Robust Routing in Changing Topologies

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Nous proposons dans cet article une nouvelle version du problème de plus court chemin robuste. L'incertitude, qui ne concerne que les poids des arcs d'un graphe dans la version initiale, s'applique ici également à la structure même du graphe. Le problème consiste alors à trouver une séquence de chemins minimisant la somme des poids des chemins plus la somme des coût de transformation de deux chemins consécutifs. Après avoir formalisé le problème et établi sa complexité, nous donnons sa formulation mathématique ainsi qu'une relaxation. Nous proposons un algorithme polynômial pour les instances constituées de deux graphes et dans lesquelles le coût de désinstallation d'un arc est inférieur à son poids.

Keywords: robustness, robust shortest path, networks

1 Introduction

Recently the introduction of the uncertainty has appeared in numerous applications. This notion captures the natural evolution of general systems. The complexity lies in the number of parameters that are allowed to vary in order to cope with the characteristics of the problem studied.

In this context, the Absolute Robust Shortest Path problem consists in finding a path corresponding to the minimum maximum weight over a set of scenarii. Each scenario corresponds to a pre-determined set of edge weights. The motivation for studying this problem comes from telecommunications where a communication network is used to send packets from a given source to a given destination. The aim is to determine a shortest path between some given source and destination under some criteria (total delay, congestion, ...). The network manager has to choose a robust path where the total delay is acceptable regardless of the realized congestion [KY97].

The common approach of the robustness consists in finding a robust solution which minimizes the maximum regret, *i.e.*, the worst case scenario. Several approaches have been proposed. Averbakh [Ave04] focuses on minmax regret for optimization over uniform matroid, Ben-Tal, El Ghaoui and Nemirovski [BTN00, BT, BTNEGR00] work on ellipsoidal uncertainty, Bertismas and Sim [BS04] investigate robust optimization with control of the conservation of a solution, Kouvelis and Yu [KY97] focus on minmax regret robust optimization and Yaman and Karasan [PKY03] also work on absolute and relative robustness for spanning tree problem.

In Robust Shortest Path problem, the uncertainty is only related to link weights and does not take into account the possible evolutions of the network topology. Based on the market quick evolutions, telecommunication networks may increase and sometimes decrease when operators sell a part of their networks. Thus, an existing route between two protagonists may appear less profitable after adding new connections or nodes, and it may have to be reconsidered with the network evolution. Further cost may appear during this evolution, especially for resource desallocation and re-allocation. Considering this other definition of robustness, we introduce an alternative problem in which the uncertainty is also related to the structure of the network itself. Here, routing consists in finding an optimal sequence of paths taking into account the network topology variations. Moreover, changing existing routing may create local disturbance which can be formulated in terms of cost. Thus, a sequence of paths will be optimal if it minimizes its total weight cost and the number of disturbances. We call this problem Robust Routing in Changing Topologies, *RRCT* for short.

We show in Section 2 that this problem is NP-complete even with simplified assumptions and we give its mathematical formulation and a combinatorial relaxation. In Section 3 we give a polynomial algorithm for an instance of a two-graph sequence, where the first graph is a simple path and disturbance cost is simplified. Finally, we draw a conclusion and address open questions in Section 4.

2 Definition

Throughout this paper, networks will be considered as weighted graphs.

2.1 RRCT problem

We consider a sequence of n directed and weighted graphs $(G_i)_{i\in\{1,..,n\}}$. Each graph is defined by its set of edges E_i and its set of vertices V_i. This sequence of graphs is such that $E_i \subset E_{i+1}$ and $V_i \subseteq V_{i+1}$. We note $G_i \subset G_{i+1}$. Let n_i be the number of nodes of graph G_i and m_i its number of edges. In addition, an edge has an installation cost and an uninstallation cost. We define $w : E \to \mathbb{N}$ as the weight function, $\iota : E \to \mathbb{N}$ as the installation cost function and $\delta : E \to \mathbb{N}$ as the uninstallation cost function. Let s and t be a source node and a destination node respectively. Note also that s and t belong to G_i for any i.

The problem consists in finding a collection of paths $(X_i)_{i\in\{1,...n\}}$, one per graph, each linking source node s to destination node t. In the rest of the paper, S_n denotes the collection $(X_i)_{i\in\{1,...,n\}}$. The aim of *RRCT* problem is to minimize total weight plus transition cost between consecutive paths. The transition cost is composed of the uninstallation cost of the subset of edges of X_i which are not in X_{i+1} plus the installation cost of the subset of edges of X_{i+1} that are not in X_i. Formally, we have to minimize the following function:

$$\varphi(S_n) = \sum_{i=1}^{n}\sum_{x\in X_i} w(x) + \sum_{x\in X_1}\iota(x) + \sum_{i=1}^{n-1}\Big(\sum_{x\in X_i, x\notin X_{i+1}}\delta(x) + \sum_{x\in X_{i+1}, x\notin X_i}\iota(x)\Big) \tag{1}$$

The Robust Routing in Changing Topologies problem can be formulated as a decision problem:
Robust Routing in Changing Topologies problem (*RRCT*)

Instance:

n graphs $G_i = (V_i, E_i)$ such that $G_i \subset G_{i+1}$ $\forall i \in \{1 \dots n-1\}$
$w : E \to \mathbb{N}$, $\iota : E \to \mathbb{N}$, $\delta : E \to \mathbb{N}$
two vertices s and t; and an integer b

Question:

Does there exist a sequence S_n of n paths between s and t such that $\varphi(S_n) \le b$?

Theorem 1 ([BGL05a]) *RRCT is NP-complete even for a two-graph sequence and when the first one is a simple path.*

The proof of this theorem is given in [BGL05a]. The reduction is based on Hamiltonian Circuit Problem [GJ79].

2.2 Mathematical formulation

We now formulate *RRCT* as a quadratic program with linear constraints. Let x^k_{ij} be the binary variable equal to 1 if the edge between the nodes i and j of graph G_k is part of path X_k, and 0 otherwise. The objective function can be formulated as:

$$\varphi(S_n) = \sum_{k=1}^{n}\left(\sum_{(i,j)\in E_k} w(ij)x^k_{ij} + \sum_{(i,j)\in E_k}\delta(ij)x^k_{ij}(1-x^{k+1}_{ij}) + \sum_{(i,j)\in E_{k+1}}\iota(ij)x^{k+1}_{ij}(1-x^k_{ij})\right) \tag{2}$$

RRCT can be formulated as follows:

$$(RRCT)\begin{cases} \min \varphi_n \\ s.t. \\ \quad \sum_{j\in V} x_{ij}^k - \sum_{j\in V} x_{ji}^k = \begin{cases} 1 \; if \; i=u \\ -1 \; if \; i=v \\ 0 \; otherwise \end{cases} & \forall k\in\{1\dots n\} \quad \forall i \in V & (3) \\ \quad \sum_{(i,j)\in E(H)} x_{ij}^k \le |H|-1 & \forall H \subset G_k \quad \forall k\in\{1\dots n\} & (4) \\ \quad x_{ij}^k \in \{0,1\} & \forall i,j \in V_k \quad \forall k\in\{1\dots n\} \end{cases}$$

Constraints 3 represent the classical flow constraints and ensure that the solution contains a path from s to t within each graph G_i. Constraints 4 eliminate potential circuits. Indeed, some circuits can appear since the objective function contains negative terms. However, Constraints 4 introduce an exponential number of constraints.

In the following, we assume that Constraints 4 can be relaxed in order to solve $RRCT$ by direct methods. It can however be handled within a decomposition scheme where cuts are added if a circuit is detected. The model then becomes:

$$(RRCTr)\begin{cases} \min \varphi_n \\ s.t. \\ \quad \sum_{j\in V} x_{ij}^k - \sum_{j\in V} x_{ji}^k = \begin{cases} 1 \text{ if } i=u \\ -1 \text{ if } i=v \\ 0 \text{ otherwise} \end{cases} & \forall k\in\{1\dots n\} \quad \forall i \in V & (5) \\ \quad x_{ij}^k \in \{0,1\} & \forall i,j \in V_k \quad \forall k\in\{1\dots n\} \end{cases}$$

$RRCTr$ is a quadratic 0-1 program problem. It can be solved by several approaches. Among all, tight relaxation can be used to obtain good lower bounds. Recent studies, such as Goemans [Goe97] and Rendl [BT-NEGR00], show the efficiency of semidefinite relaxation for solving quadratic 0-1 programming, namely, Quadratic Knapsack problem, Max Cut problem.... However, we show in [BGL05b] that semidefinite relaxation can only be used for small instances.

3 Polynomial cases

In this section, we identify two cases in which $RRCT$ turns to be polynomially solvable. The first one is based on the basic formulation given in the previous section, and the second one presents a case in which a more efficient algorithm can be used.

3.1 Link between both formulations

Theorem 2 *If the optimal solution of $RRCTr$ problem is composed of n paths between s and t, then this solution is optimal for $RRCT$ problem.*

Proof: Let $\tilde{S}$ be the optimal solution of $RRCTr$ and S the optimal solution of $RRCT$. It is easy to see that $\tilde{S}$ is an upper bound of $RRCT$ since it is a feasible solution of $RRCT$ problem. It is also a lower bound of $RRCT$ since it is a feasible solution of $RRCTr$. □

This theorem provides a simple way to determine whether our instance belongs to the polynomially solvable cases. Since the constraints are simply flow constraints and disjointed linearization constraints, the linearization of the problem induces that the resulting linear program is unimodular, and consequently the binary solution can be obtained in polynomial time by solving the linear relaxation

We define the following algorithm:

1. Solve the relaxed program $RRCTr$ (5). Let P_i denote part of the solution which is a path between s and t in graph G_i and C_i denote the part of the solution which is an independent cycle in graph G_i.

2. Return $(P_i)_{i\in\{1,\dots,n\}}$

3.2 A path and a graph

We showed that *RRCT* problem is NP-complete in the general case. In this section, we consider the case in which G_1 is a simple path and G_2 is a general graph. Since this sub-problem is NP-complete, we consider furthermore that the uninstallation cost of any edge is lower than its weight.

Let us consider now the following algorithm. First, let us define a new function w' on the edges of G_2 as:

- $w'(e) = w(e) + \iota(e)$ if $e \notin X_1$, i.e., $e \notin G_1$;
- $w'(e) = w(e) - \delta(e)$ otherwise.

At this step, let X_2 be the shortest path in G_2 using this new function. Since all the weights are positive, X_2 exists and can be found by the classical Dijkstra algorithm in $O(mn \log n)$ time. It is quite easy to see that solution $(X_1 = G_1, X_2)$ is optimal for the *RRCT* problem. Note that, if the weight function w' is negative, the previous problem may be solved using the Bellman-Ford algorithm in $O(n^3)$. However, this algorithm may not find a solution when there exists an absorbent circuit.

4 Conclusion

In this paper, we have shown that the Robust Routing in Changing Topologies problem is NP-complete and have provided a formulation based on 0-1 quadratic program as well as a combinatorial relaxation. However, some questions remain open. Indeed, the problem is polynomial when the transition cost are equals to zero or infinite. On the other hand, we have seen in Theorem 1 that the problem becomes difficult when the transition costs have a large amplitude. It would be of interest to determine the impact of the transition functions on the complexity of the problem, e.g., when these functions are constant.

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