Interval Analysis in Coq

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Interval arithmetic

Tool used to handle inaccuracies in computations.

$$-\pi * \sqrt{2} \approx -3.14 * 1.41 = -4.4274$$

$$[-3.15, -3.14] * [1.41, 1.42] = [-4.473, -4.4274]$$

If we know the bounds on the input data we can compute the bounds on the result.

Interval arithmetic, more formally

Definition

interval := closed, bounded, connected, nonempty subset of $\mathbb{R}$

$$x := [\underline{x}, \overline{x}] = \{\tilde{x} \in \mathbb{R} \mid \underline{x} \leq \tilde{x} \leq \overline{x}\}, \quad \text{where } \underline{x}, \overline{x} \in \mathbb{R}, \underline{x} \leq \overline{x}$$

Notation $\mathbb{IR}$ – set of intervals

Classification

- thin interval $\underline{x} = \overline{x}$
- thick interval $\underline{x} < \overline{x}$

Associated quantities

midpoint $x_c := \frac{\underline{x}+\overline{x}}{2}$ radius $\Delta_x := \frac{\overline{x}-\underline{x}}{2}$

$$x = [x_c - \Delta_x, x_c + \Delta_x]$$

Basic interval operations

$$x + z := \Box\{\tilde{x} + \tilde{z} \mid \tilde{x} \in x, \tilde{z} \in z\}$$

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$$= [\underline{x} + \underline{z}, \overline{x} + \overline{z}]$$

$$-x := \Box\{-\tilde{x} \mid \tilde{x} \in x\} = \{-\tilde{x} \mid \tilde{x} \in x\} = [-\overline{x}, -\underline{x}]$$

$$xz := \Box\{\tilde{x}\tilde{z} \mid \tilde{x} \in x, \tilde{z} \in z\} = \{\tilde{x}\tilde{z} \mid \tilde{x} \in x, \tilde{z} \in z\} =$$

$$= [\min(\underline{xz}, \underline{x}\overline{z}, \overline{x}\underline{z}, \overline{xz}), \max(\underline{xz}, \underline{x}\overline{z}, \overline{x}\underline{z}, \overline{xz})]$$

Issues

Principle:
Correctness is more important than accuracy.

$$\pi - \pi = 0$$

$$[3.14, 3.15] - [3.14, 3.15] = [-0.01, 0.01]$$

Techniques to increase accuracy (avoid decorrelation)

- e.g., bisection

Rounded interval arithmetic

Usage

- in theory: $[\underline{x}, \overline{x}]$ with $\underline{x}, \overline{x} \in \mathbb{R}$
- in practice: $[\underline{x}, \overline{x}]$ with $\underline{x}, \overline{x} \in M$,
  where M is a machine representable subset of $\mathbb{R}$

Outward rounding

$$\Diamond x := [\nabla \underline{x}, \Delta \overline{x}]$$

$$x \subseteq \Diamond x$$

$$x +^{\Diamond} z = \Diamond(x + z)$$

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Example

$$[-3.15, -3.14] * [1.41, 1.42] = [-4.473, -4.4274]$$

M : decimal numbers with 2 digits

$$[-3.15, -3.14] *^{\Diamond} [1.41, 1.42] = [-4.48, -4.42]$$

Issues with rounded arithmetic

Ideal arithmetic

$$x + z = \{\tilde{x} + \tilde{z} \mid \tilde{x} \in x, \tilde{z} \in z\} = [\underline{x} + \underline{z}, \overline{x} + \overline{z}]$$

Rounded arithmetic

$$x +^{\Diamond} z = \Diamond[\underline{x} + \underline{z}, \overline{x} + \overline{z}]$$

$$\{\tilde{x} + \tilde{z} \mid \tilde{x} \in x, \tilde{z} \in z\} \subseteq x +^{\Diamond} z$$

Interval arithmetic in proof assistants

Interval arithmetic in proof assistants

Nature of interval methods

- interval arithmetic was born to safely deal with errors

Usage

- interval arithmetic appears in critical software
- certified computation

Formalizations

- Coq, PVS, Isabelle
- focus on computation efficiency and automation of techniques

Computation driven formalizations

- basic operations
- elementary functions
- techniques to increase accuracy
- rounded interval arithmetic
- automated procedures to compute and prove bounds for expressions
- computations by external tools

Formalizing more “theoretical” results

Formalizing more “theoretical” results

- solving systems of linear equations with interval coefficients

Exercise

Consider the following system:

$$\begin{cases} [1,2]x_1 & + & [2,4]x_2 & = [-1,1] \\ [2,4]x_1 & + & [1,2]x_2 & = [1,2] \end{cases}$$

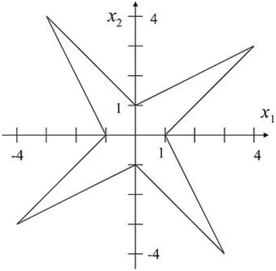
Find a box that contains all pairs $(x_1, x_2) \in \mathbb{R}^2$ that satisfy the equations for some choice of coefficients in their respective intervals.

- correctness of methods for solving these systems is based on more involved theoretical results
- application: robot movement

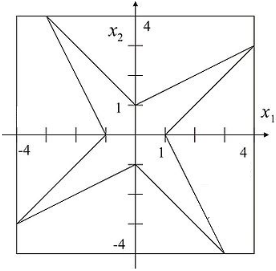
Two steps:

1. checking regularity of the associated interval matrix
2. computing bounds of the solution set

exact solution



bounds for the solution set



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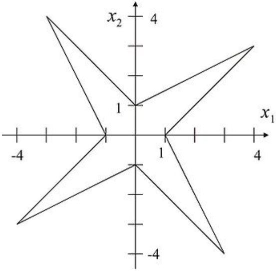

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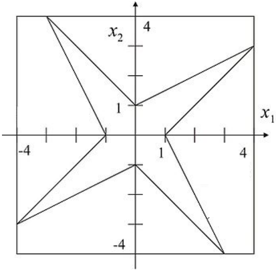

Interval matrices

Definition

$$A = [A_{ij}]_{m\times n},\ \ A_{ij} \in \mathbb{IR}.$$

Characterization

$$A = \{\tilde{A} \in M(\mathbb{R})_{m\times n} \mid \tilde{A}_{ij} \in A_{ij}, i = 1, \ldots, m, j = 1, \ldots, n\}.$$

Associated real matrices

$$\underline{A} := [\underline{A}_{ij}] \qquad \overline{A} := [\overline{A}_{ij}]$$

$$A_c := [(A_{ij})_c] \qquad \Delta_A := [\Delta_{A_{ij}}]$$

Operations on interval matrices

Addition

$$A + B := \square\{\tilde{A} + \tilde{B} \mid \tilde{A} \in A, \tilde{B} \in B\}$$

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$$(A + B)_{ij} = A_{ij} + B_{ij}$$

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$$(A + B)_{ij} = A_{ij} + B_{ij}$$

Multiplication

$$AB = \square\{\tilde{A}\tilde{B} \mid \tilde{A} \in A, \tilde{B} \in B\} \neq \{\tilde{A}\tilde{B} \mid \tilde{A} \in A, \tilde{B} \in B\}$$

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$

Special case: multiplication by a scalar vector

$$A\tilde{x} = \{\tilde{A}\tilde{x} \mid \tilde{A} \in A\}$$

Regularity of interval matrices

An interval matrix A is called regular iff $\forall \tilde{A} \in A, \det \tilde{A} \neq 0$

and it is called singular otherwise ($\exists \tilde{A}, \tilde{A} \in A \wedge \det \tilde{A} = 0$).

! Notice the classical nature of the concepts we manipulate.

Systems of linear interval equations

A system of linear interval equations with coefficient matrix $A \in M(\mathbb{IR})_{m\times n}$ and right-hand side $b \in \mathbb{IR}^m$ is defined as the family of linear systems of equations

$$\tilde{A}\tilde{x} = \tilde{b} \text{ with } \tilde{A} \in A, \tilde{b} \in b$$

The *solutions set* of such a system is given by:

$$\Sigma(A, b) := \{\tilde{x} \in \mathbb{R}^n \mid \exists \tilde{A} \in A, \exists \tilde{b} \in b \text{ such that } \tilde{A}\tilde{x} = \tilde{b}\}$$

Proof example

Theorem

$$\Sigma(A,b) = \{\tilde{x} \in \mathbb{R}^n \mid A\tilde{x} \cap b \neq \emptyset\}$$

Proof excerpt.

We show: $\{\tilde{x} \in \mathbb{R}^n \mid A\tilde{x} \cap b \neq \emptyset\} \subseteq \Sigma(A,b)$.

Consider $\tilde{x}$ such that $A\tilde{x} \cap b \neq \emptyset$.

Then $A\tilde{x} \cap b$ contains some $\tilde{b} \in \mathbb{R}^m$.

Clearly $\tilde{b} \in b$.

Also, $\tilde{b} \in A\tilde{x}$ and by relation (1), $\tilde{b} = \tilde{A}\tilde{x}$ for some $\tilde{A} \in A$.

Therefore $\tilde{x} \in \Sigma(A,b)$. □

$$A\tilde{x} = \{\tilde{A}\tilde{x} \mid \tilde{A} \in A\} \tag{1}$$

Setting up the formalization

We need to talk about

- real numbers
- matrices

We use

- Coq standard library Reals
- SSReflect library matrix

Mix SSReflect and standard Coq !

Mix SSREFLECT and COQ

in SSREFLECT

- types with decidable equality and a choice operator
- hierarchy of algebraic structures
- abstract matrices, but operations when elements are from a ring

in COQ's Reals library

- axiom of trichotomy $\Rightarrow$ decidable equality

```
Axiom total_order_T : ∀ r1 r2 : R, {r1 < r2} + {r1 = r2} + {r1 > r2}.
```

- choice operator by choice and extensionality axioms (for now)
- ring structure

Definition

$x := [\underline{x}, \overline{x}] = \{\tilde{x} \in \mathbb{R} \mid \underline{x} \leq \tilde{x} \leq \overline{x}\}, \quad$ where $\underline{x}, \overline{x} \in \mathbb{R}, \underline{x} \leq \overline{x}$

```
Structure IR: Type := ClosedInt
  { inf: R ; sup: R ; leq_proof: inf ≤_b sup }.
```

Intervals as sets

- coerce IR to R $\rightarrow$ bool

Equality of intervals

```
Lemma eq_intervalP :
  ∀ x z : IR, x = z ↔ inf x = inf z ∧ sup x = sup z.
```

leq_proof: inf $\leq_b$ sup

```
Lemma Rle_dec: ∀ r1 r2, {r1 <= r2} + {~ r1 <= r2}.

Definition Rleb r1 r2 :=
  match (Rle_dec r1 r2) with
    |left _ ⇒ true
    |right _ ⇒ false
  end.
```

inf $\leq_b$ sup $\rightsquigarrow$ Rleb inf sup $\rightsquigarrow$ is_true (Rleb inf sup) $\rightsquigarrow$

$\rightsquigarrow$ Rleb inf sup = true

Boolean equality is decidable and therefore proof irrelevant.

Interval properties

$$\{\tilde{x} + \tilde{z} \mid \tilde{x} \in x, \tilde{z} \in z\} = [\underline{x} + \underline{z}, \overline{x} + \overline{z}]$$

Interval addition

- associative
- commutative
- has $[0, 0]$ as neutral element

$\Rightarrow$ intervals with addition form a monoid

good news for work with big operators!

Interval matrices

- use SSREFLECT library
- vectors are column matrices
- redefine operations on matrices as intervals do not have a ring structure

```
Definition mmul_i (A: 'M[IR]_(m, n)) (x: 'M[IR]_(n, 1)) :=
    \col_i \big[add_i / 0 ]_j mul_i (A i j) (x j).
```

- prove specific properties

$$A\tilde{x} = \{\tilde{A}\tilde{x} \mid \tilde{A} \in A\}$$

- associated real matrices

```
Definition minf (A: 'M[R]_(m, n)) := \matrix_(i, j) inf (A i j).
```

Results on real matrices

- norm for real matrices
- properties for symmetric and positive definite matrices
- eigenvalues for real matrices
  - Rayleigh quotients
- spectral radius
  - Perron Frobenius theorem

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eigenvalues for real matrices:

- roots of the characteristic polynomial
- they can be complex
- Rayleigh quotient: $\frac{x^T A x}{x^T x}$, $x \neq 0$, $A -$ symmetric

$$\forall x \in \mathbb{R}^n, x \neq 0, \lambda_{\min}(A) \leq \frac{x^T A x}{x^T x} \leq \lambda_{\max}(A)$$

spectral radius: $\rho(A) = \max\{|\lambda(A)|\}$

Theorem (Perron Frobenius)

If $A \in \mathbb{R}^{n \times n}$ is nonnegative then the spectral radius $\rho(A)$ is an eigenvalue of A, and there is a real, nonnegative vector $x \neq 0$ with $Ax = \rho(A)x$.

Formalized criteria of regularity

Criterion

A is regular if and only if $\forall \tilde{x} \in \mathbb{R}^n, 0 \in A\tilde{x} \Rightarrow \tilde{x} = 0$.

Criterion

A is regular if and only if $\forall \tilde{x} \in \mathbb{R}^n, |A_c\tilde{x}| \leq \Delta_A|\tilde{x}| \Rightarrow \tilde{x} = 0$.

Criterion (using positive definiteness)

If the matrix $(A_c^T A_c - \|\Delta_A^T \Delta_A\| I)$ is positive definite for some consistent matrix norm $\|\cdot\|$, then A is regular.

Criterion (using the midpoint inverse)

If the following inequality holds $\rho(|I - RA_c| + |R|\Delta_A) < 1$ for an arbitrary matrix R, then A is regular.

Criterion (using eigenvalues)

If the inequality $\lambda_{max}(\Delta_A^T \Delta_A) < \lambda_{min}(A_c^T A_c)$ holds, then A is regular.

How far from the real world

- adapt results for rounded rounded arithmetic
- treat methods for bounding the solution set
- finish proving the admitted results

Interesting References

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