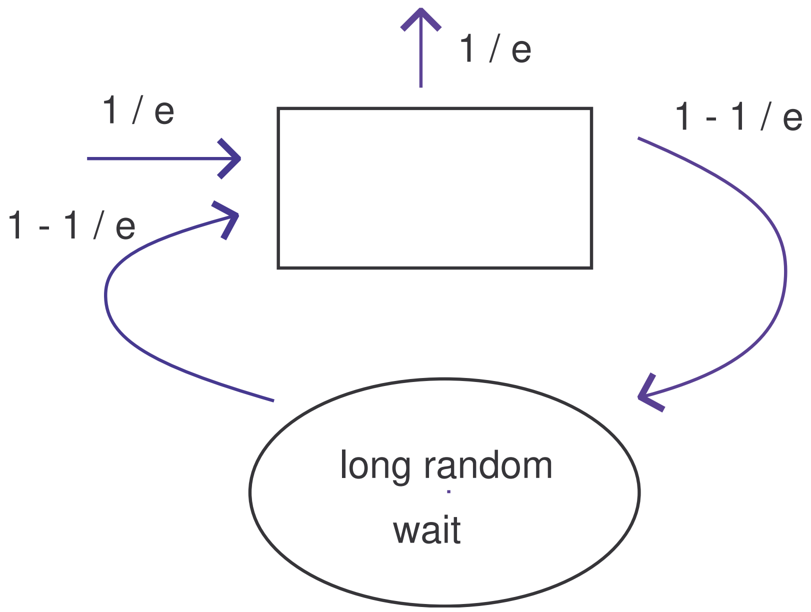
$$\frac{\text{\# successes up to time } t}{t} \to 0 \text{ as } t \to \infty$$

Also, the corresponding Markov chain is transient

Aldous conjectures same negative result for *all* backoff protocols

Heuristic evidence: why we should be able to achieve rates up to $\lambda = 1/e$.





$\mu e^{-\mu}$: max is $1/e$ at $\mu = 1$.

Heuristics: identifying "stable" regions for small λ

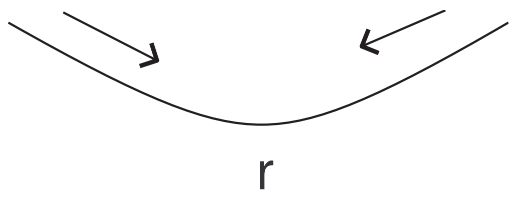
Fixed Poisson arrival rate λ

In a (fairly) steady state, suppose r is the expected number of tries for each packet before succeeding

Distribution of tries is roughly Poisson with mean $r\lambda$

Net decrease in population: $r\lambda e^{-r\lambda} - \lambda$

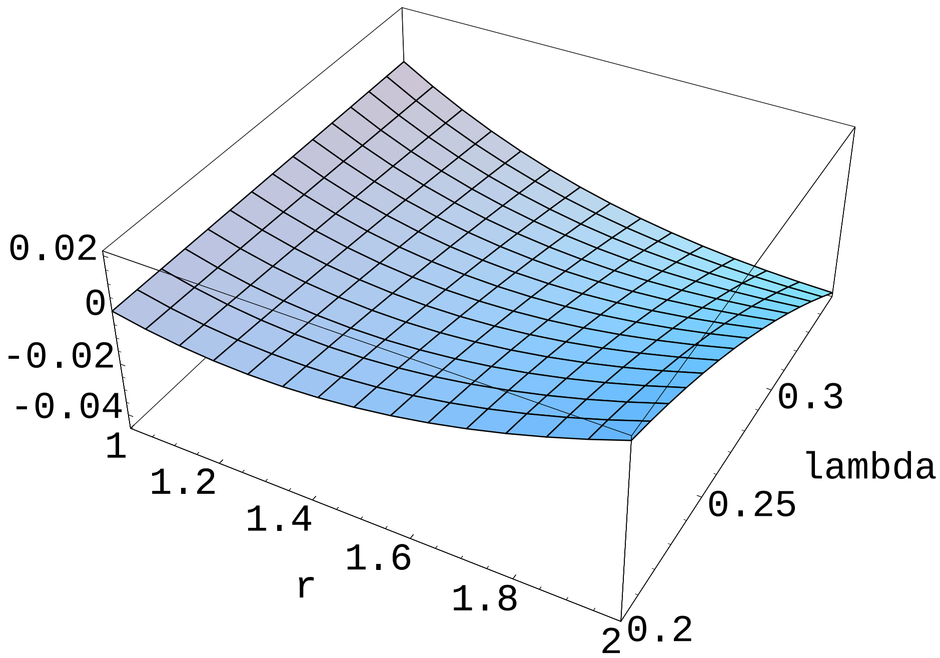
Plot a function $F(r)$ with derivative $dF/dr = r\lambda e^{-r\lambda} - \lambda$



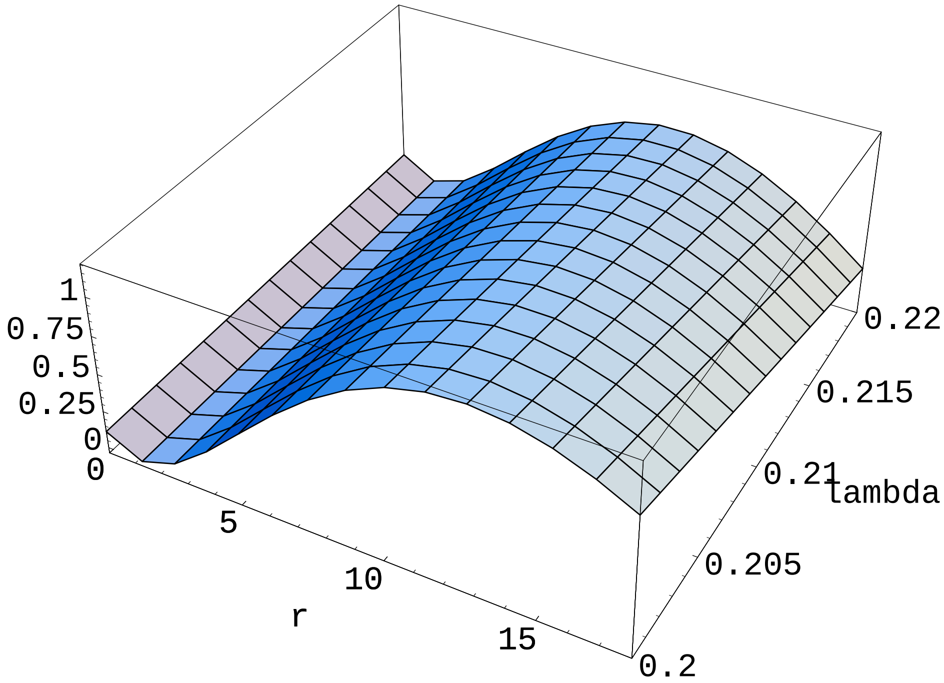
Population tends to move down along the curve, stabilising r.

$\lambda \in [0.2, 0.35]$

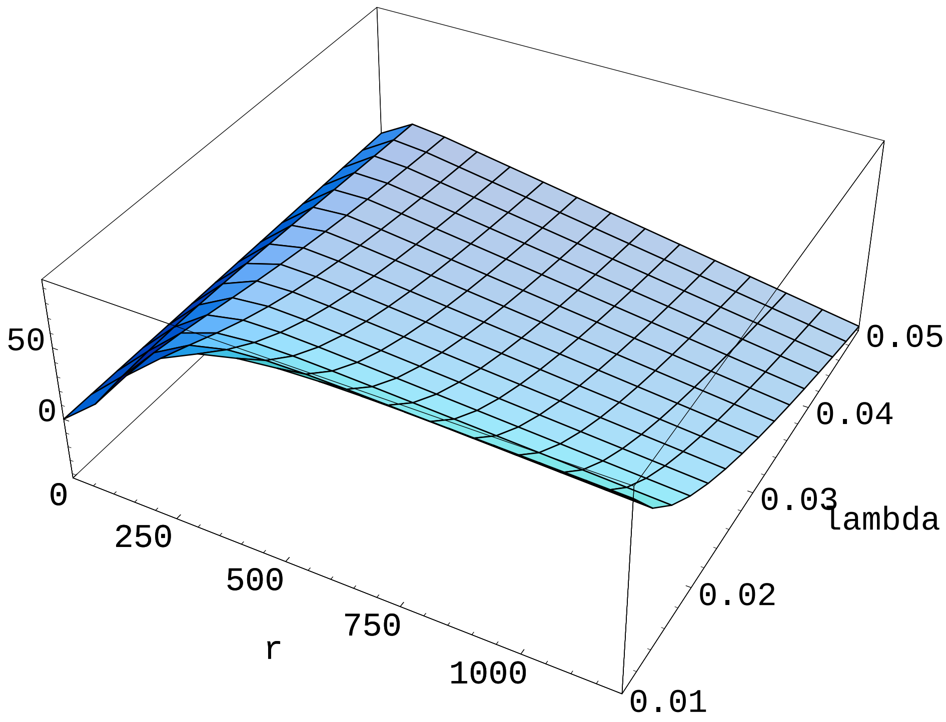
We seem to have a stable region for small λ in this range



But the picture becomes more clear if we look over a wider range of r . . .



And we seem to have bi-stability even for arbitrarily small arrival rates.



Open question: Is there any backoff protocol which is recurrent for any positive arrival rate λ?

Conjecture: No

Best known (not much!)
(G, Jerrum, Kannan, Paterson, 2000)
No backoff protocol is recurrent for any arrival rate exceeding 0.42.

NB: $1/e \approx 0.368$.

Fayolle, Malyshev, Menshikov

A Markov chain is **transient** if there is a function f from the state space to $\mathbb{R}^+$ and constants C, d and ϵ such that

1. f can change by at most d in one step, and

2. for some state x, $f(x) > C$, and

3. if $f(x) > C$ then during one step from state x, the expected change in f is at least ϵ.

Lyapounov function, potential function, drift

Modelling a backoff protocol as a Markov chain

Backoff Protocol

- backoff counter for each packet: Initially 0, incremented by 1 after each collision

- packet with backoff count b sends with probability p_b

Protocol: Sequence $p_0, p_1, \ldots$

state $x = (x_1, x_2, \ldots)$, where
$x_b =$ number of messages with backoff count b.

transition:

1. $x_0 \leftarrow$ Poisson(λ).

2. Each of the x_b messages with backoff count b sends (independently) with probability p_b.

3. new state.

Aldous's potential function

$f(x) =$ expected number of attempted sends.

There are counter-examples (protocols) for which this potential function does not always have the right drift.

The "Obvious" Potential Function

$f(x) =$ number of messages backlogged in state x

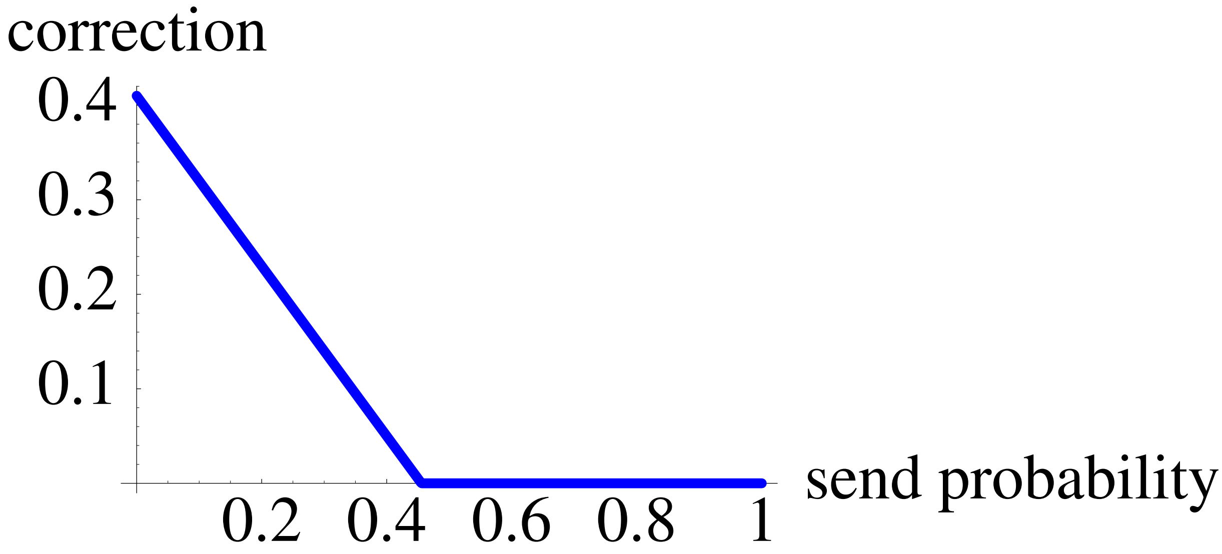
Counter-example

b	0	1	2	3	4	...	
p_b	1	1	tiny	tiny	tiny	...	
x_b		1	big	0	0	0	...

$$
\begin{aligned}
E[f(X_1) - f(X_0) \quad | \quad X_0 = x] & \\
&\approx e^{-\lambda}(-1) + \sum_{j=1}^{\infty} \frac{\lambda^j}{j! e^{\lambda}} j \\
&= -e^{-\lambda} + \lambda \\
&< 0.
\end{aligned}
$$

A potential function that works

$f(x)$ = number of messages backlogged in state x plus...



"send probability" = probability that at least one backlogged message sends on the next step

The counter-example is ok now.

b	0	1	2	3	4	...	
p_b	1	1	tiny	tiny	tiny	...	
x_b		1	big	0	0	0	...

$$
\begin{aligned}
E[f(X_1) - f(X_0) \quad | \quad X_0 = x] & \\
&\approx e^{-\lambda}(-1 + 0.41) + \sum_{j=1}^{\infty} \frac{\lambda^j}{j! e^{\lambda}} j \\
&= -0.59 e^{-\lambda} + \lambda \\
&> 0.
\end{aligned}
$$

And actually, the potential function does have a positive drift

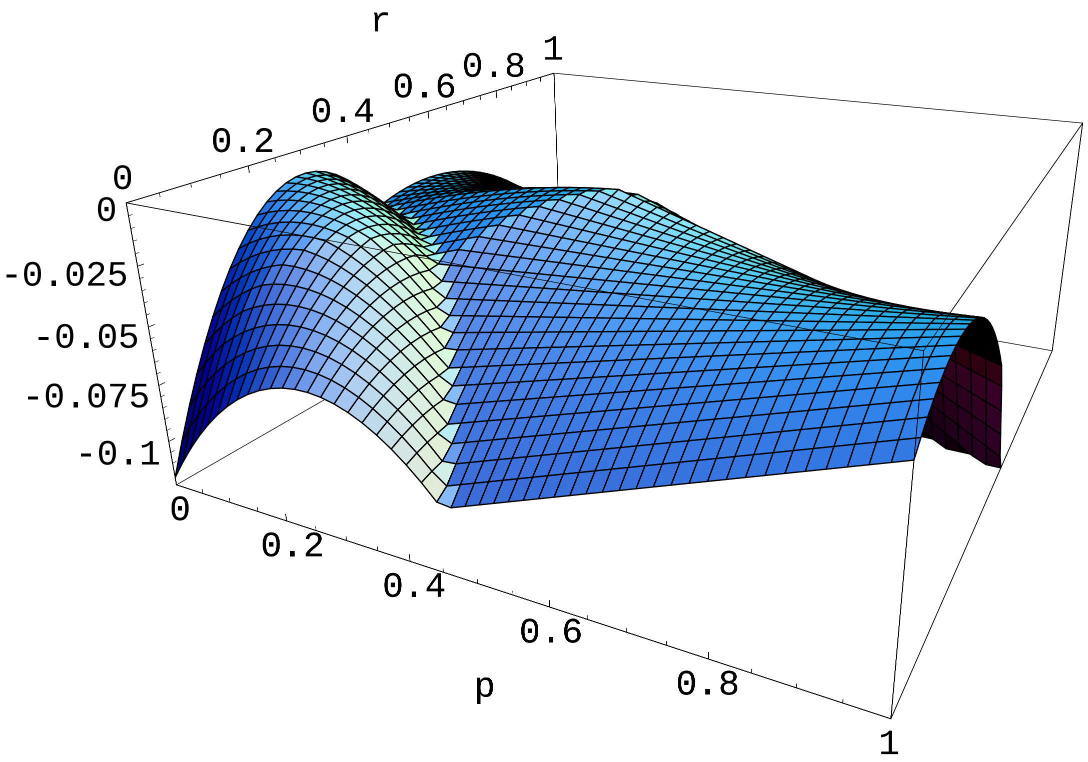
To see this, define...

$p(x)$ = largest p_b for backlogged message in state x; say packet w.

$r(x)$ = probability that at least one of the other backlogged messages tries to send.

Expected increase in f is at least $h(r,p)$.

Picture of $-h(r,p)$.



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Open Question: Is there any backoff protocol which is recurrent for any positive arrival rate λ?

The quadratic backoff protocol

$p_b = (b+1)^{-2}$.

Our Model

Kelly and MacPhee Transient for any $\lambda > 0$.

Queueing Model

N queues.

Just before each step, each queue gains a message (independently) with probability λ/N.

Only the heads of the queues participate in contention resolution.

Håstad, Leighton, and Rogoff Recurrent for any $\lambda < 1$.

Recall: Kelly and MacPhee

Given protocol, calculate λ^* such that for $\lambda > \lambda^*$, the expected number of successes is finite.

For quadratic backoff, $\lambda^* = 0$.

Håstad, Leighton, and Rogoff
Very strong positive result

$\lambda < 1$

$p_b = (b+1)^{-\alpha}$, $\alpha > 1$.

positive recurrent: For all i,
$E[\min\{n \geq 1 \mid X_n = i\} \mid X_0 = i] < \infty$.

W_i = delay of ith message

$$W_{\text{avg}} = \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} W_i$$

$E[W_{\text{avg}}] < \infty$.

Proving Positive Recurrence
General method
Foster's Theorem

A Markov chain is positive recurrent if and only if there is a positive Lyapounov function which is expected to decrease by a positive amount from every state except some finite subset of the state space.

Håstad, Leighton, and Rogoff's method

$$\Phi(s) = \sum_{i=1}^{N} q_i + \sum_{i=1}^{N} (b_i + 1)^{\alpha+1/2}$$

1. For any state s with big potential, there is a stopping time $T_s \le d$ such that

$$E[(\Phi(X_{T_s}))^2 - (\Phi(X_0))^2 \mid X_0 = s] \le -\delta\Phi(s).$$

2. Any state with small potential cannot get very big in a single move.

3. There are not too many states with small potential.

4. One more technical condition.

Summary: The Queueing Model

Polynomial backoff

$p_b = (b+1)^{-\alpha}$, $\alpha > 1$.

positive recurrent for $\lambda < 1$

Binary exponential backoff

Not positive recurrent $\lambda > 1/2$.

Why does anybody ever use Binary exponential backoff?

Meta-stability: Binary exponential backoff looks good in practice for $\lambda \leq 1/e$.

Exception: Multiple-server systems!

The Queueing Model
Binary Exponential Backoff

Håstad, Leighton, and Rogoff
$\lambda > 1/2$
not positive recurrent.

Al-Ammal, G, MacKenzie
$\lambda < \epsilon n^{-0.75}$
positive recurrent.

The Full-Sensing Model
Every user listens to the channel at every step.

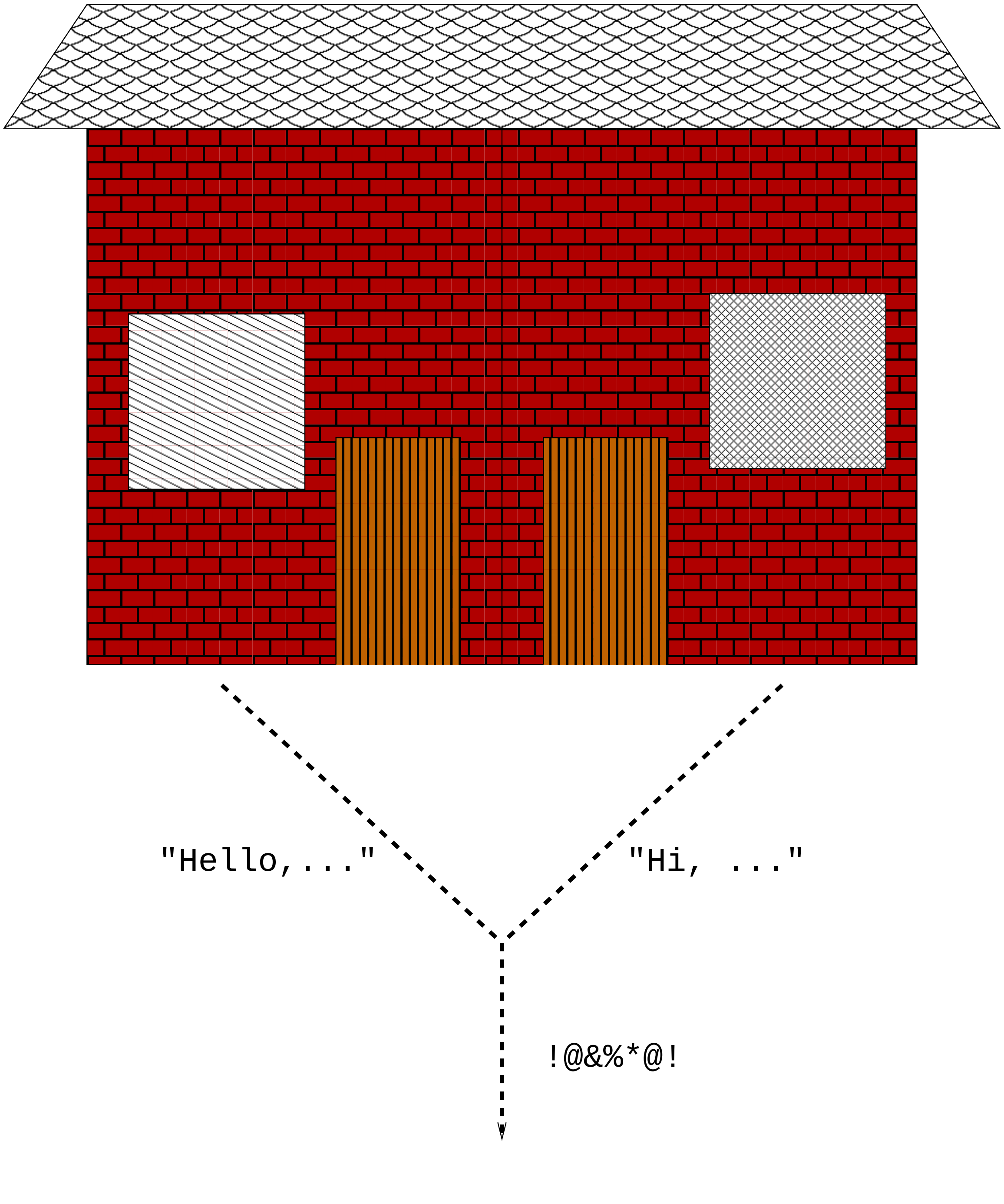
Mosely and Humblet: A protocol with capacity 0.48776

tree protocol in the sense of Capetanakis, Tsybakov, Mikhailov

Improvement: Vvedenskaya and Pinsker

Upper bound: Tsybakov and Likhanov

Warm-up: A toy problem



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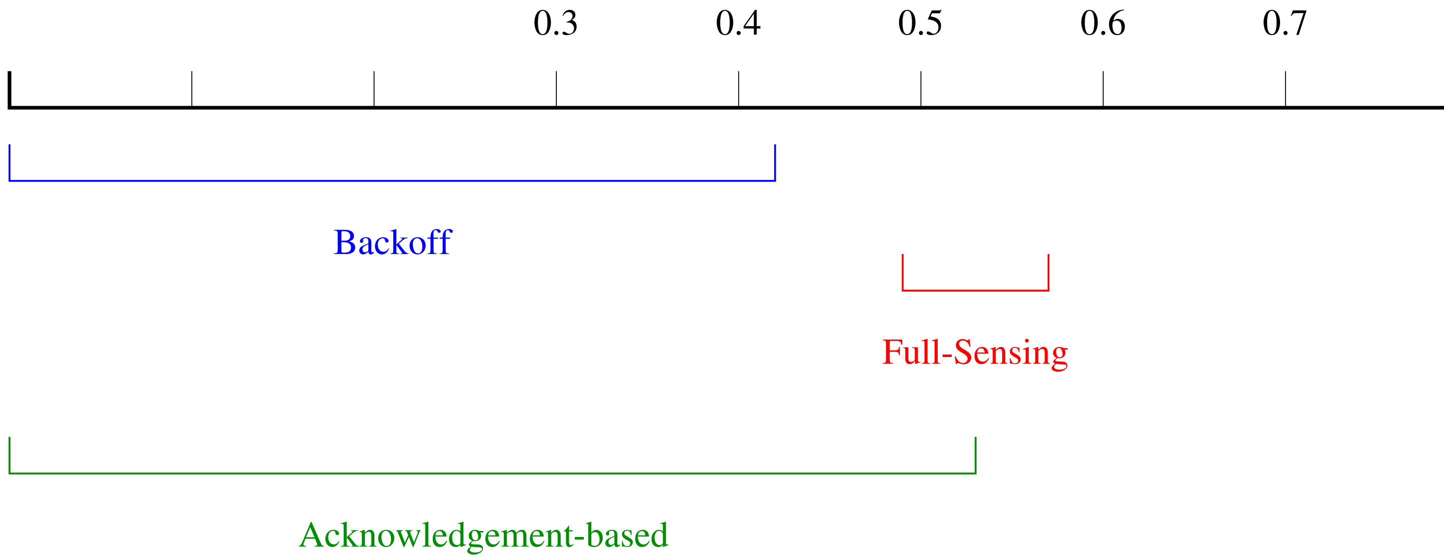

Upper and lower bounds on achievable arrival rates

Acknowledgement-based protocol

‘Send’ decision can depend arbitrarily on collision history

Open Questions

In the ordinary model, is there a backoff protocol which is recurrent for any positive arrival rate?

In the queueing model, is binary exponential backoff stable for any positive arrival rate?

In the ordinary model, is there an acknowledgement-based protocol which is recurrent for any positive arrival rate?

Paterson and Srinivasan

Suppose all packets share a clock!

$\lambda \leq 1/e$

first attempt: $1, 3, 5, 7, 9, \ldots$

second attempt: $2, 6, 10, 14, 18, 22, \ldots$

third attempt: $4, 12, 20, \ldots$

Expected waiting time of each message: $O(1)$.