Graphs with low forwarding index and few extra edges

Frédéric Giroire
frederic.giroire@inria.fr
http://www-sop.inria.fr/members/Frederic.Giroire/

Stéphane Perennes
stephane.perennes@sophia.inria.fr
http ://www-sop.inria.fr/members/Stephane.Perennes/

Presentation of the subject

– We will consider graphs and digraphs.
– Given a (di)graph $G = (V, E)$, the *edge (resp. arc) forwarding index* is defined as the minimum edge (arc) load induced by a set of (fractional) (di)paths connecting all the n^2 couples of vertices. In other words, it is the minimum congestion of the routing of the *ALL-to-ALL* traffic matrix. We will denote it $\gamma(V, E)$. When one assumes uniform traffic demands at a given rate λ, one can prove that whatever be the routing policy the average load of an edge (arc) will be at least $\lambda \cdot \gamma(V, E)$. Moreover there exist routing policies that ensure that the maximum load of an arc will be close $\lambda \cdot \gamma(V, E)$. This makes the edge forwarding index of great practical importance.

We will give the definitions for graphs, the ones for digraphs are analogous. Our aim is to study how the number of edges of a graph impacts on its edge forwarding index. This induces several design problems :

a) Min congested k-subgraph : Given $G = (V, E)$ and $k \in \mathbb{N}$, find $E' \subset E, |E'| \leq k$ such that $\gamma(V, E')$ is minimum. We will denote this minimum by $\gamma^*(V, E, k)$.
b) Min congested (n, e)-graph : Given $n, e \in \mathbb{N}$, find a graph (V, E) with $|V| = n$ vertices and $|E| = e$ edges such that $\gamma(V, E)$ is minimum. We denote this number $\gamma^*(n, e)$ (when $e \leq n - 1$ we take as convention $\gamma^*(n, e) = \infty$).

We intend to study both problems under degree restriction or other constraints (as example classical graph properties like planarity, genus, ...). Those two problems belong to the family of design problems. In both cases, given some constraints, one wishes to construct “good” (or a best) *spanners* (i.e. spanning subgraph) of a given graph. The problem (b) in which one design the graph itself can be seen as spanner design problem for which the original graph G is the complete graph K_n. Our measure of goodness for a graph is its edge forwarding index. This quantity is strongly related to distance properties of the graph. Indeed, a usual naive lower bound on γ is :

$$\gamma(V, E) \geq \frac{\sum_{(u,v)\in V^2} D(u, v)}{|E|} = \frac{\overline{D(G)}|V|^2}{E} = 2|V|\frac{\overline{D(G)}}{\overline{d}(G)}$$

where $D(u, v), \overline{D}(G)$ and $\overline{d}(G)$ denote respectively the distance from u to v, the average distance in the graph G, and the average degree of the graph G. This bound is attained if and only if there exists a shortest paths routing that is balanced on the edges, which is the case for highly symmetric graphs such as cycles, toruses, complete bipartite graphs, and other edge-transitive graphs. When no assumption is made about the graph, optimal solutions are quite easy to construct since they look like stars, as example when $e = n - 1$ it is easy to show that the best graph is simply a star with n nodes and that $\gamma^*(n, n - 1) = 2(n - 1)$. This extends to the case $e = x(n - x), x \in N$, for which complete bipartite graphs $K_{x,n-x}$ are optimal with $\gamma^*(n, x(n - x)) \sim \frac{2(n-k)}{k}(1 + o(1))$.

Project team Mascotte, INRIA, I3S, CNRS, University of Nice Sophia
2004, route des Lucioles – B.P. 93 – F-06902 Sophia Antipolis Cedex – France

The problem gets more complex when one assume that the graph has bounded degree. Then, solving the minimally congested graph problem means finding a rather symmetric spanning (sub)graph with small average distance, and a good (n, e)-graphs should ressemble (Δ, D) *graphs*. The (Δ, D) problem has received considerable attention and is still unsolved (see the references). In this problem, one wishes to construct a graph with the largest number of vertices given a maximum degree Δ and diameter D. One define $N(\Delta, D)$ as this largest number, a simple upper bound on $N(\Delta, D)$ (known as the Moore's bound) is obtained by counting the number of nodes in a tree with degree Δ and depth D. This leads to :

$$N(\Delta, D) \geq 1 + \Delta + \Delta(\Delta - 1) + \cdots + \Delta(\Delta - 1)^{D-1} = 1 + \Delta(\frac{\Delta^D - 1}{\Delta - 2})$$

Graphs with $N'(\Delta, D) \sim \Delta^{D-1}/D$ are known, but their existence is proven using random constructions (namely random regular graphs with degree Δ since one can prove that a such a random graph with N nodes has, with high probability, diameter $(1+o(1)) \log_{\Delta-1}(N)$)). Such graphs are likely to also have the smallest forwarding index among graphs with maximum degree Δ. Notably, deterministic constructions are very far to perform as well as random ones since the best ones only have diameter of order $\log_{\Delta/2}(N)$ instead of $\log_{\Delta}(N)$). So the relationship between (Δ, D) graphs and graphs with small forwarding index and bounded degree has to be explored.

More on Distances and Edge forwarding index

Duality of linear programming implies that the relationship between distance and edge forwarding index is exact. One can compute the edge forwarding index of a graph by solving the next path packing problem. We let $\mathcal{P}$ denote the set of all paths and $\mathcal{P}_{uv}$ denote the set of all paths from u to v.

$$\begin{array}{llll} & Minimize\ \gamma & & \\ \forall P \in \mathcal{P} & w(P) & \geq 0 & \\ \forall e \in E & \sum_{P \in \mathcal{P}, e \in P} w(P) & \leq & \gamma \\ \forall (u,v) \in V^2 & \sum_{P \in \mathcal{P}_{xy}} w(P) & \geq & 1 \end{array}$$

The dual of this problem is the following distance maximization problem. First, we need the following definition : Given a positive length function $l : E \to \mathbb{R}^+$, $l(P) = \sum_{e \in P} l(e), P \in P$ is the length of P. Then the dual of the edge forwarding index problem can then be seen as :

$$\begin{array}{llll} & Maximize\ D_{tot} & & \\ \forall P \in \mathcal{P}_{xy} & l(P) & \geq & l(x,y) \\ & \sum_{e \in E} l(e) & = & |E| \\ & \sum_{(x,y) \in V^2} l(x,y) & = & D_{tot} \end{array}$$

If we denote D^*_{tot} the maximum of the above problem, Farkas's lemma (strong duality) implies that :

$$\gamma(V, E) = D^*_{tot}(V, E)$$

In other words, the edge forwarding index is indeed an average distance on G but taken over a distorted metric. This distorted metric is often a "bad cut" that is a set of edges $[S, V \setminus V]$ that is congested, the dual then set $l(e) = 0, e \in [S, V \setminus S]$ and $l(e) = 0$ otherwise and proves that $\forall S \in 2^V, S \notin \{\emptyset, V\}, \gamma(V, E) \geq \frac{2|S||V \setminus S|}{[S, V \setminus S]}$.

Objectives

The scope is very wide, so the goal is to solve one or several of the following problems. As example for problem (b) :

- Determine exactly the edge forwarding index of a graph with n vertices and e edges. We currently know the general rough behavior, but details have to be worked on.
- Determine minimally congested graphs given a maximum degree constraint. This will mean (re)-using ideas developed when dealing with the well-studied (Δ, D) problem. Note that in the (Δ, D) problem one deal with graphs having $\frac{\Delta|V|}{2}$ edges, so it does not cover the case of graphs with n nodes and $n+k$ edges. So determining the behavior of $\gamma(n, e)$ when $e = n + k$ when k is a fixed constant will be particular interest (in other words how decrease the load of the edges when we add a few edges to a tree).
- Determine the minimum value of $\gamma(V, E)$ when $G = (V, E)$ is planar, and when G is planar with max degree 3, 4. This part will probably rely on the Tarjan's isoperimetric inequality that state that planar graphs do have congested cuts.
- Consider possible extensions to the case of graphs with bounded genus or with excluded minor.

Similar questions can be asked for problem (a) in which the solution must be included into an input graph G. Then one need to either provide algorithms (finding a good spanner) or to prove bounds that would be expressed from parameters of the input graph G. As example one can wonder what is the forwarding index (i.e. the load of the edges) when one remove half the edges. Should this load roughly double ?

Bibliography

Francesc comellas from the UPC in Barcellona maintains a web page about the (Δ, D) problem, with a lot of bibliography : `http://www-ma4.upc.es/~comellas/delta-d/taula_delta_d.html`, see also his personal page `http://maite71.upc.es/%7Ecomellas/`

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Information

Hosting team	Coati-Mascotte Team, joint team of the CNRS/INRIA/I3S-University of Nice.
Advisors	Stéphane Perennes (CNRS), Frédéric Giroire (CNRS).
Location	INRIA Sophia Antipolis
Duration	2-6 months
Contact	stephane.perennes@inria.fr

Prerequisites :

Taste and competencies in Discrete mathematics, Graph Theory and Algorithmics.