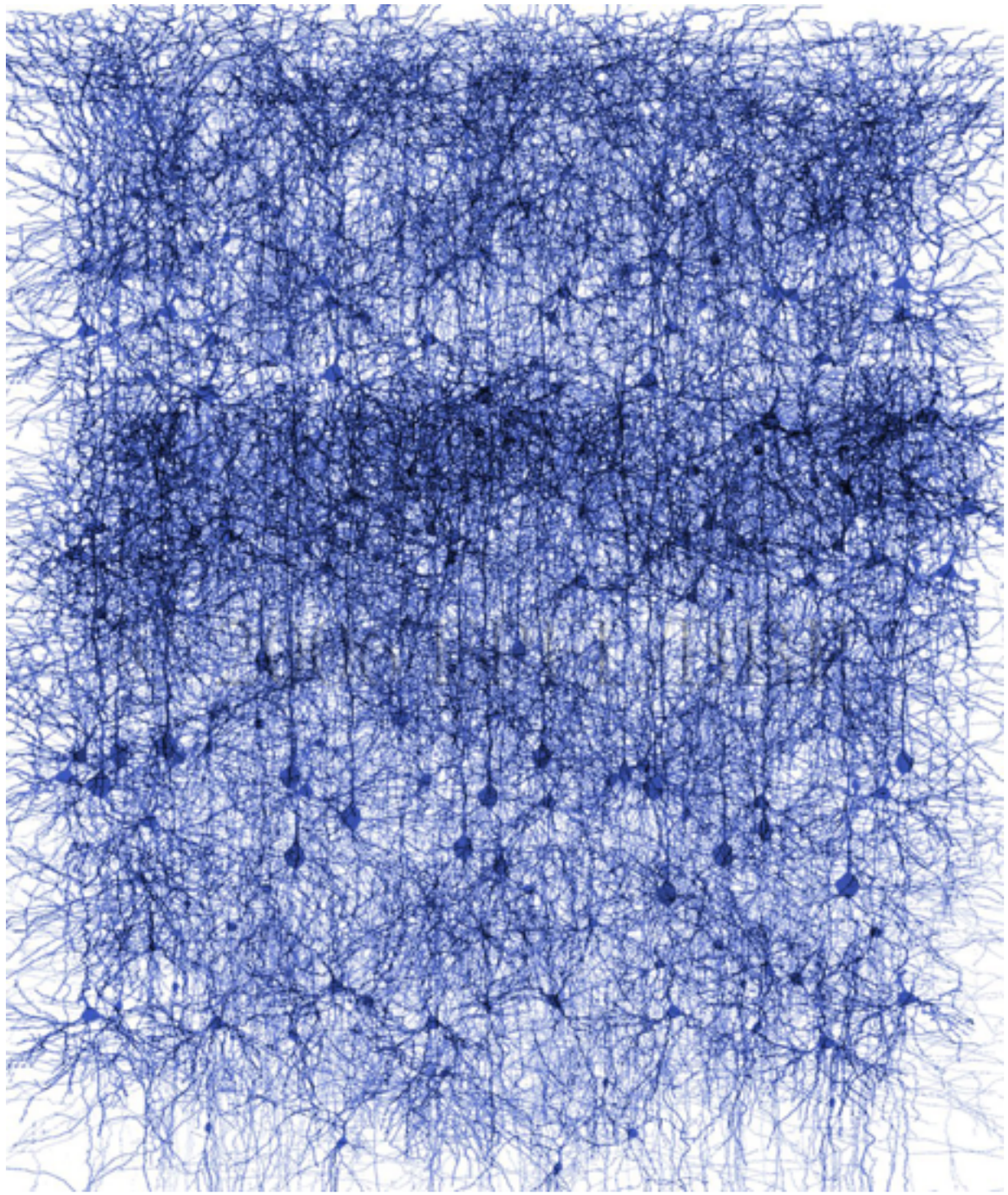
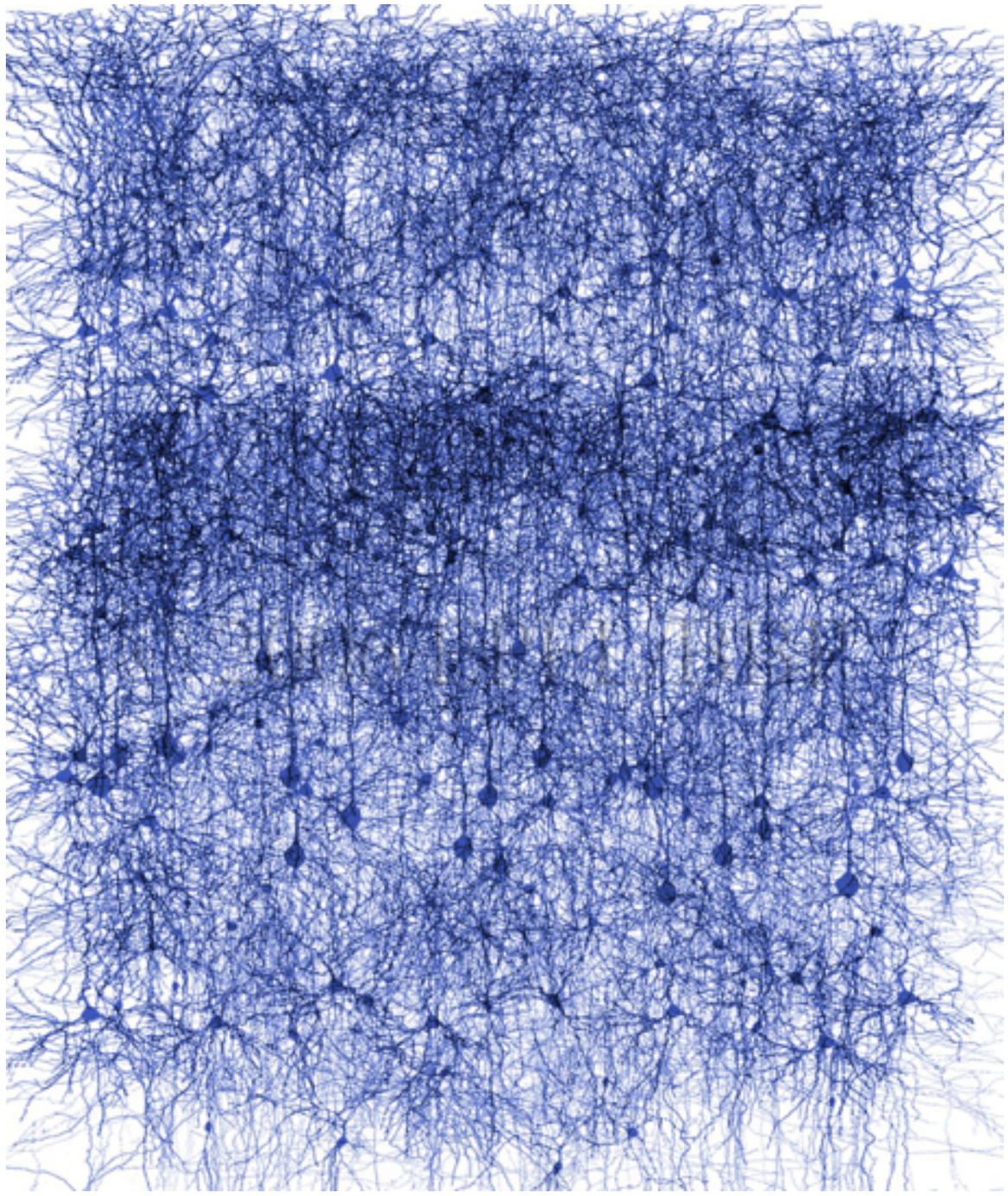
Finite size effects in neural networks

Michael Buice, Carson Chow



mean field theory

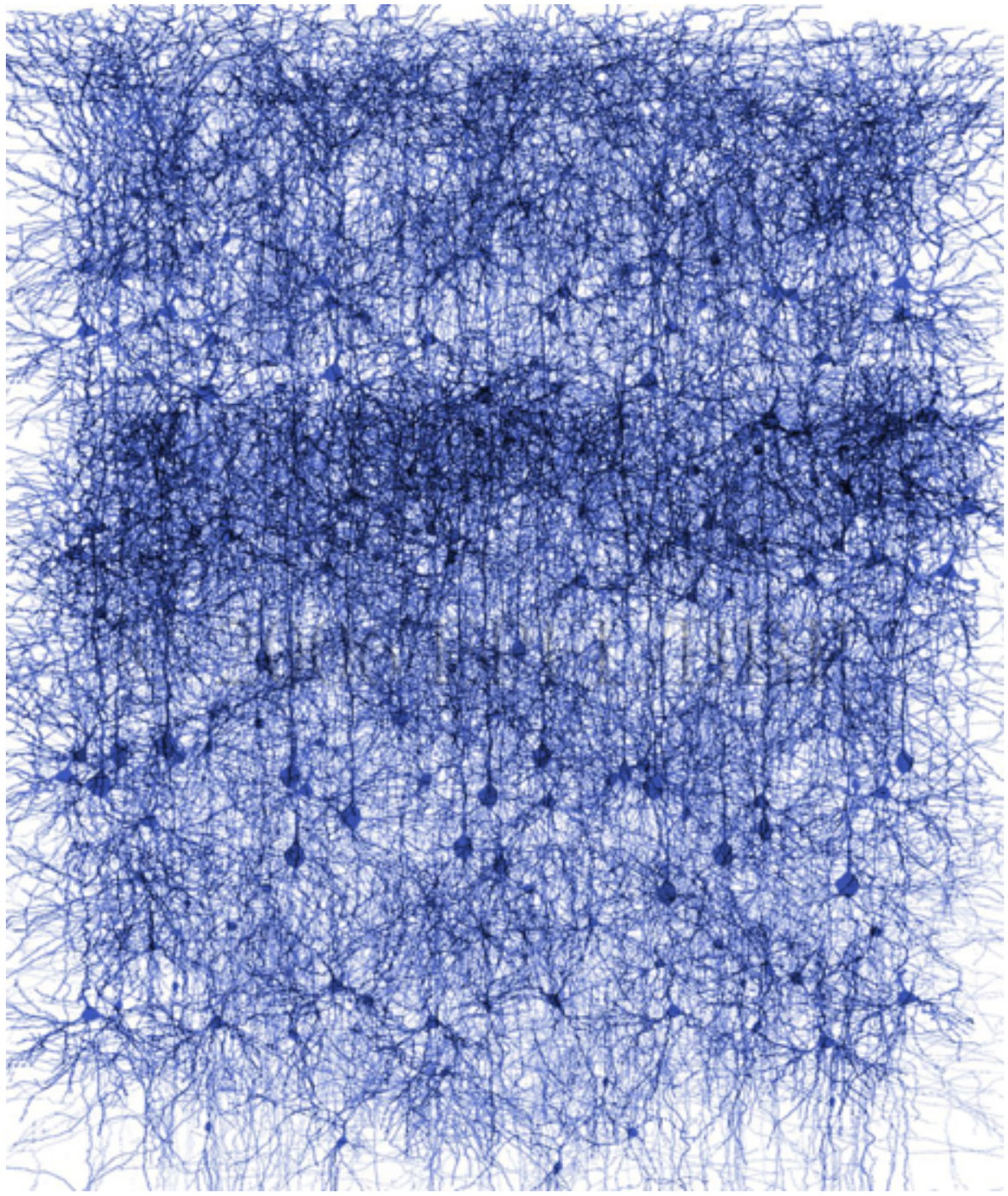
$$N \to \infty$$



mean field theory

$$N \to \infty$$

What about finite N?

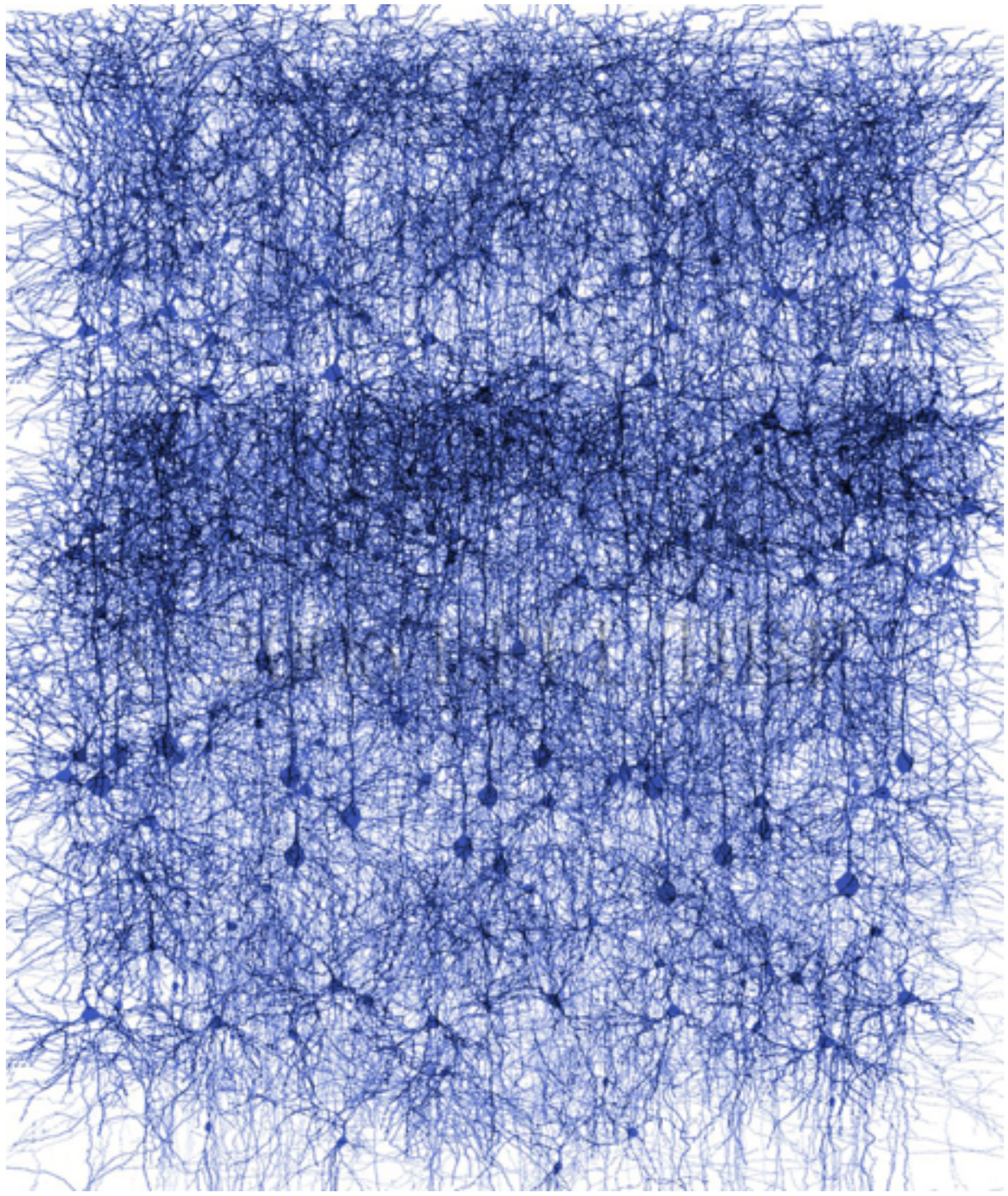


mean field theory

$$N \to \infty$$

What about finite N?

Kinetic Theory



mean field theory

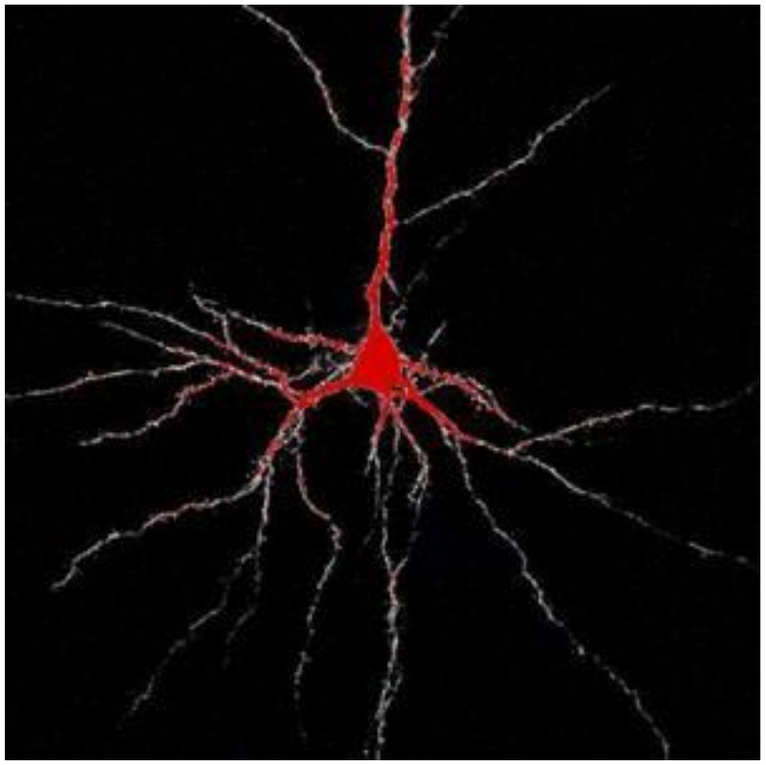
$$N \to \infty$$

What about finite N?

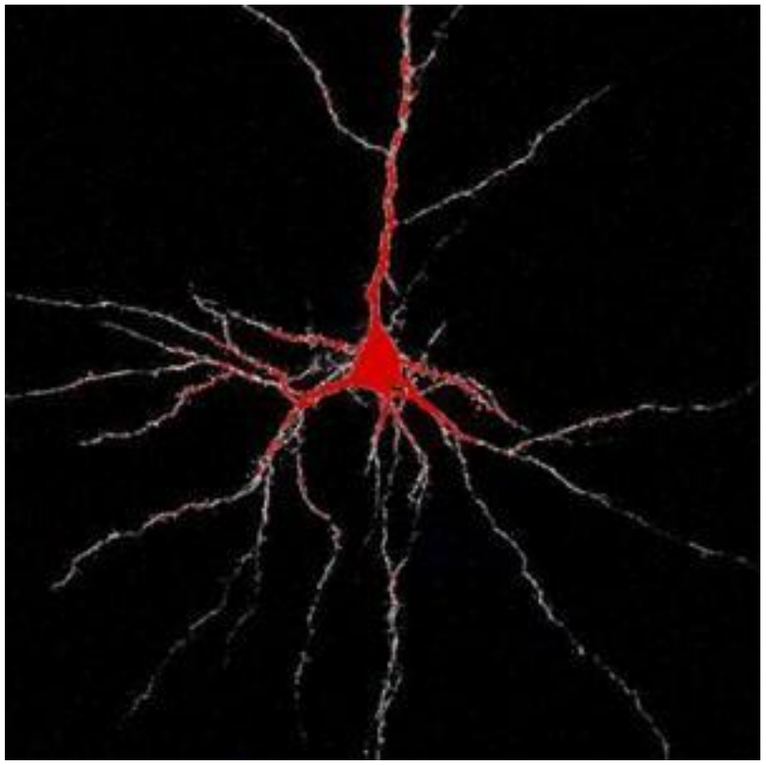
Kinetic Theory

microscopic $\to$ macroscopic

Neuron



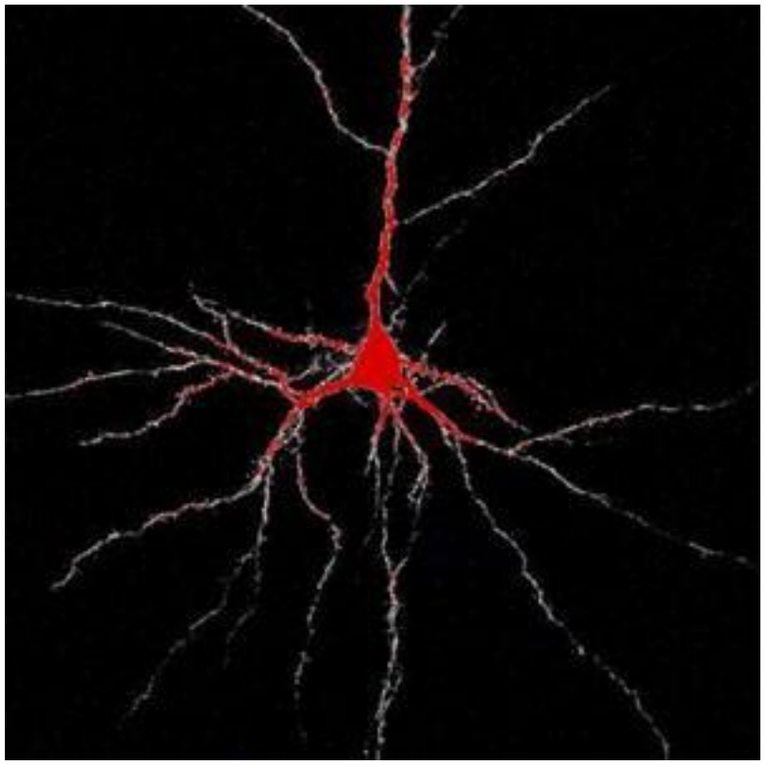
Neuron



$$C\frac{dV}{dt} = -\sum_{r=1}^{n} g_r(x_r)(V - v_r)$$

$$\tau_r \frac{dx_r}{dt} = f(x, V) - x_r$$

Neuron



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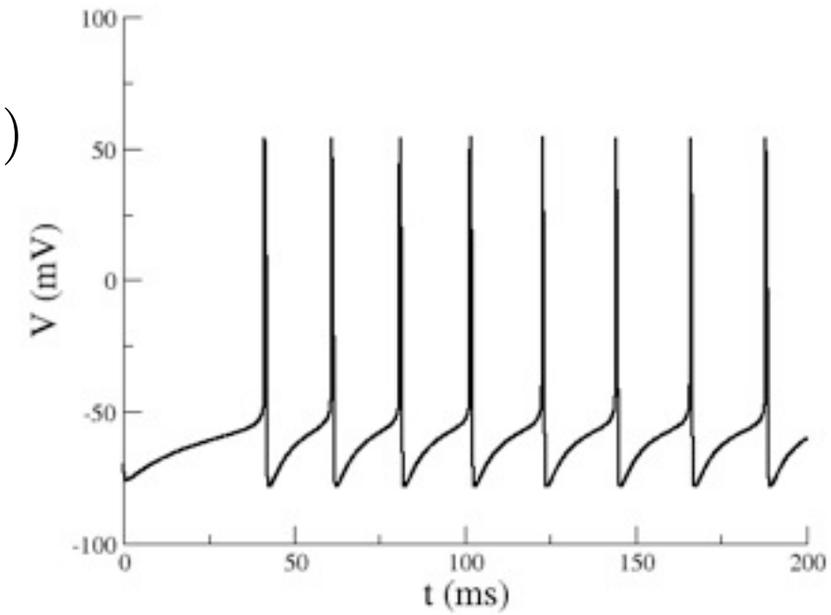
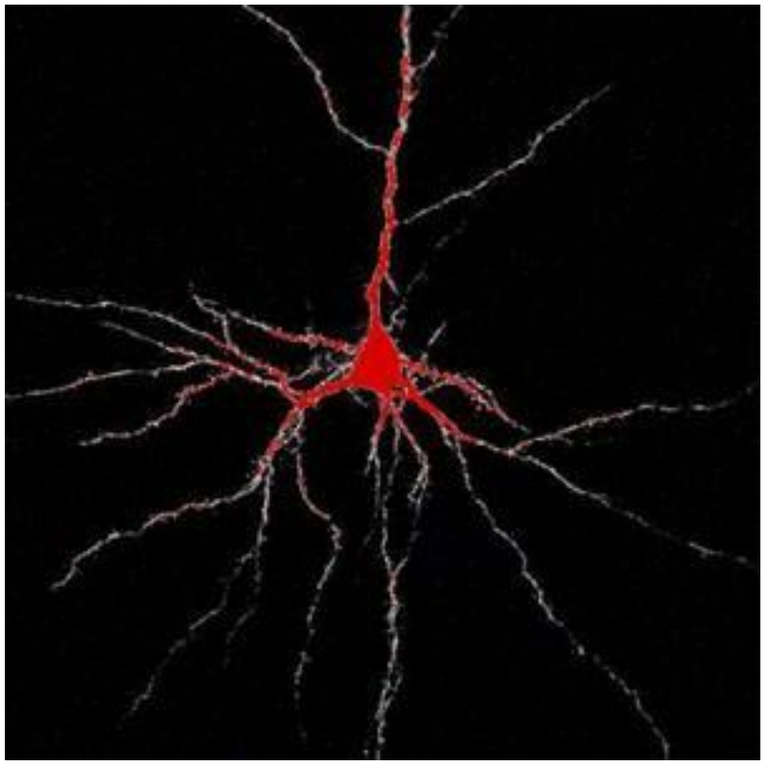


Network



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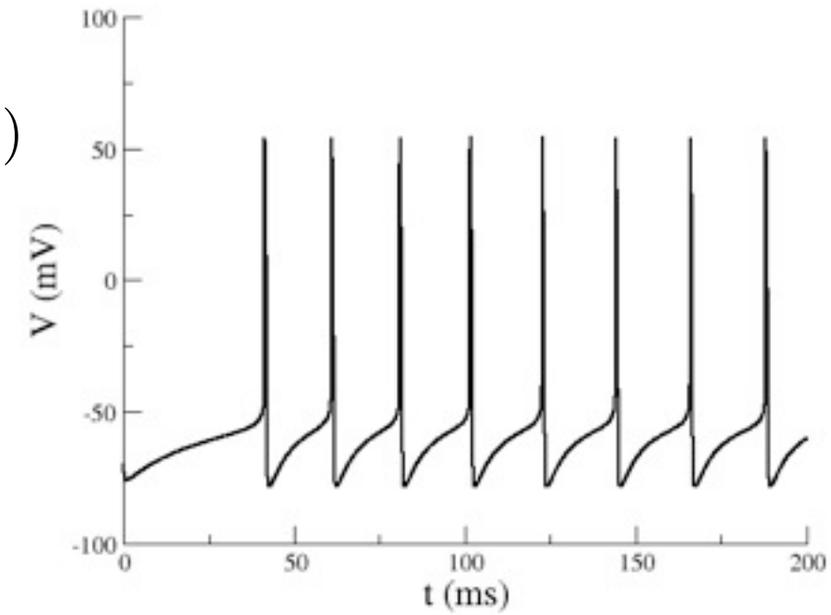

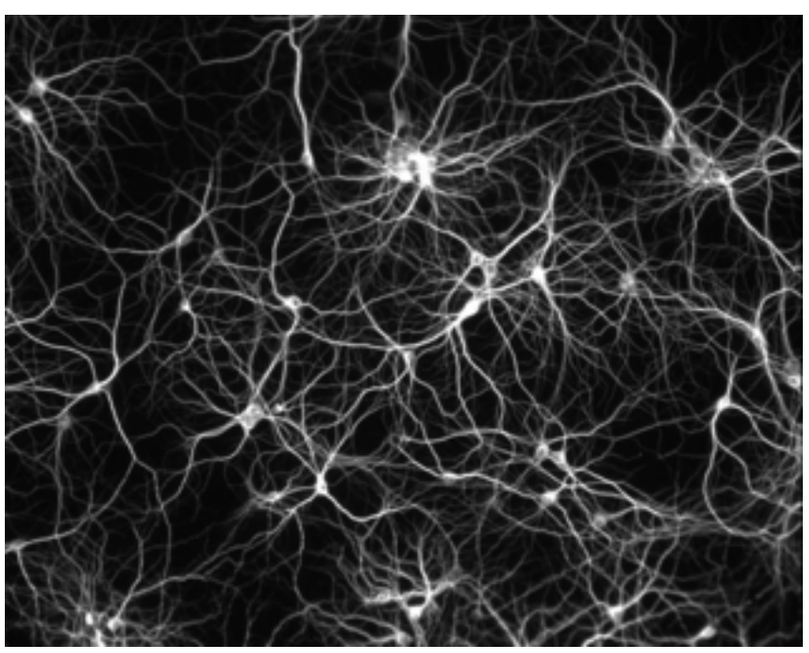

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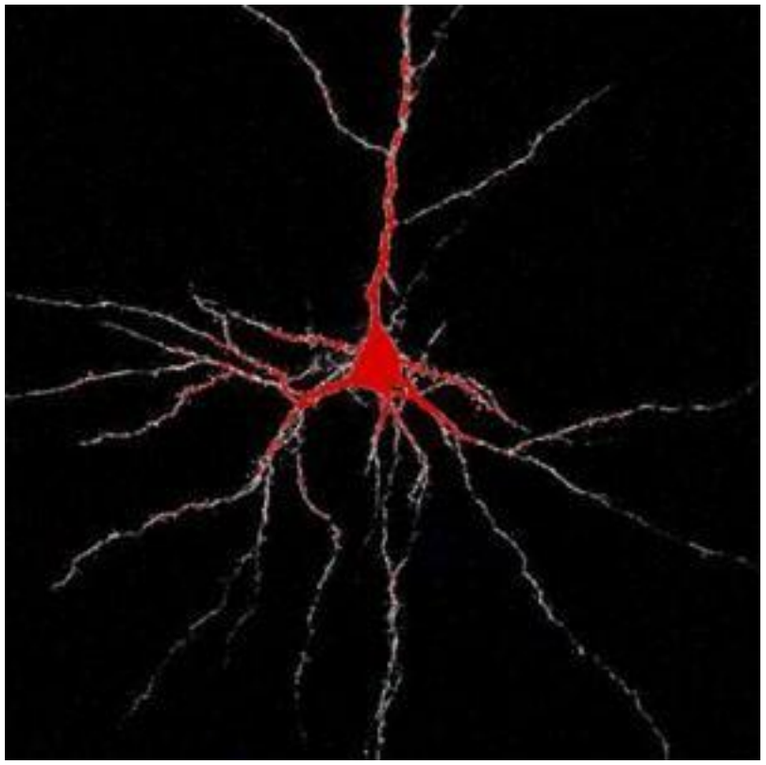

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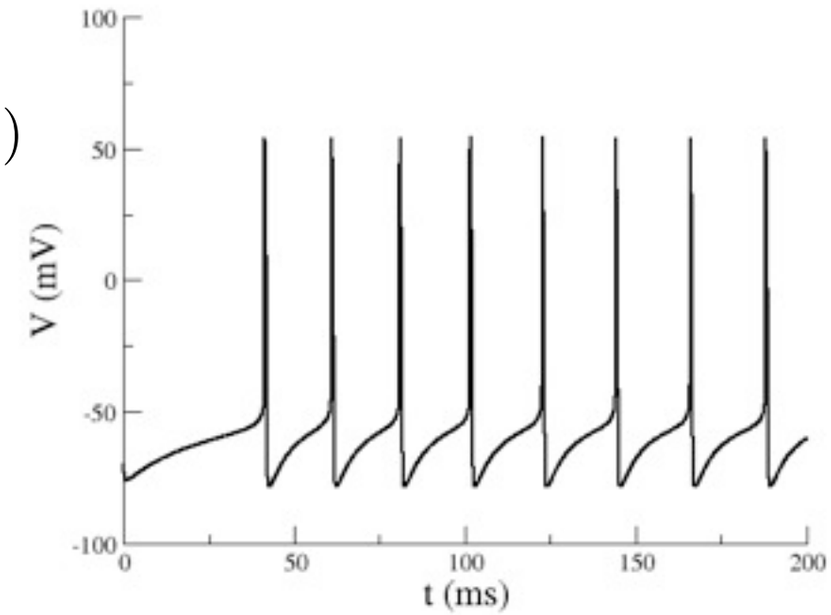
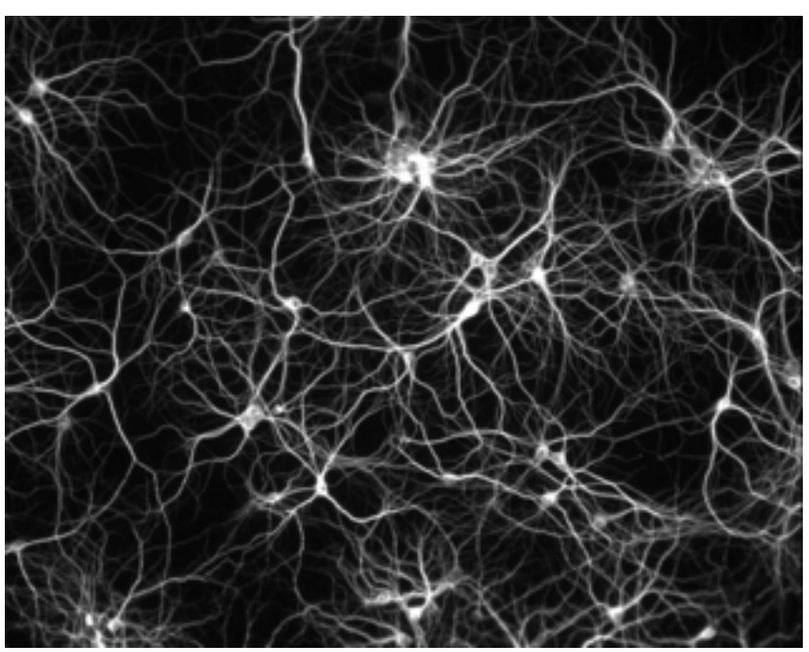
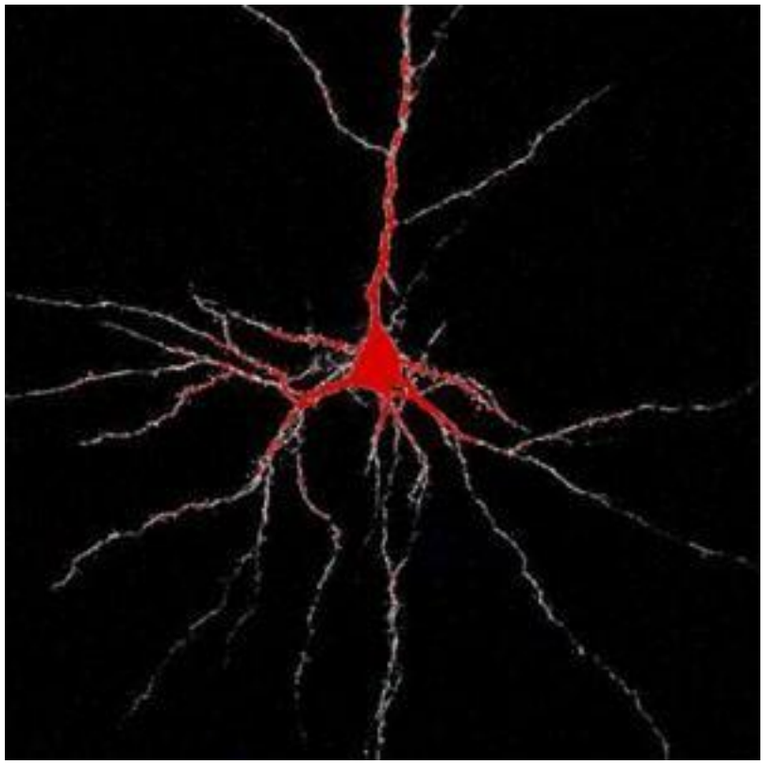




$$C\frac{dV_i}{dt} = -\sum_{r=1}^{n} g_r(x_i^r)(V_i - v_r) + \sum_{j=i}^{N} g_{ij} s_j(t)$$

Network



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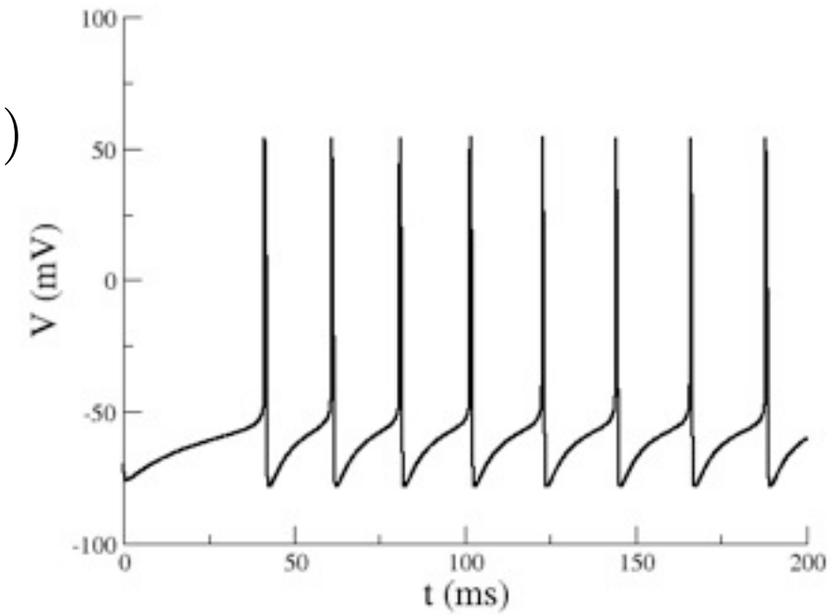

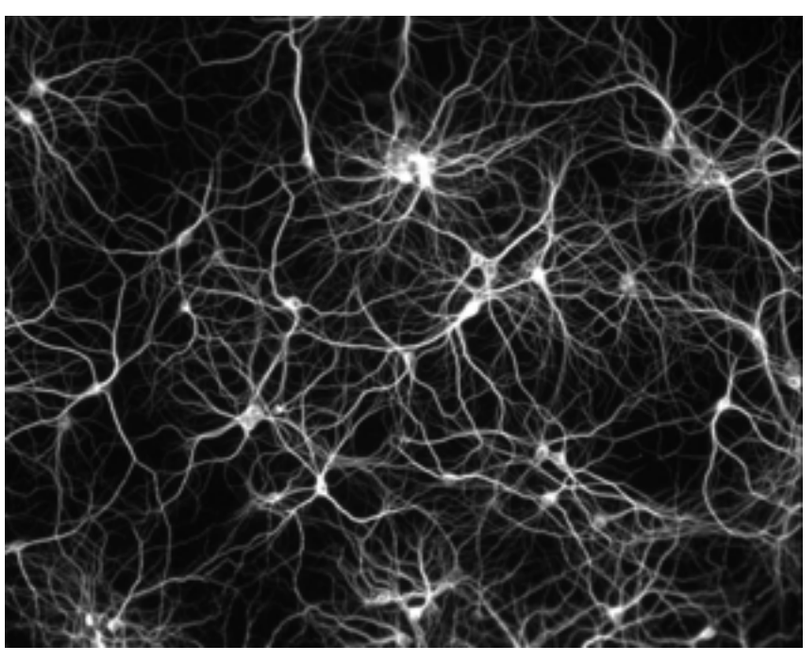

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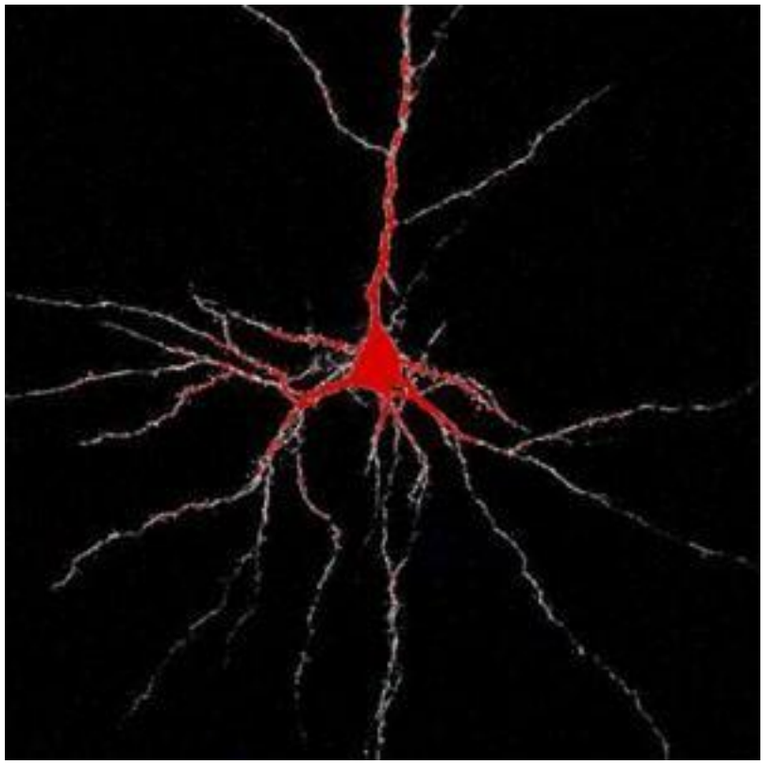

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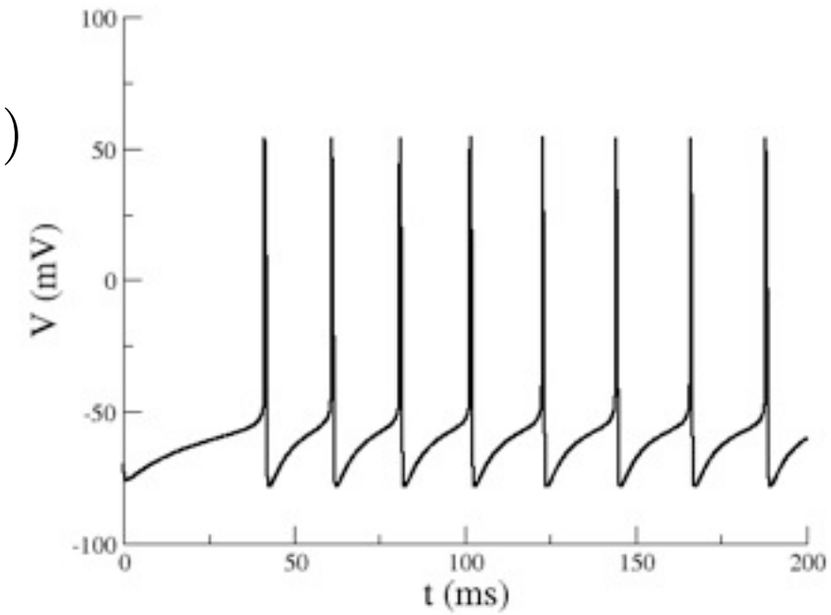

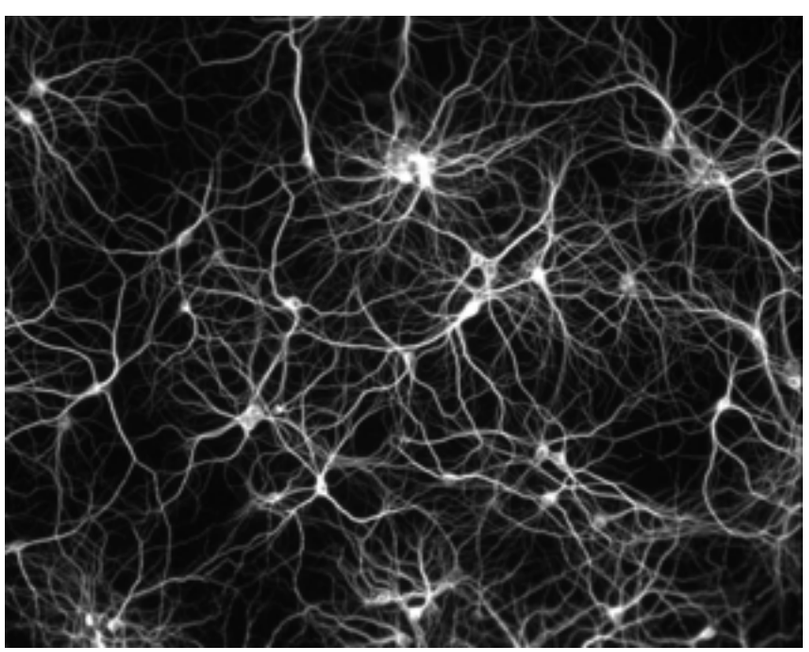

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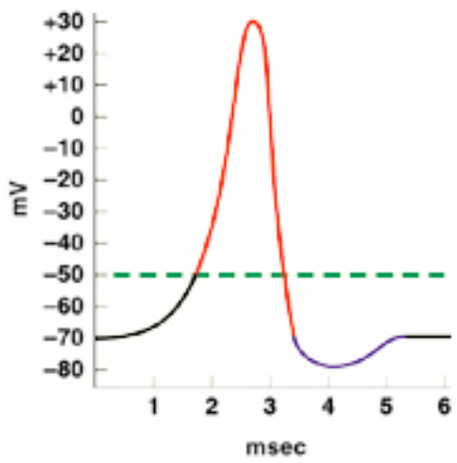
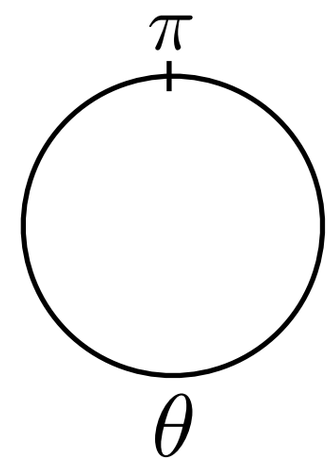
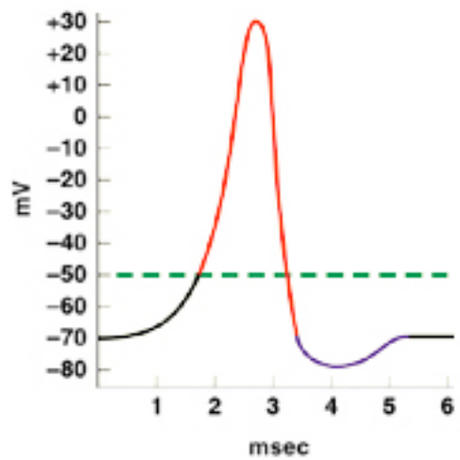
Calculate moments

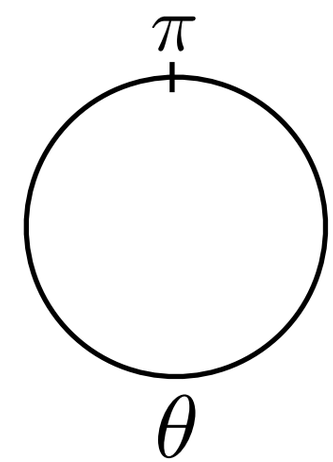
Phase models





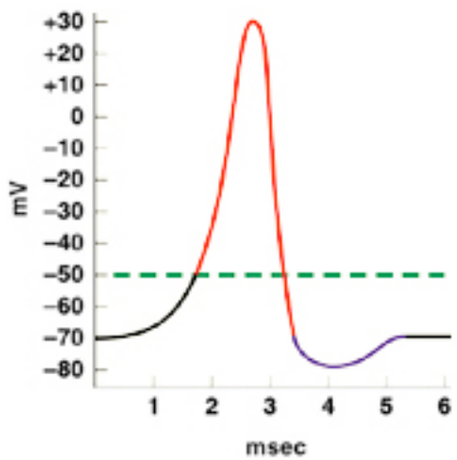
Phase models

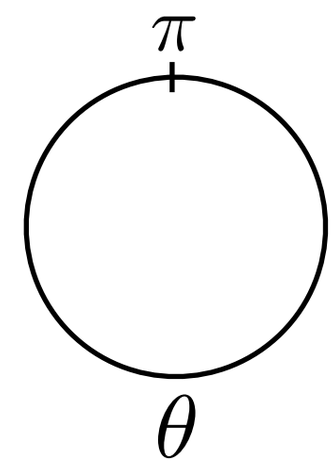




$$\frac{dv}{dt} = I + v^2 \quad \xrightarrow[v = \tan(\theta/2)]{} \quad \frac{d\theta}{dt} = 1 - \cos\theta + I(1 + \cos\theta)$$

Phase models





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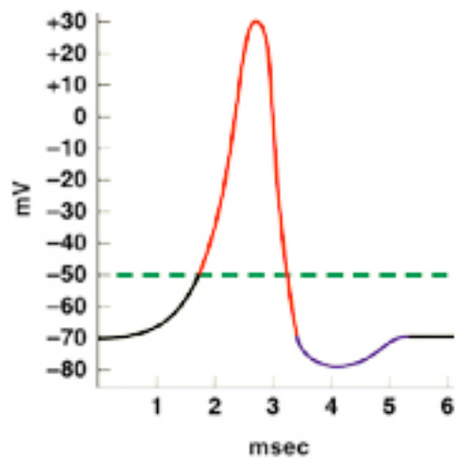
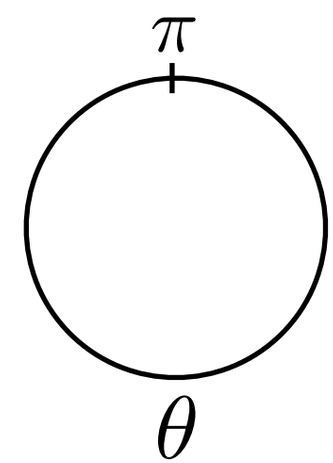
Theta model

Phase models





$$\frac{dv}{dt} = I + v^2 \xrightarrow[v = \tan(\theta/2)]{} \frac{d\theta}{dt} = 1 - \cos\theta + I(1 + \cos\theta)$$

Theta model

Simple phase model

$$\frac{d\theta}{dt} = I$$

Synaptic coupling

$$\dot{\theta}_i = f_i(\theta) + \alpha_i u(t)$$

$$\dot{u}_i + \beta u_i = \frac{\beta}{N} \sum_j \delta(t - t_j^s)$$

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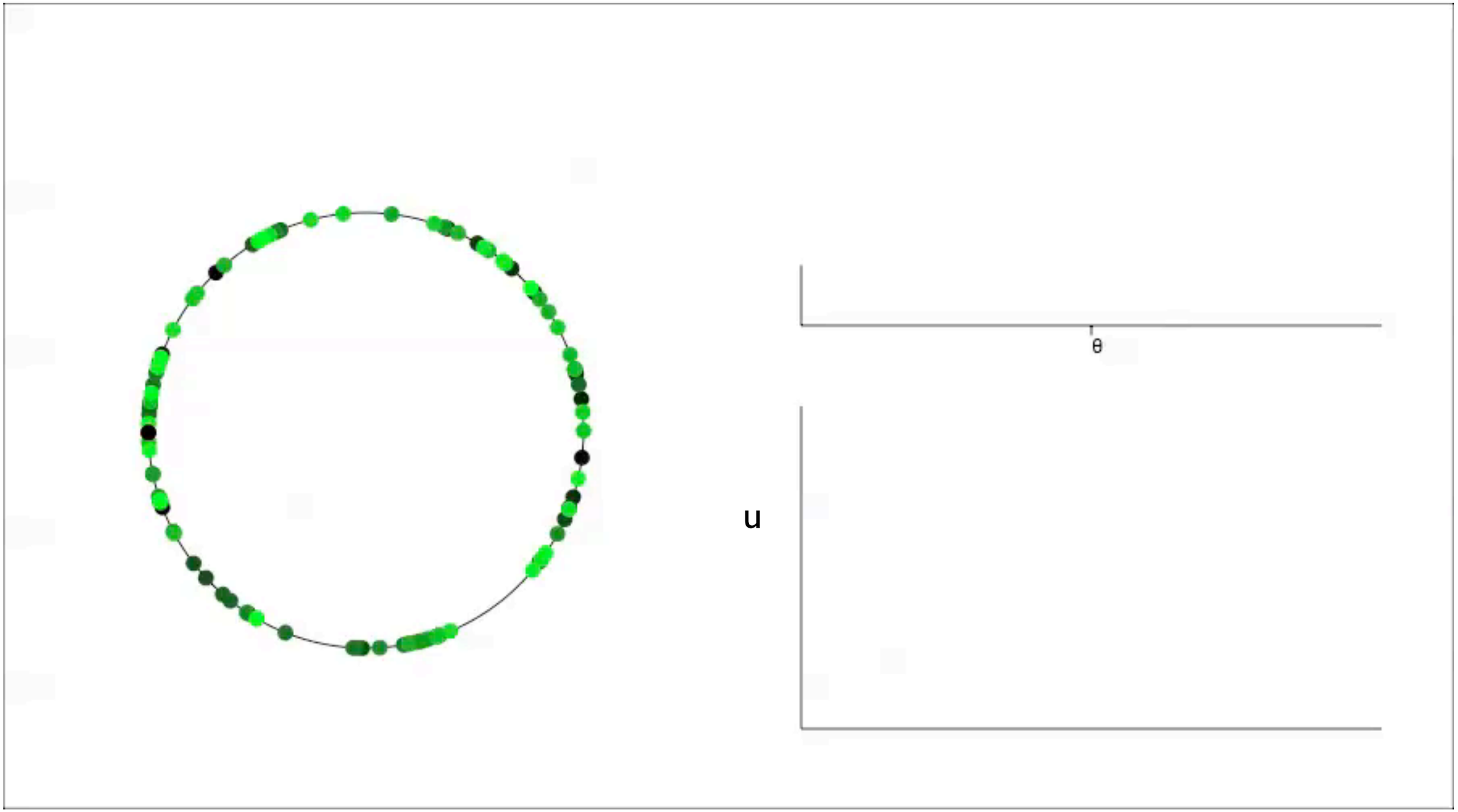
spike times of neuron j

Finite size effects



Three prongs of kinetic theory

Liouville

Three prongs of kinetic theory

Liouville

Klimontovich

Three prongs of kinetic theory

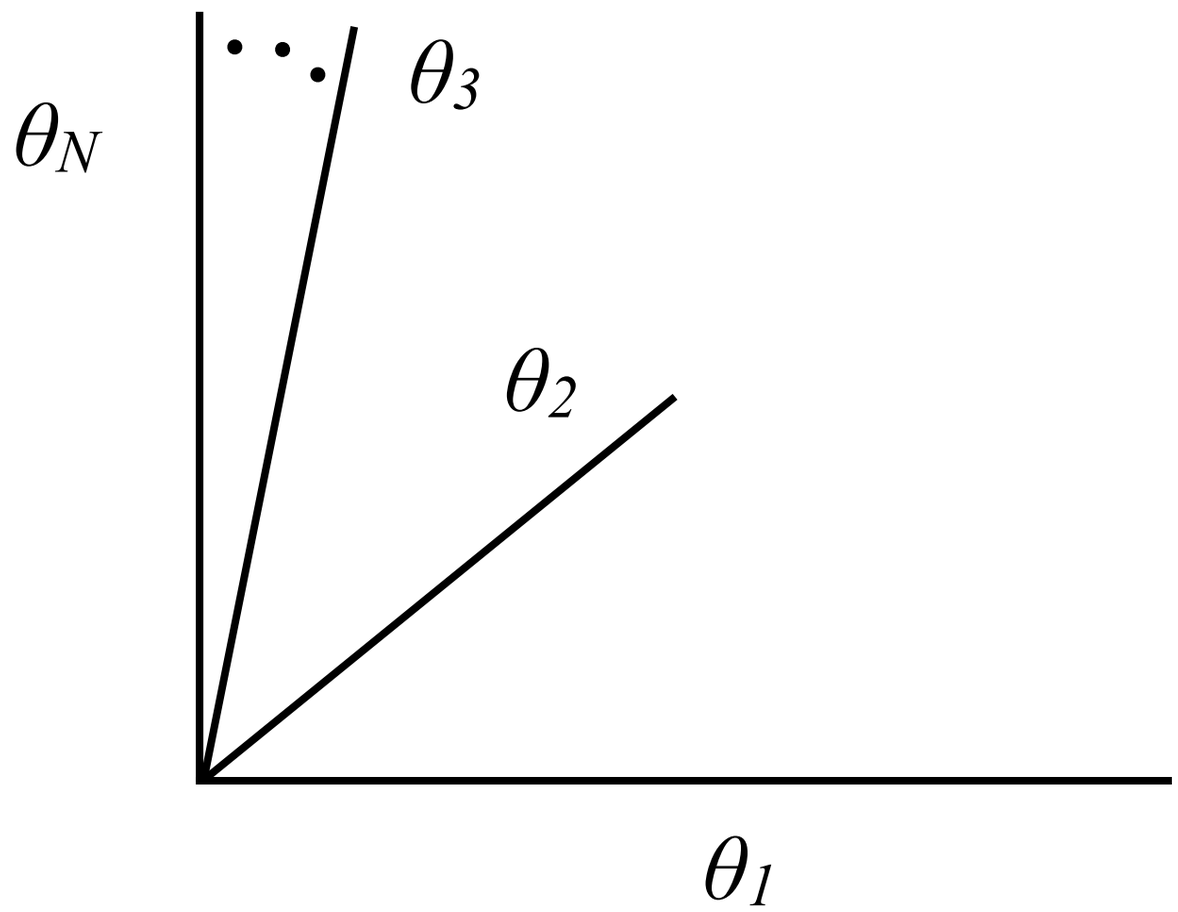
Density functional

Liouville

Klimontovich

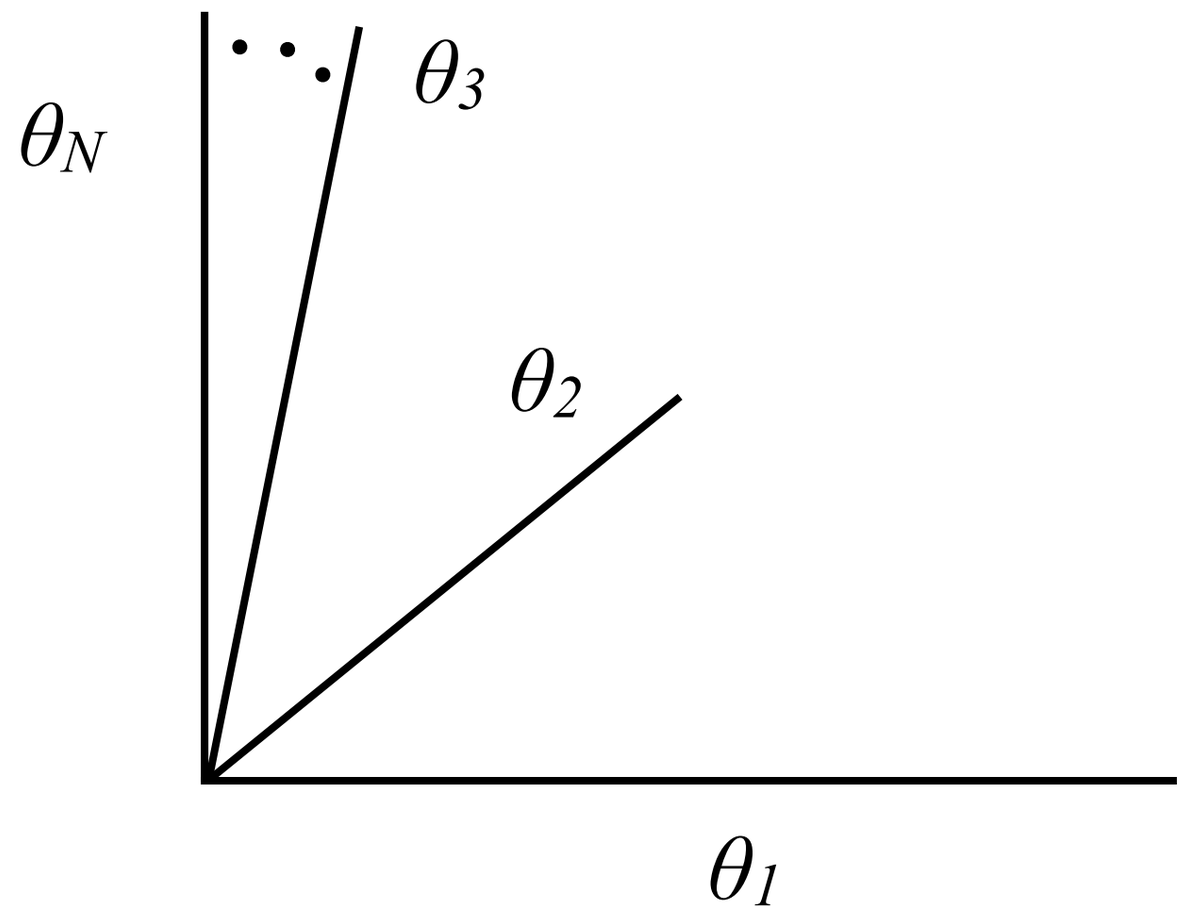
Three prongs of kinetic theory

Liouville



Liouville

Orbit in N^D phase space



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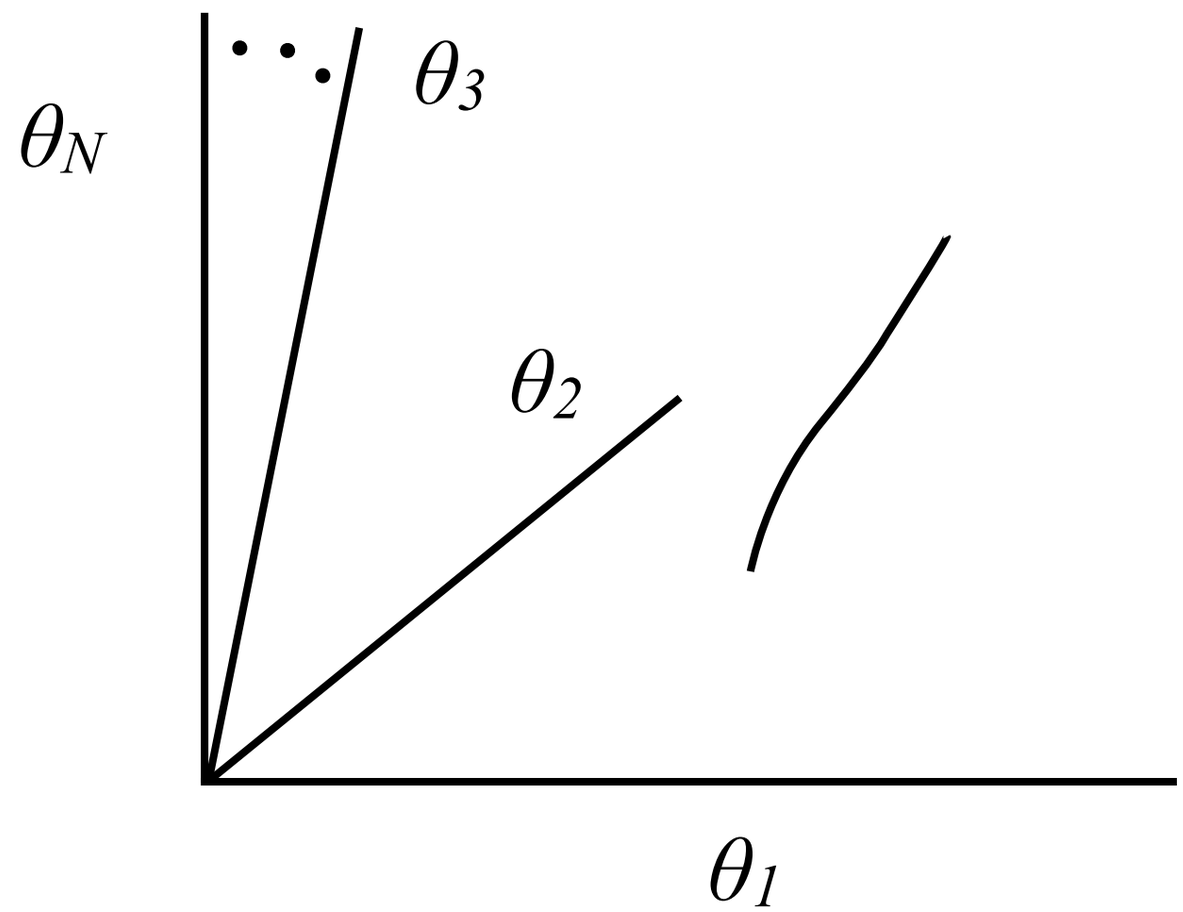
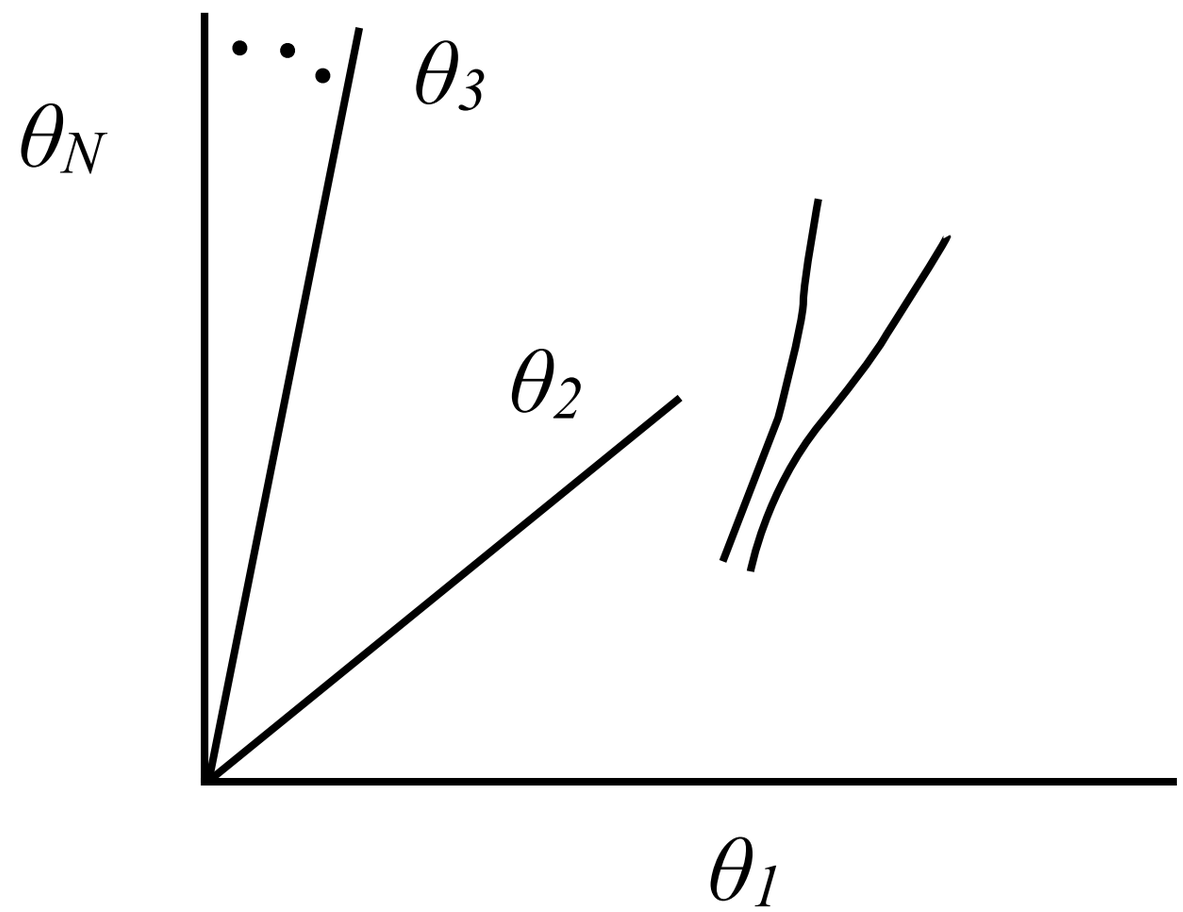


Liouville

Orbit in N^D phase space

Different initial data, parameters



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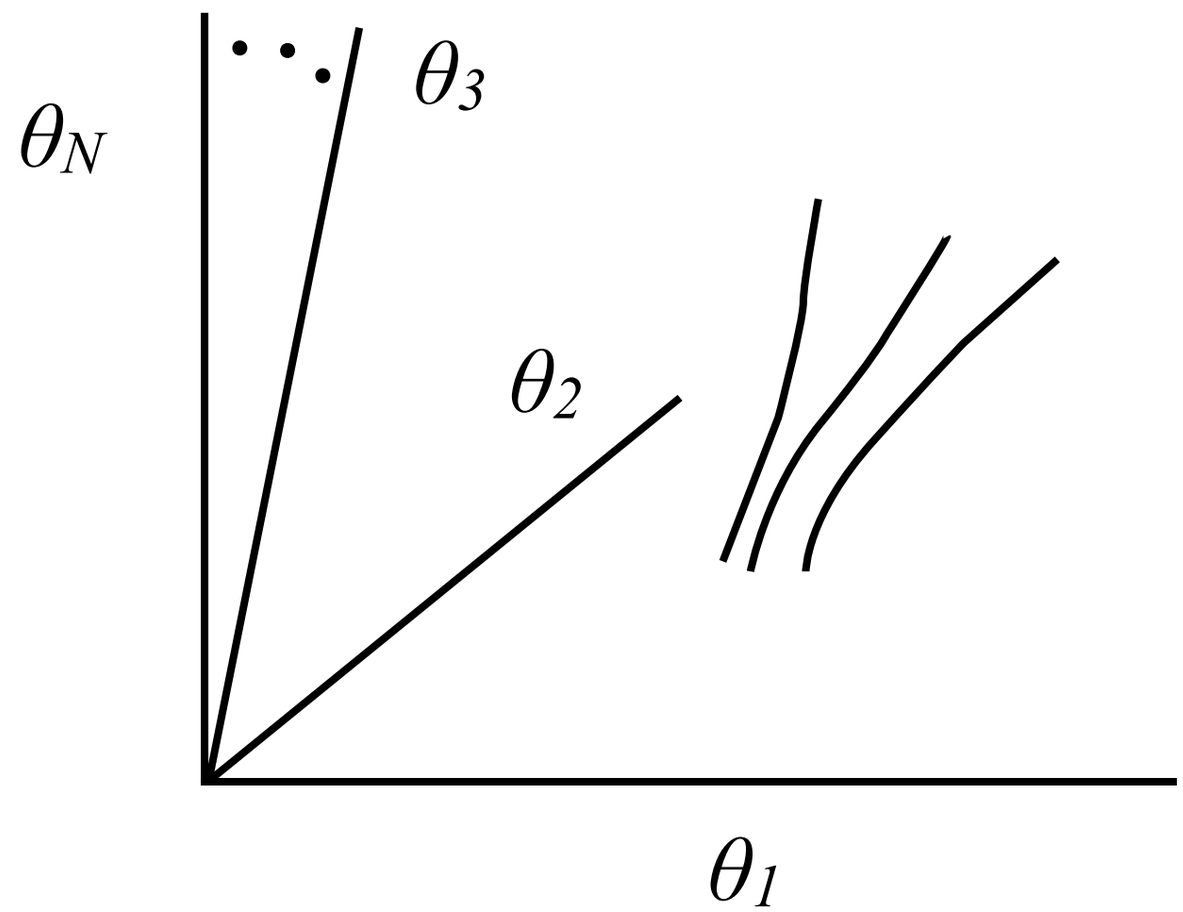

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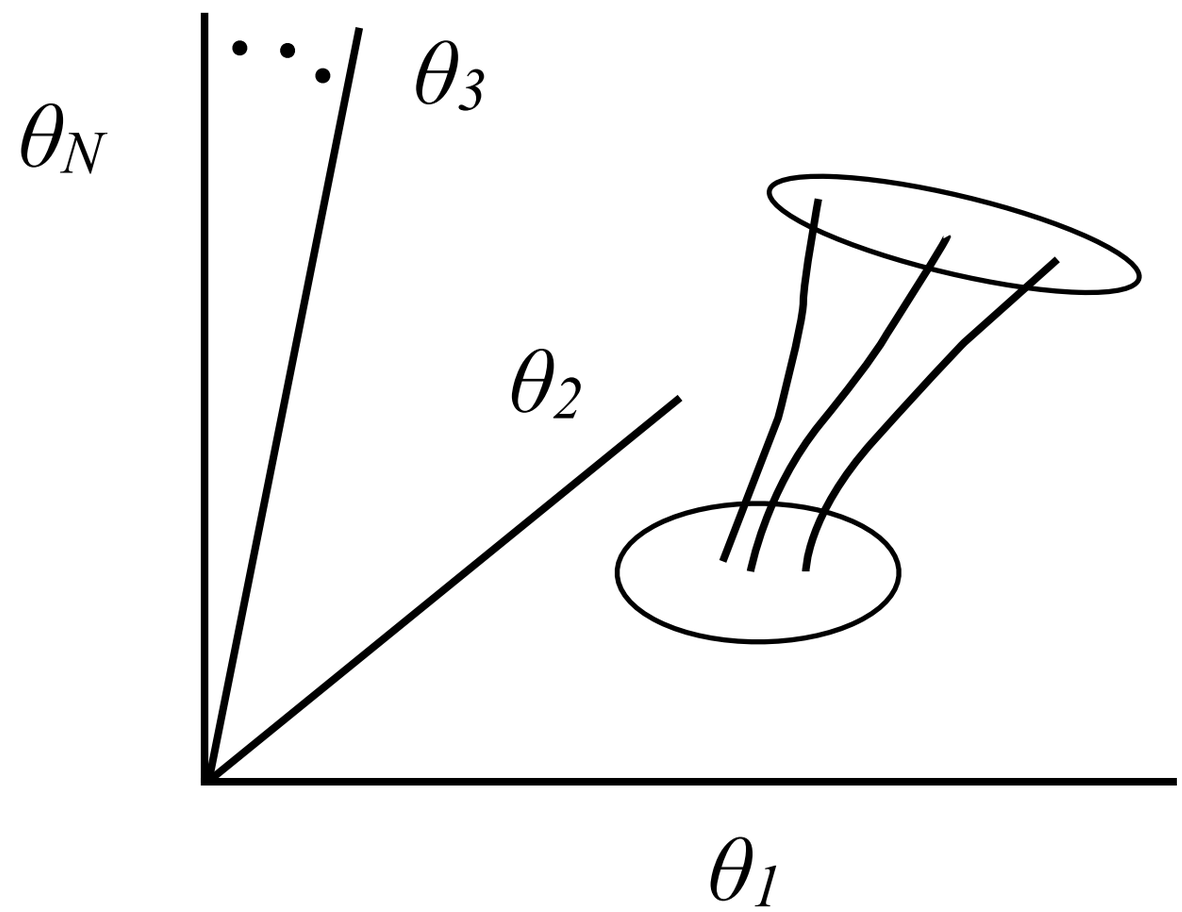

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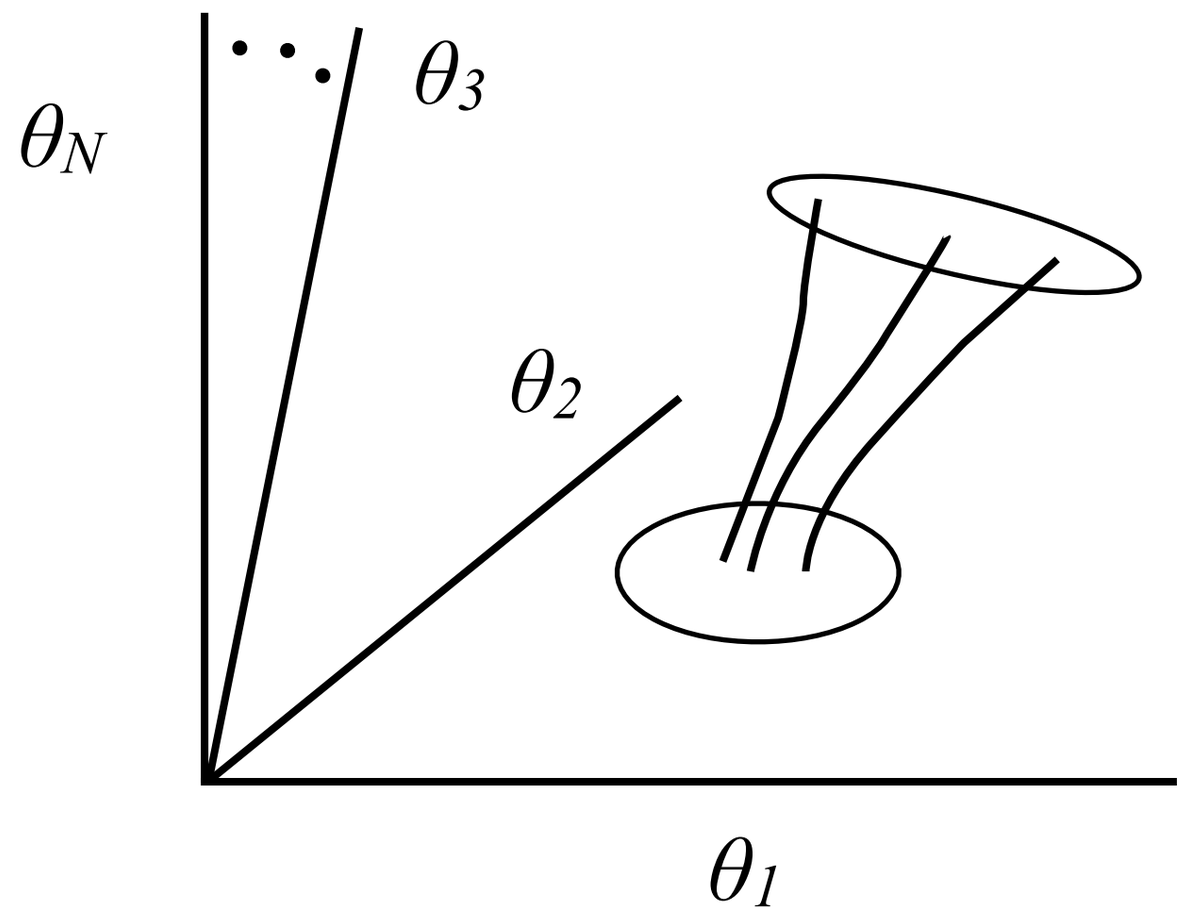
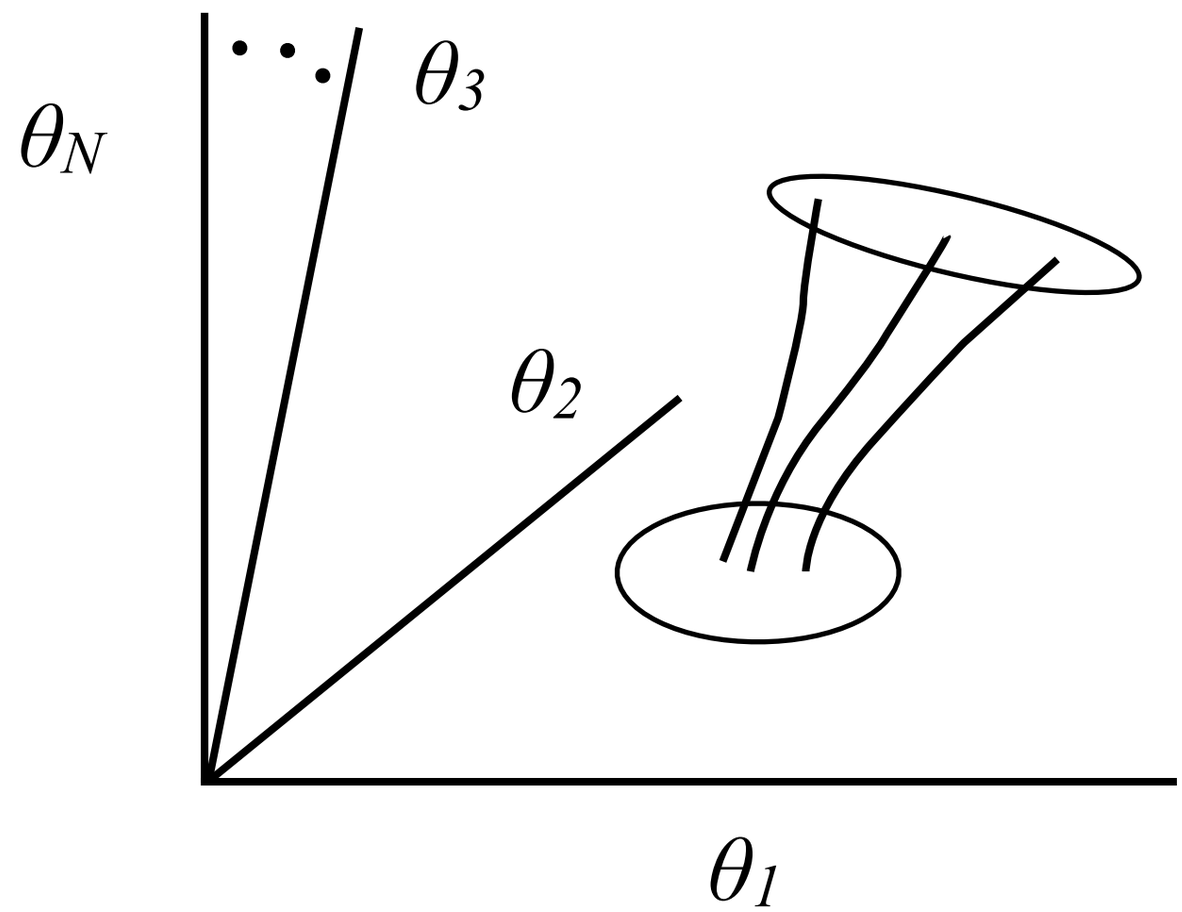


$$P_N(\vec{\theta}, t)$$

Liouville

Orbit in N^D phase space

Different initial data, parameters



$P_N(\vec{\theta}, t)$

Probability density evolution in N^D

Liouville

$$\dot{\theta}_i = f_i(\vec{\theta}, t) \qquad \vec{\theta} = \{\theta_1, \theta_2, \ldots, \theta_N\}$$

Liouville

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Probability conservation

Liouville

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Probability conservation

$$\frac{\partial P_N(\vec{\theta})}{\partial t} = -\frac{\partial}{\partial \theta_i} f_i P_N(\vec{\theta})$$

Liouville

$$\dot{\theta}_i = f_i(\vec{\theta}, t) \qquad \vec{\theta} = \{\theta_1, \theta_2, \ldots, \theta_N\}$$

Probability conservation

$$\frac{\partial P_N(\vec{\theta})}{\partial t} = -\frac{\partial}{\partial \theta_i} f_i P_N(\vec{\theta})$$

(Einstein summation convention)

Fokker-Planck

$$\dot{\theta}_i = f_i(\vec{\theta}, t) + g_i(\vec{\theta})\eta(t)$$

Fokker-Planck

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White noise

Fokker-Planck

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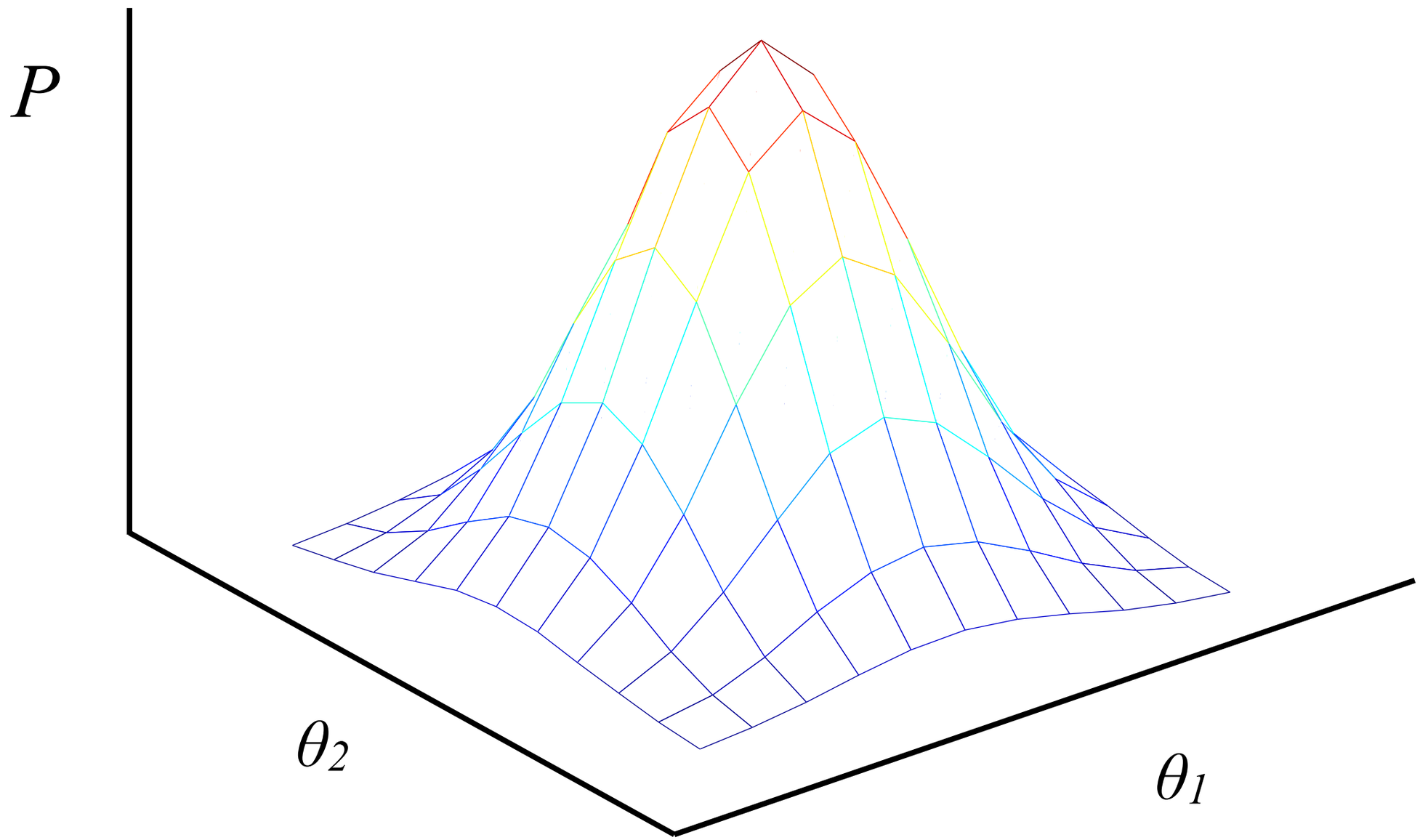
White noise

$$\frac{\partial P_N(\vec{\theta})}{\partial t} = -\frac{\partial}{\partial \theta_i} f_i P_N(\vec{\theta}) + \frac{1}{2}\frac{\partial^2}{\partial \theta_i \partial \theta_j} g_{ik} g_{jk} P_N(\vec{\theta})$$

Marginalize



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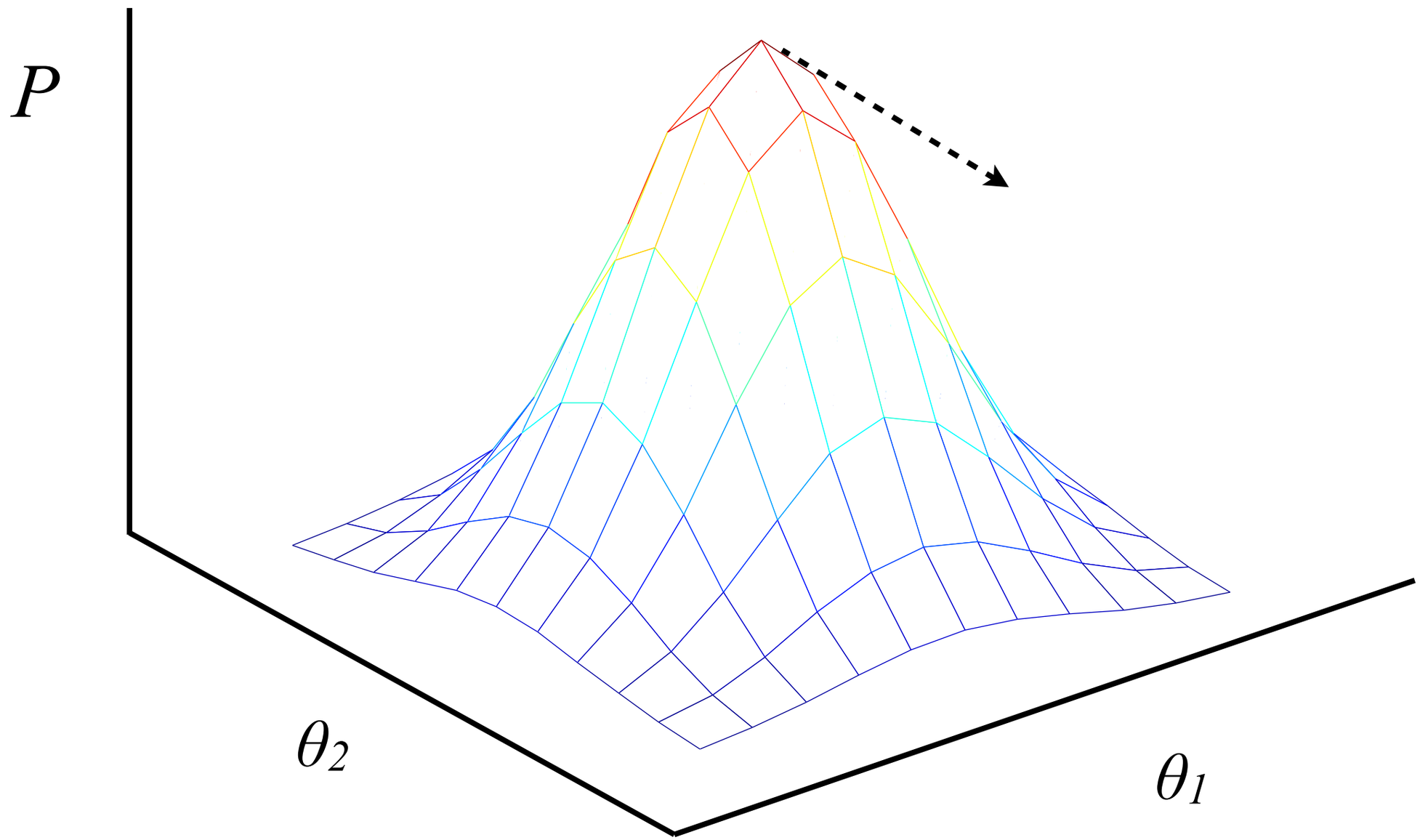

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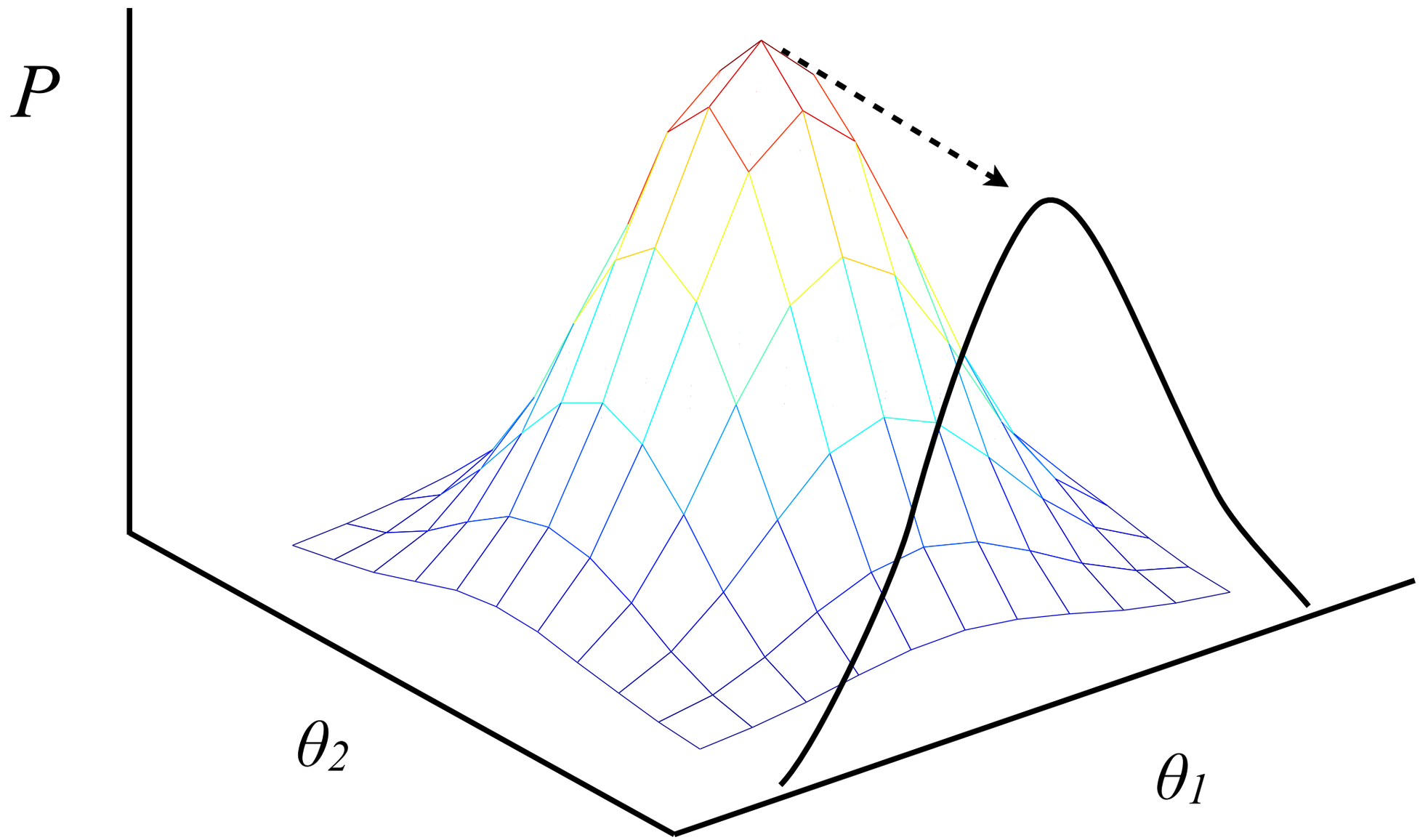

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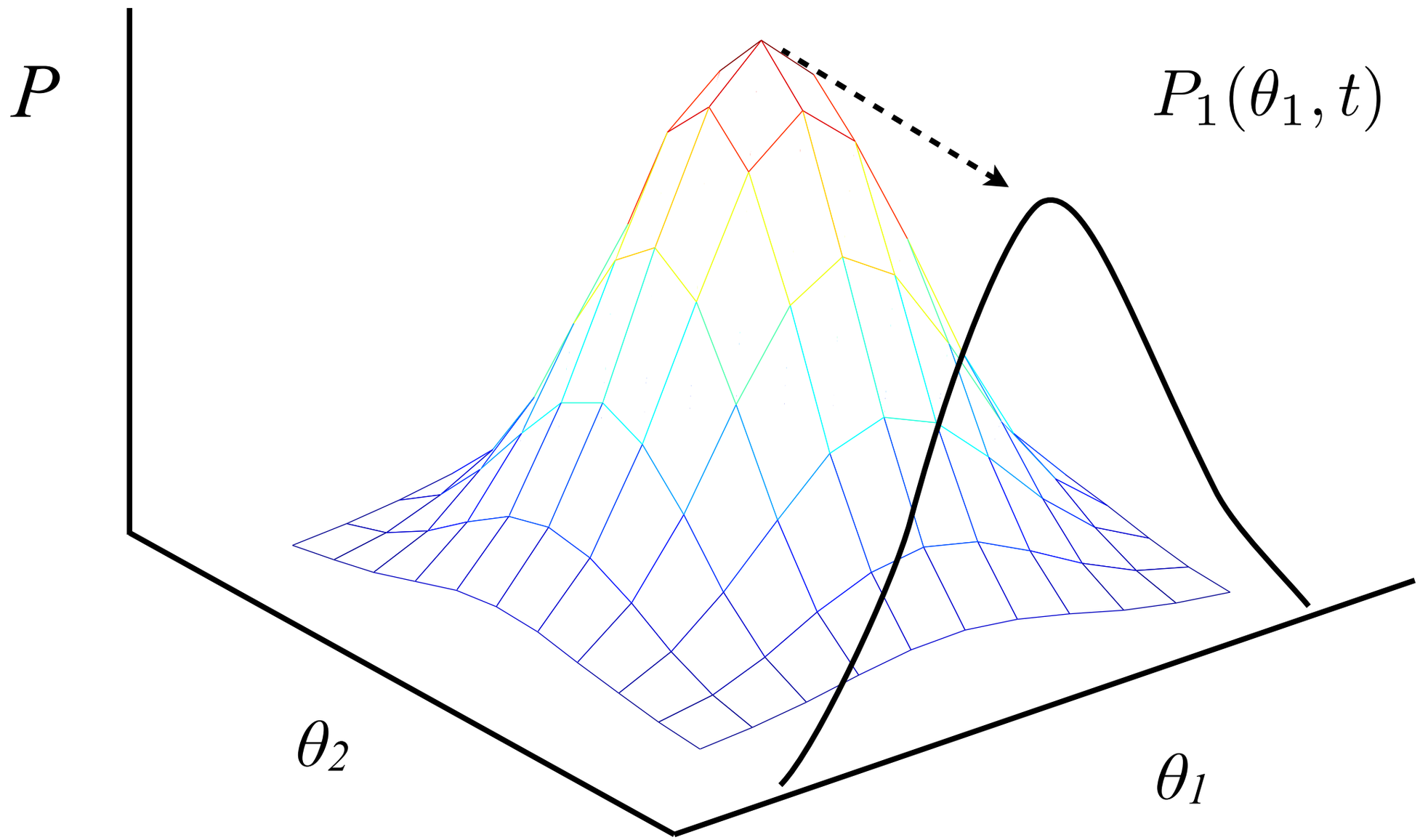

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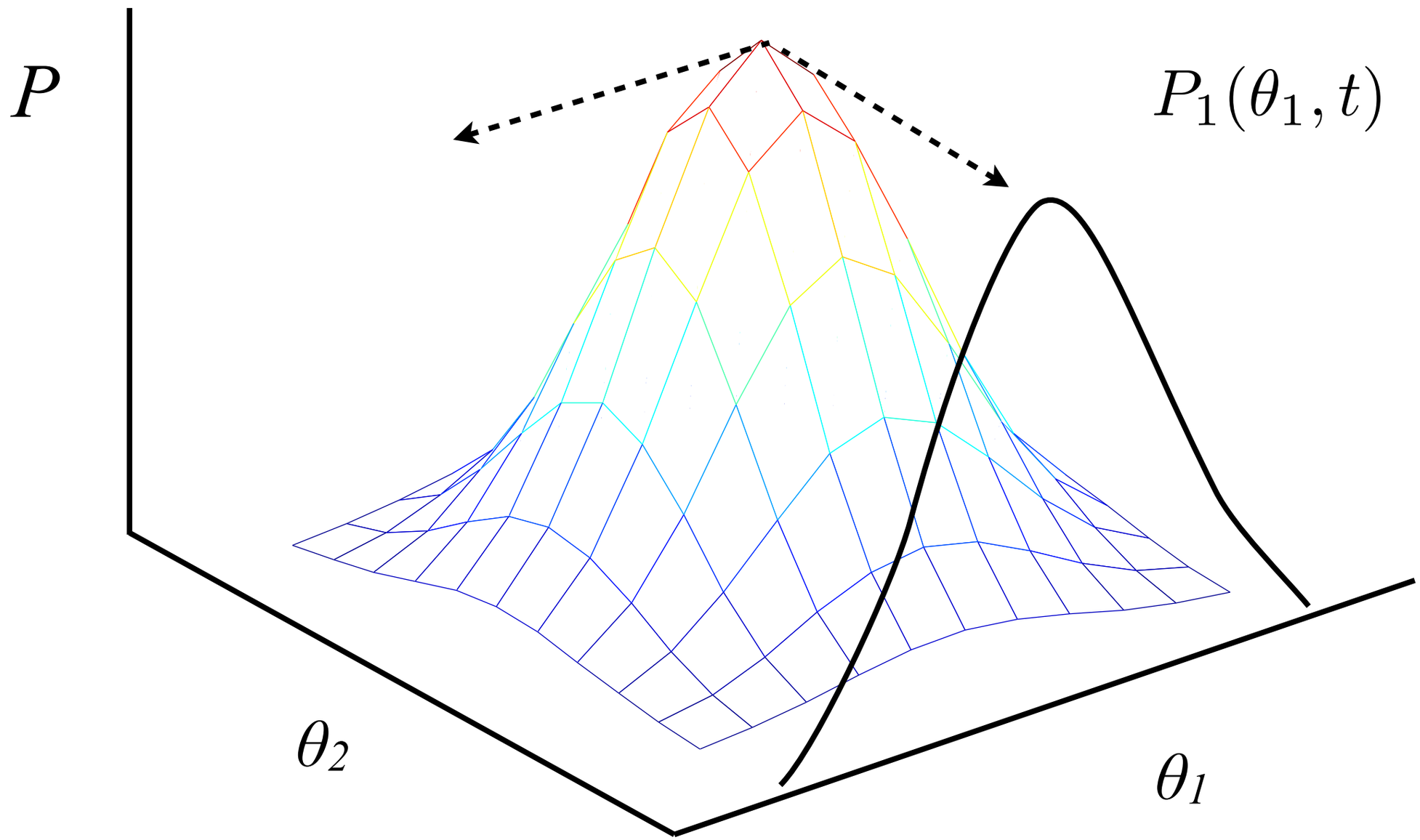

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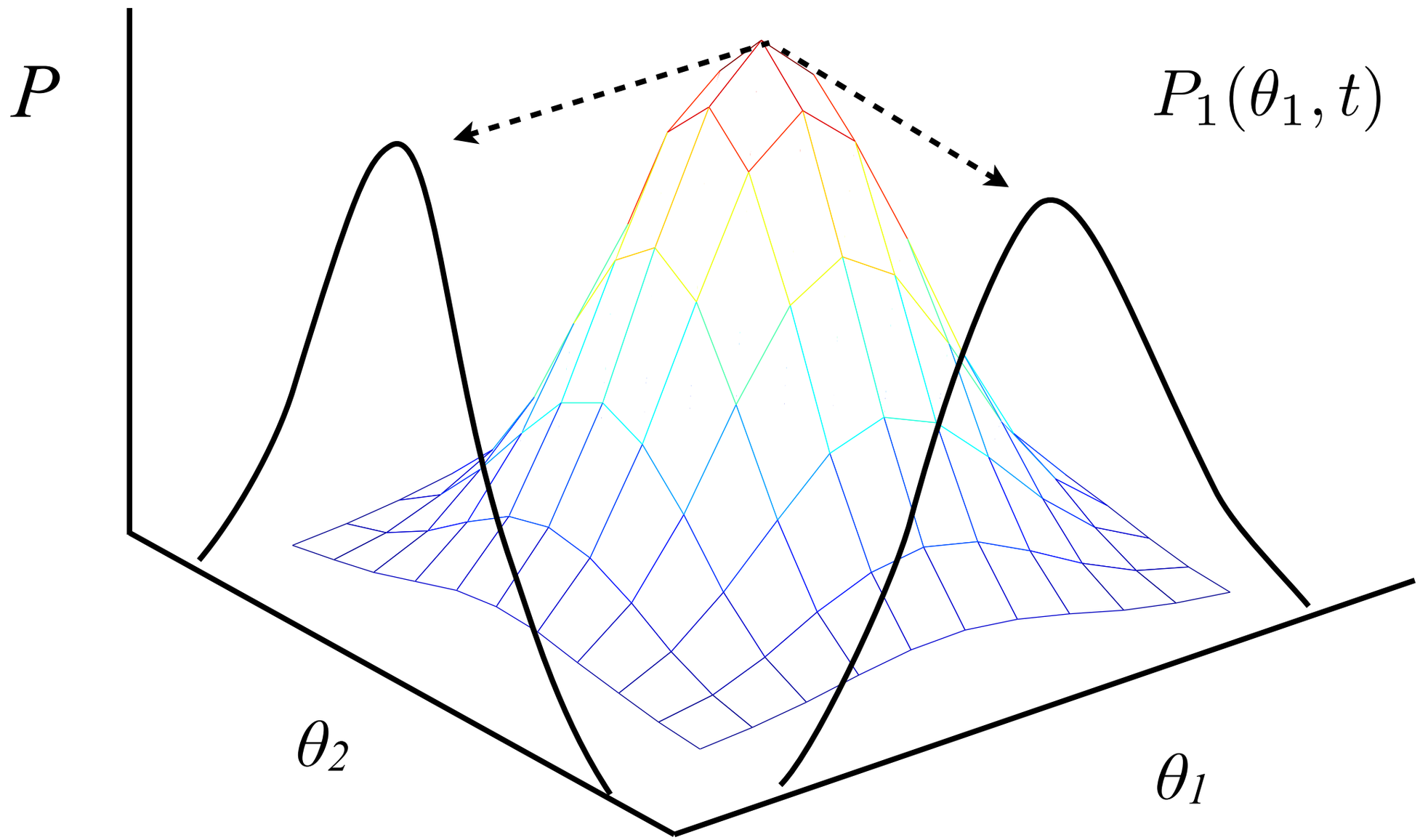

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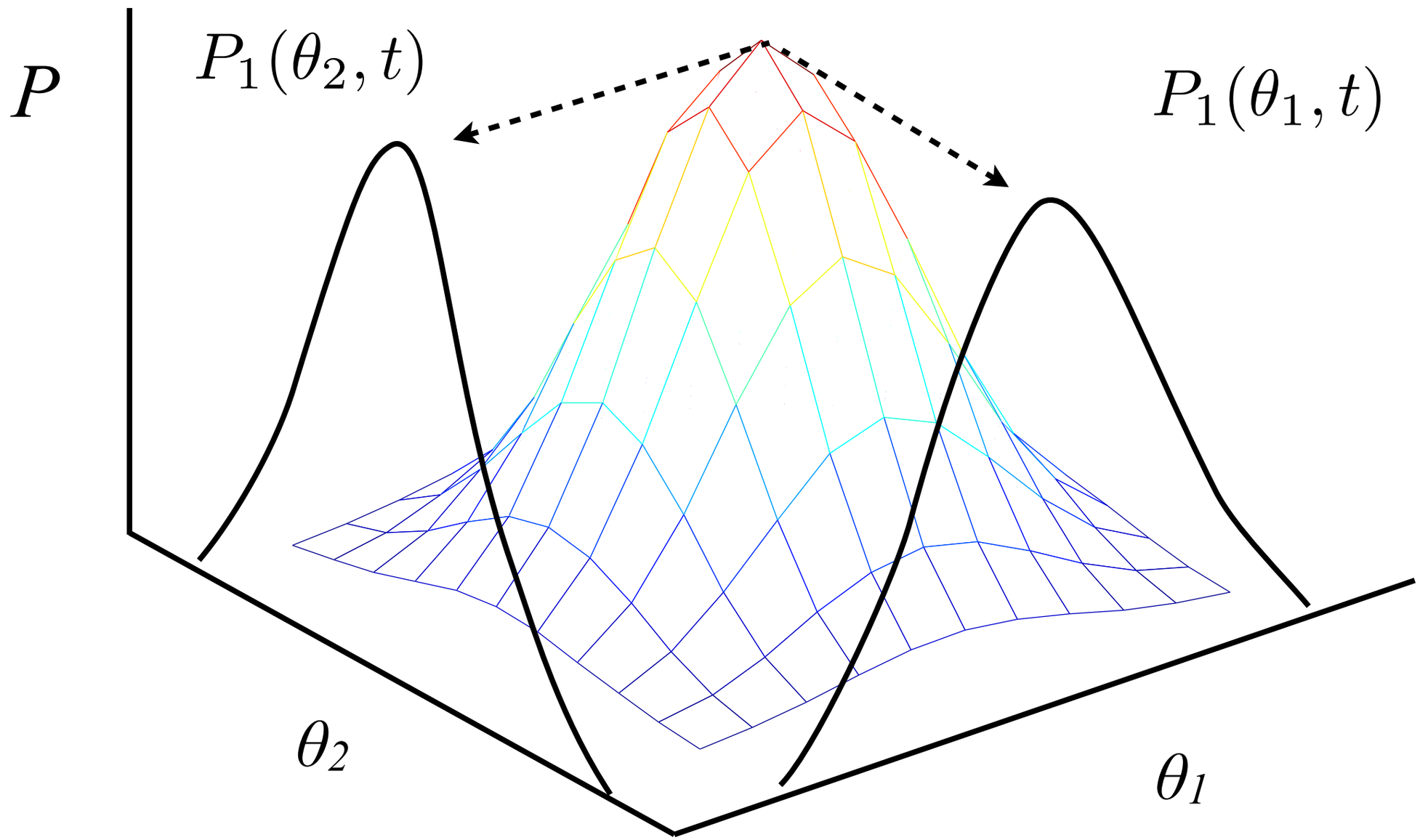

Marginalize

$$P_k(\theta_1, \cdots, \theta_k) = \int \prod_{i=k+1}^{N} d\theta_i P_N(\vec{\theta})$$

Marginalize

$$P_k(\theta_1, \cdots, \theta_k) = \int \prod_{i=k+1}^{N} d\theta_i P_N(\vec{\theta})$$

Exchange symmetry

Marginalize

$$P_k(\theta_1, \cdots, \theta_k) = \int \prod_{i=k+1}^{N} d\theta_i P_N(\vec{\theta})$$

Exchange symmetry $\quad P_N(\cdots, \theta_i, \cdots, \theta_j, \cdots)$

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Exchange symmetry $\quad P_N(\cdots, \theta_i, \cdots, \theta_j, \cdots)$

$$P_1(\theta_1) = P_1(\theta_2) = \cdots = P_1(\theta)$$

$$\frac{\partial P_N(\vec{\theta})}{\partial t} = -\frac{\partial}{\partial \theta_i} f_i(\vec{\theta}) P_N(\vec{\theta})$$

$$\int \prod_{i=2}^{N} d\theta_i \;\; \frac{\partial P_N(\vec{\theta})}{\partial t} = -\frac{\partial}{\partial \theta_i} f_i(\vec{\theta}) P_N(\vec{\theta})$$

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For pairwise interactions, e.g. $f_i(\vec{\theta}) = \sum_{j=1}^{N} f(\theta_i, \theta_j)$

$$\int \prod_{i=2}^{N} d\theta_i \; \frac{\partial P_N(\vec{\theta})}{\partial t} = -\frac{\partial}{\partial \theta_i} f_i(\vec{\theta}) P_N(\vec{\theta})$$

For pairwise interactions, e.g. $f_i(\vec{\theta}) = \sum_{j=1}^{N} f(\theta_i, \theta_j)$

$$\frac{\partial P_1(\theta_1)}{\partial t} = -\frac{\partial}{\partial \theta_1} \sum_{j=1}^{N} \int d\theta_j f(\theta_1, \theta_j) P_2(\theta_1, \theta_j)$$

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Exchange symmetry

$$\frac{\partial P_1(\theta)}{\partial t} = -N \frac{\partial}{\partial \theta} \int d\theta' f(\theta, \theta') P_2(\theta, \theta')$$

$$P_2(\theta, \theta') = P_1(\theta')P_1(\theta) + \frac{1}{N}C_2(\theta, \theta')$$

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$$P_2(\theta,\theta') = P_1(\theta')P_1(\theta) + \frac{1}{N}C_2(\theta,\theta')$$

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Finite size effects

$$P_2(\theta,\theta') = P_1(\theta')P_1(\theta) + \frac{1}{N}C_2(\theta,\theta')$$

$$\frac{\partial P_1(\theta)}{\partial t} + N\frac{\partial}{\partial\theta}\int d\theta' f(\theta,\theta')P_1(\theta')P_1(\theta) = -\frac{\partial}{\partial\theta}\int d\theta' f(\theta,\theta')C_2(\theta,\theta')$$

Finite size effects

$$\frac{\partial P_1(\theta)}{\partial t} + N\frac{\partial}{\partial\theta}\int d\theta' f(\theta,\theta')P_1(\theta')P_1(\theta) = 0$$

Mean field theory

$$P_2(\theta,\theta') = P_1(\theta')P_1(\theta) + \frac{1}{N}C_2(\theta,\theta')$$

$$\frac{\partial P_1(\theta)}{\partial t} + N\frac{\partial}{\partial\theta}\int d\theta' f(\theta,\theta')P_1(\theta')P_1(\theta) = -\frac{\partial}{\partial\theta}\int d\theta' f(\theta,\theta')C_2(\theta,\theta')$$

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Finite size effects

$$\frac{\partial P_1(\theta)}{\partial t} + N\frac{\partial}{\partial\theta}\int d\theta' f(\theta,\theta')P_1(\theta')P_1(\theta) = \frac{1}{2}D\frac{\partial^2}{\partial\theta^2}P_1(\theta)$$

Mean field theory

White noise

$$P_2(\theta,\theta') = P_1(\theta')P_1(\theta) + \frac{1}{N}C_2(\theta,\theta')$$

$$\frac{\partial P_1(\theta)}{\partial t} + N\frac{\partial}{\partial\theta}\int d\theta' f(\theta,\theta')P_1(\theta')P_1(\theta) = -\frac{\partial}{\partial\theta}\int d\theta' f(\theta,\theta')C_2(\theta,\theta')$$

Finite size effects

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McKean-Vlasov equation

$$\frac{\partial P_1(\theta)}{\partial t} + N \frac{\partial}{\partial \theta} \int d\theta' f(\theta, \theta') P_1(\theta') P_1(\theta) = \frac{1}{2} D \frac{\partial^2}{\partial \theta^2} P_1(\theta)$$

McKean-Vlasov equation

$$\boxed{\frac{\partial P_N(\vec{\theta})}{\partial t} = -\frac{\partial}{\partial \theta_i} f_i P_N(\vec{\theta}) + \frac{1}{2}\frac{\partial^2}{\partial \theta_i \partial \theta_j} g_{ik} g_{jk} P_N(\vec{\theta})}$$

The Fokker-Planck equation

$$\frac{\partial P_1(\theta)}{\partial t} + N\frac{\partial}{\partial \theta}\int d\theta' f(\theta,\theta') P_1(\theta') P_1(\theta) = \frac{1}{2} D \frac{\partial^2}{\partial \theta^2} P_1(\theta)$$

McKean-Vlasov equation

BBGKY Hierarchy

$$\frac{\partial P_1(\theta)}{\partial t} + N\frac{\partial}{\partial \theta}\int d\theta' f_i(\theta,\theta')P_1(\theta')P_1(\theta) = -N\frac{\partial}{\partial \theta}\int d\theta' f_i(\theta,\theta')C_2(\theta,\theta')$$

BBGKY Hierarchy

$$\frac{\partial P_1(\theta)}{\partial t} + N\frac{\partial}{\partial \theta}\int d\theta' f_i(\theta,\theta')P_1(\theta')P_1(\theta) = -N\frac{\partial}{\partial \theta}\int d\theta' f_i(\theta,\theta')C_2(\theta,\theta')$$

C_2 depends on C_3 and so on

BBGKY Hierarchy

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N coupled PDEs

BBGKY Hierarchy

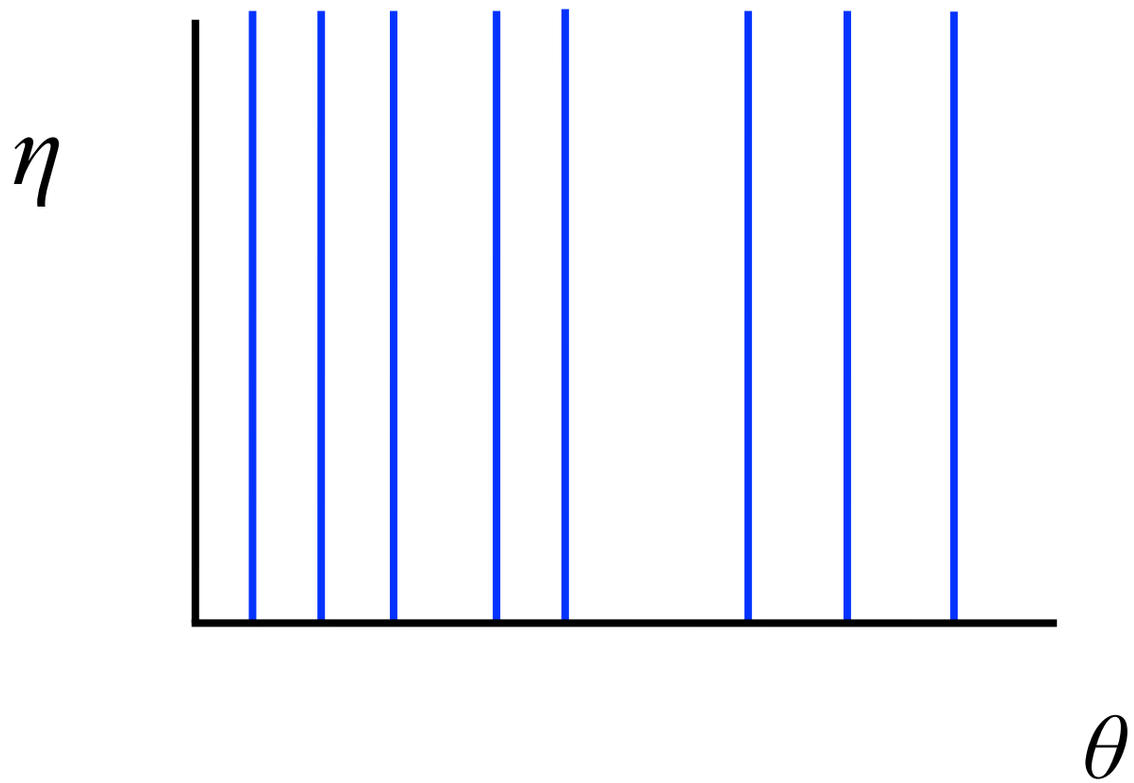
$$\frac{\partial P_1(\theta)}{\partial t} + N\frac{\partial}{\partial \theta}\int d\theta' f_i(\theta,\theta')P_1(\theta')P_1(\theta) = -N\frac{\partial}{\partial \theta}\int d\theta' f_i(\theta,\theta')C_2(\theta,\theta')$$

C_2 depends on C_3 and so on

N coupled PDEs

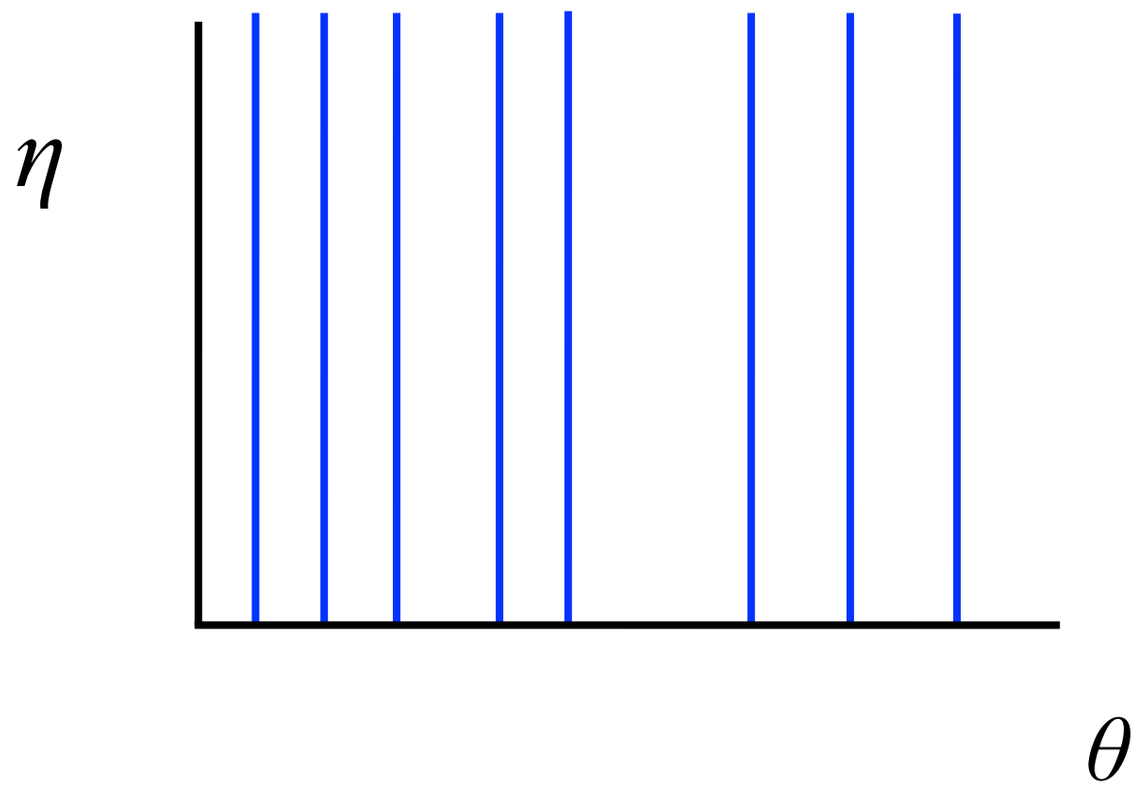
Need to truncate

Klimontovich



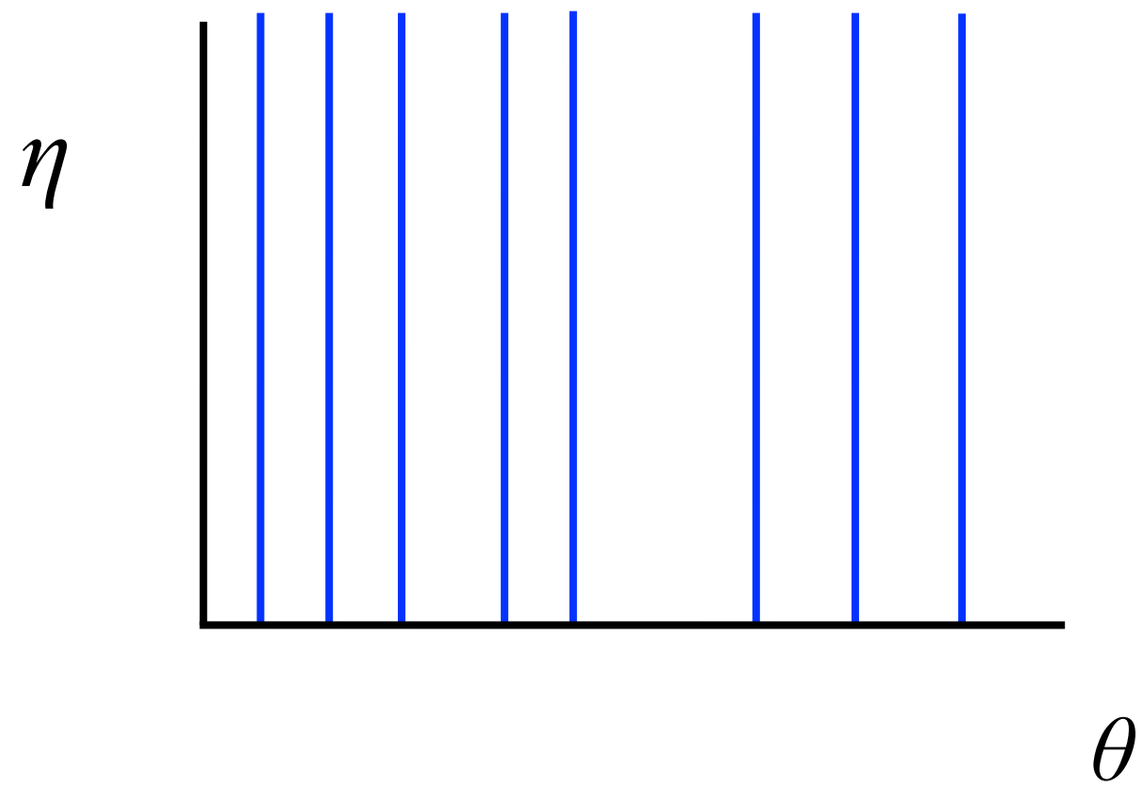
Klimontovich

Begin with exchange symmetry



Klimontovich

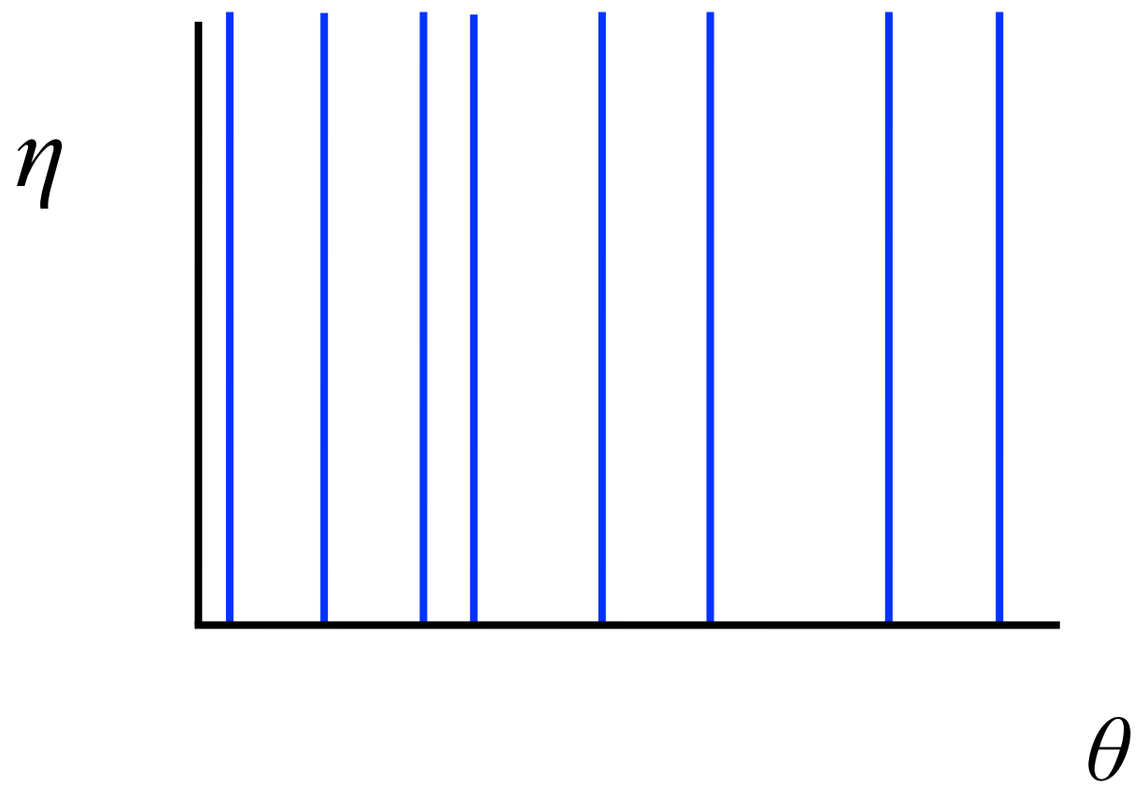
Begin with exchange symmetry



N neurons in 1^{D}

Klimontovich

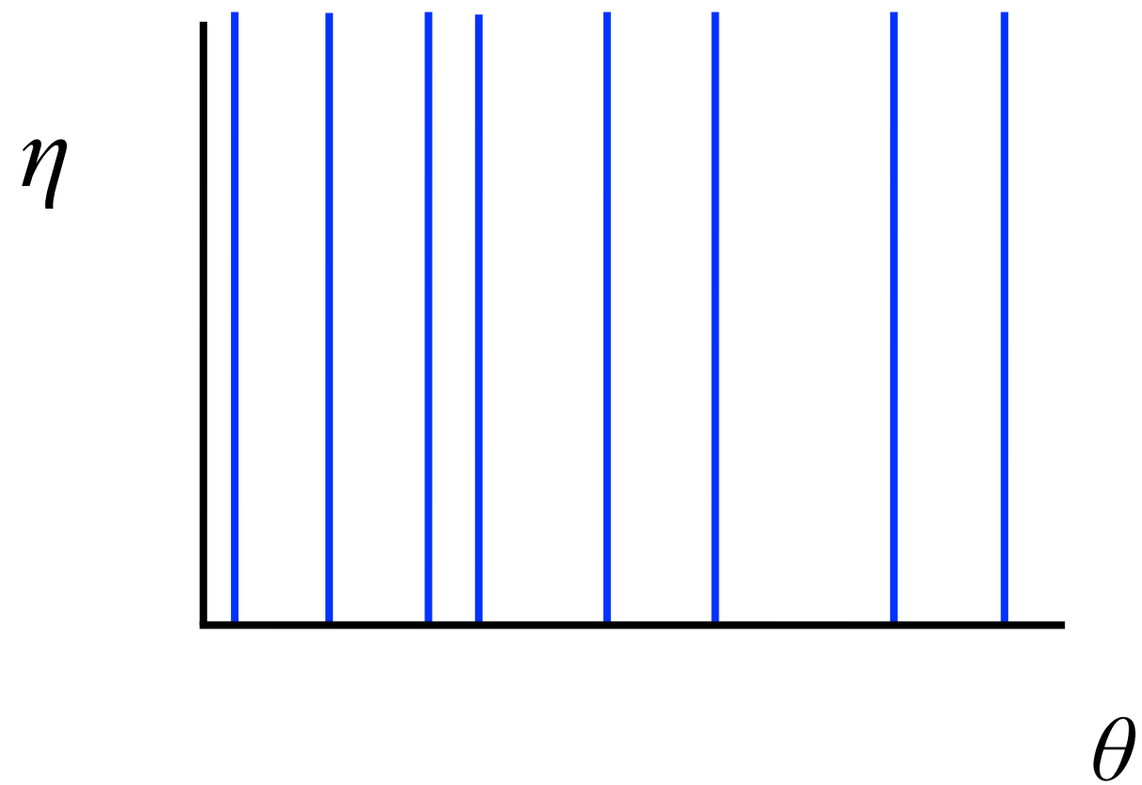
Begin with exchange symmetry



N neurons in 1^{D}

Klimontovich

Begin with exchange symmetry



N neurons in 1^{D}

density $$\eta(\theta, u, t) = \frac{1}{N} \sum_{i=1}^{N} \delta(\theta - \theta_i(t))$$

Oscillator dynamics:

$$\dot{\theta} = I(t) + \alpha u(t)$$

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Synaptic dynamics:

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Synaptic dynamics: $$\dot{u} + \beta u = \beta \nu$$

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Firing rate:

Oscillator dynamics: $$\dot{\theta} = I(t) + \alpha u(t)$$

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Firing rate: $$\nu = \frac{\beta}{N} \sum_j \delta(t - t_j^s)$$

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Synaptic dynamics: $$\dot{u} + \beta u = \beta \nu$$

Firing rate: $$\nu = \frac{\beta}{N} \sum_j \delta(t - t_j^s)$$

$$\delta(t - t_j^s) = \dot{\theta} \delta(\pi - \theta(t))$$

Oscillator dynamics: $\dot{\theta} = I(t) + \alpha u(t)$

Synaptic dynamics: $\dot{u} + \beta u = \beta \nu$

Firing rate: $\nu = \frac{\beta}{N} \sum_j \delta(t - t_j^s)$

$$\delta(t - t_j^s) = \dot{\theta} \delta(\pi - \theta(t))$$

$$\nu(t) = \frac{1}{N} \sum_i \dot{\theta}_i(t) \delta(\pi - \theta_i(t)) = (I(t) + \alpha u(t)) \eta(\pi, t)$$

Klimontovich

e.g. Hildebrand, Buice, Chow, PRL 98.054101, 2007

Oscillator conservation

$$\partial_t \eta + \partial_\theta \left[(I(t) + \alpha u(t)) \eta \right] = 0$$

Klimontovich

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Klimontovich

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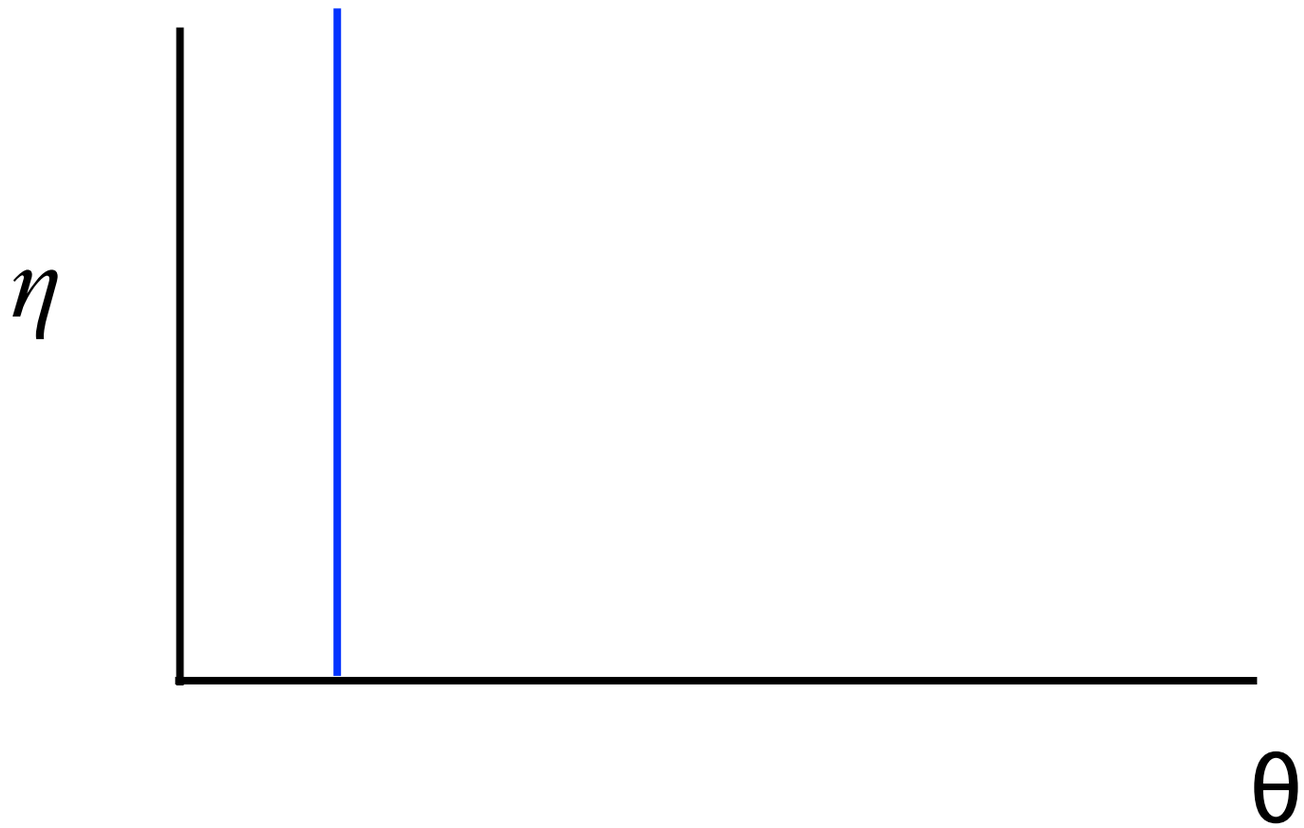
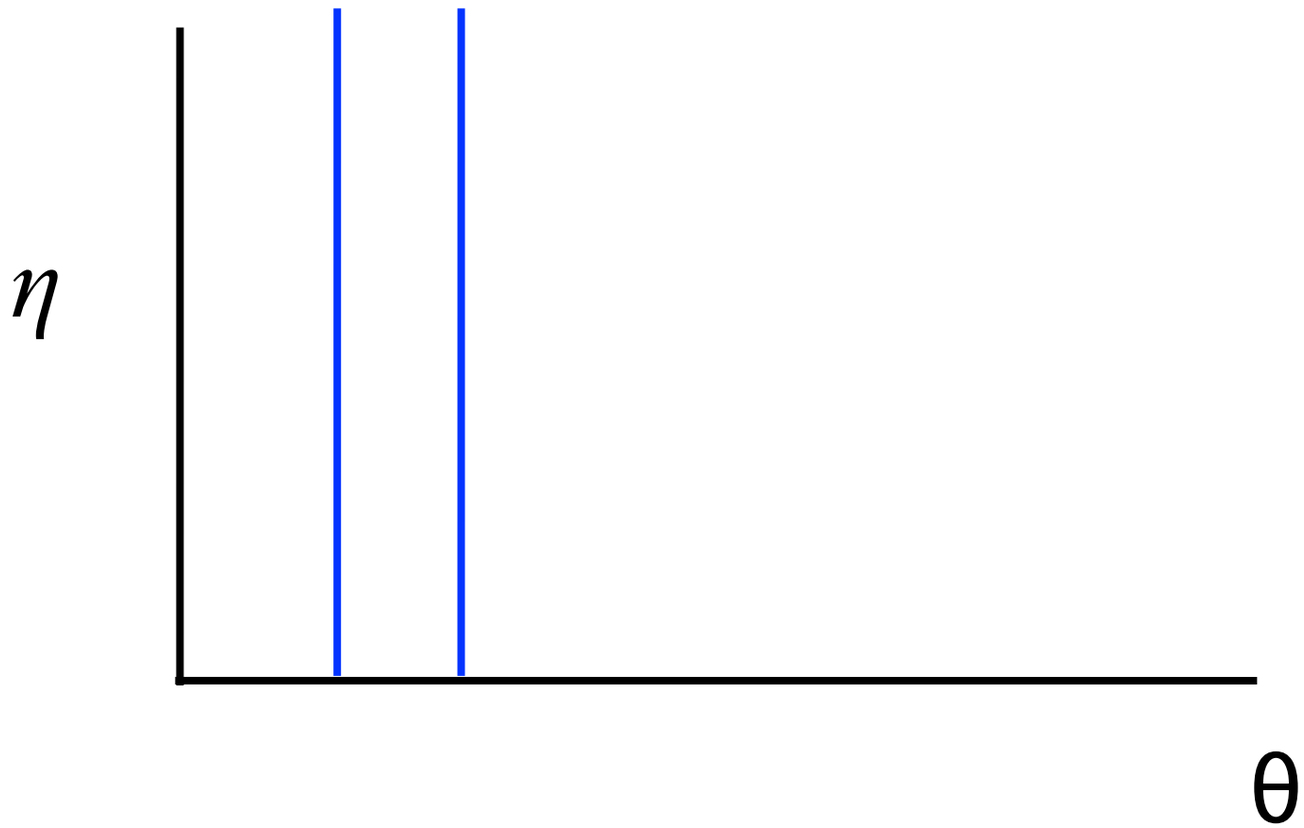
but η is not differentiable

Average over initial data



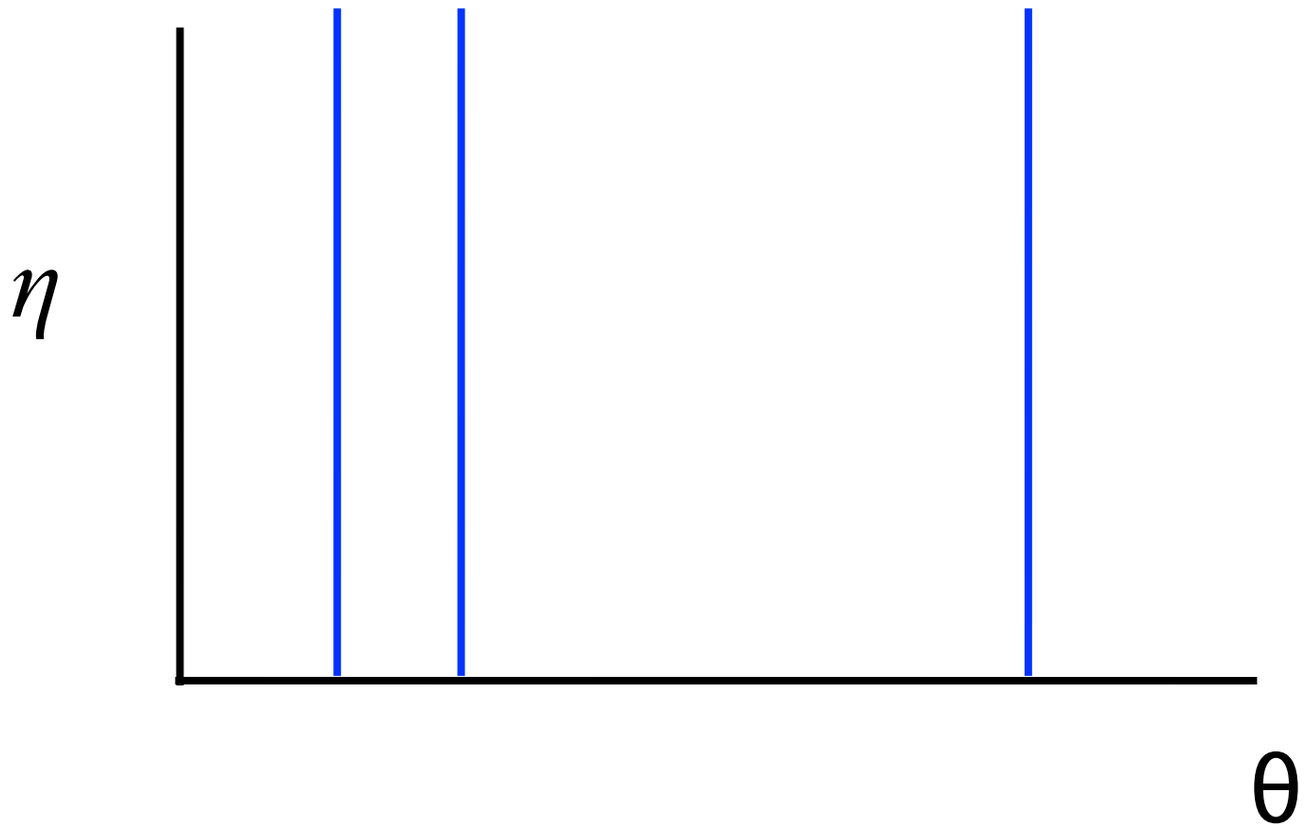
smooth by averaging

Average over initial data



smooth by averaging

Average over initial data



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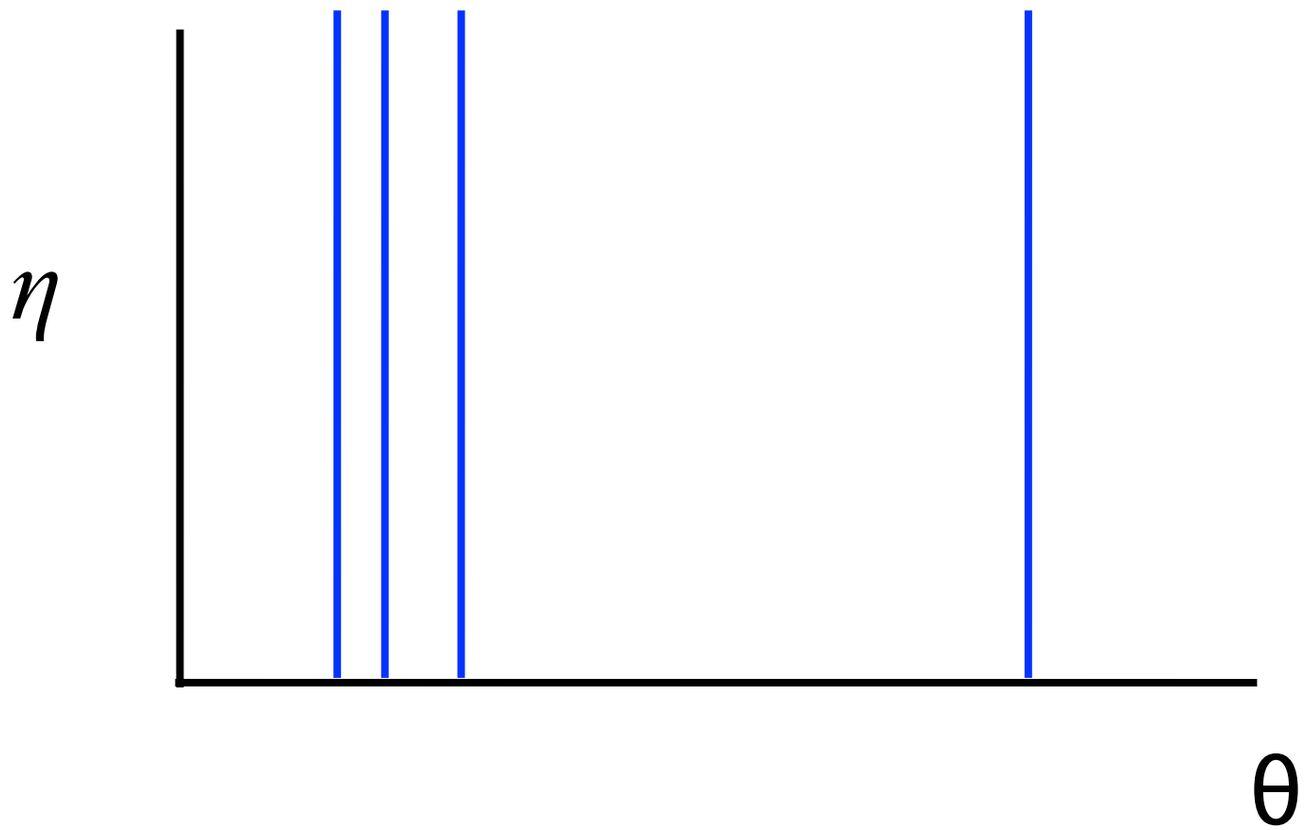
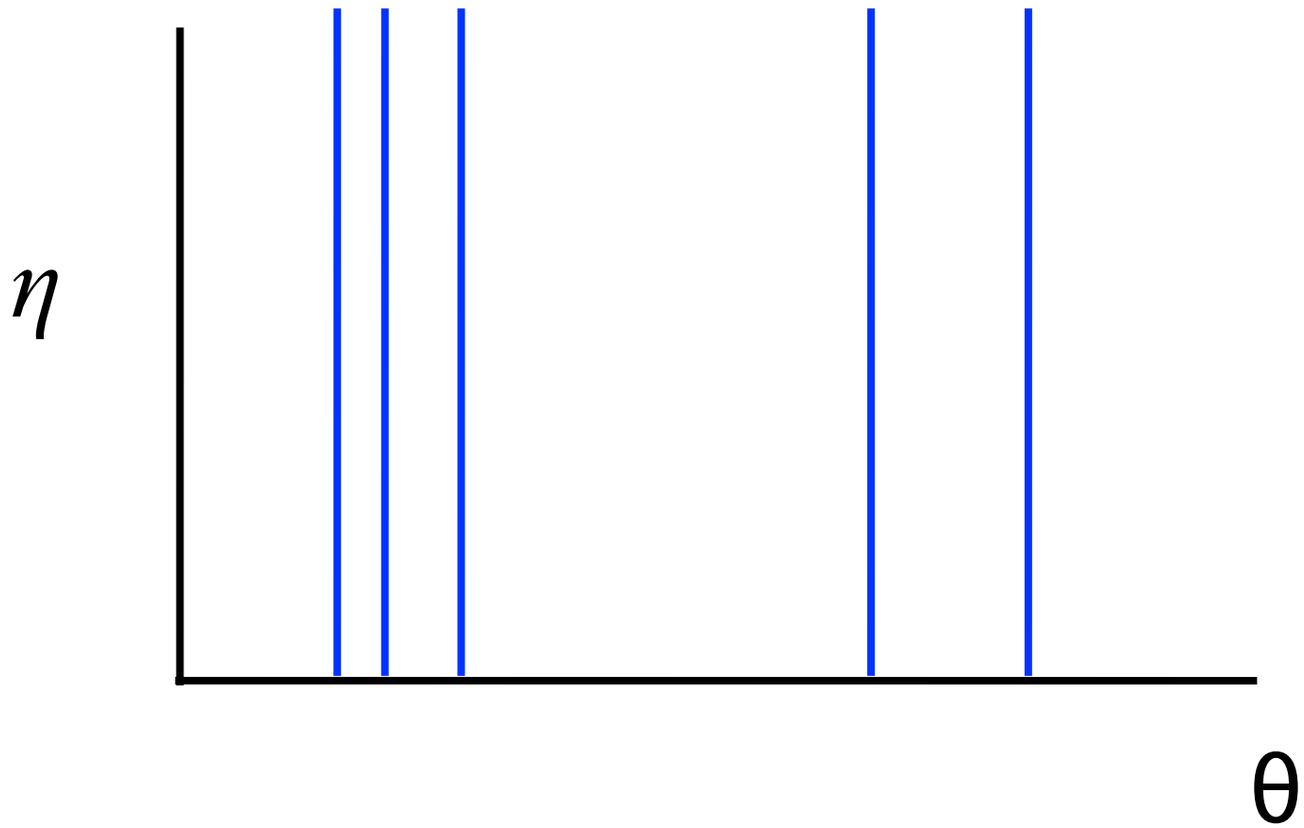
Average over initial data



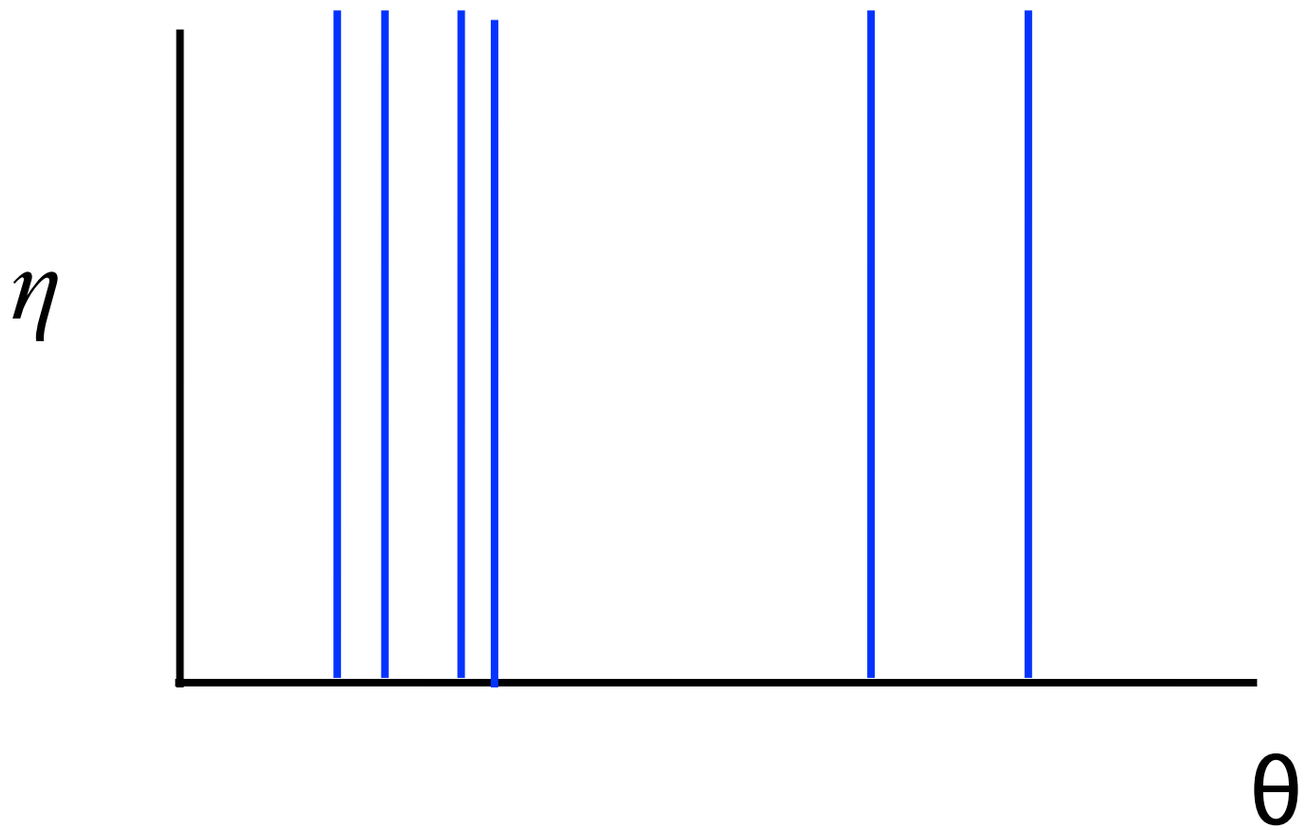
smooth by averaging

Average over initial data



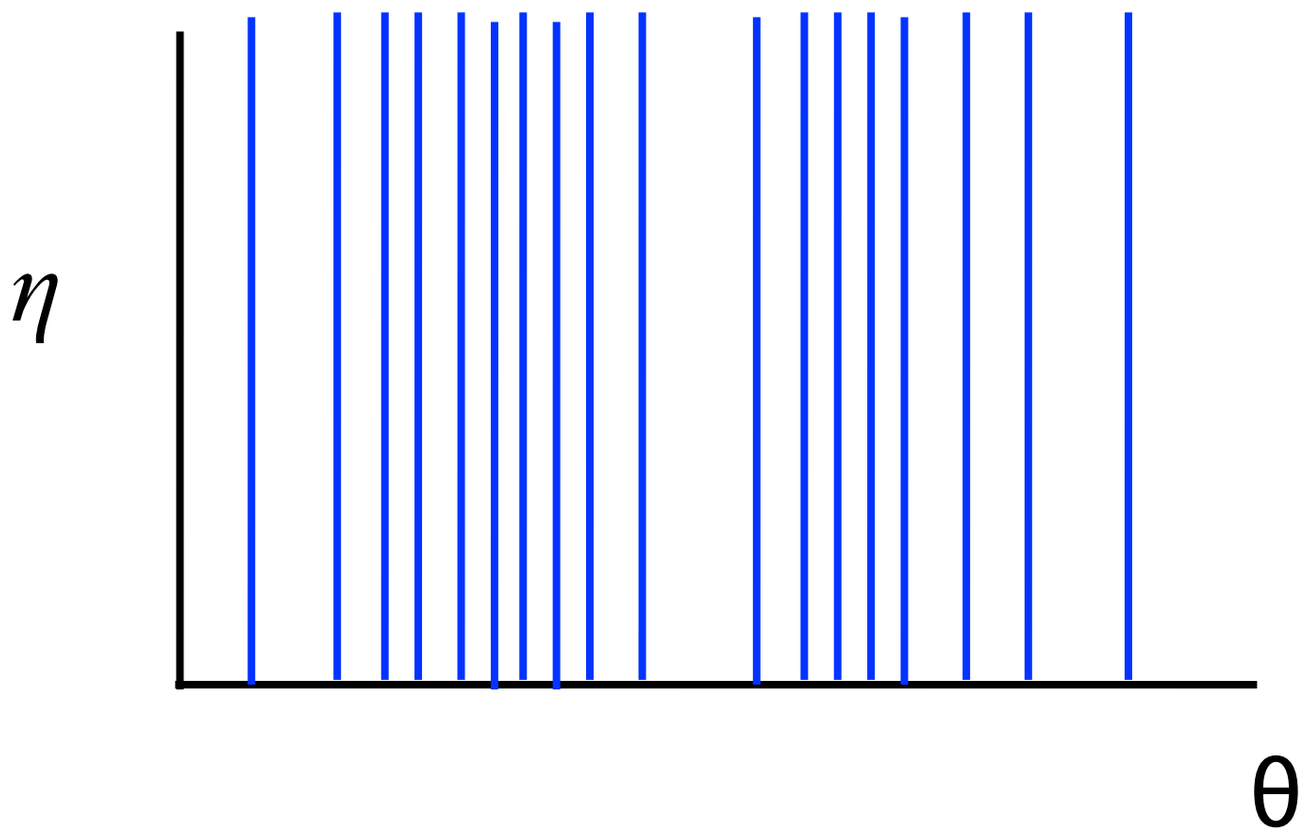
smooth by averaging

Average over initial data



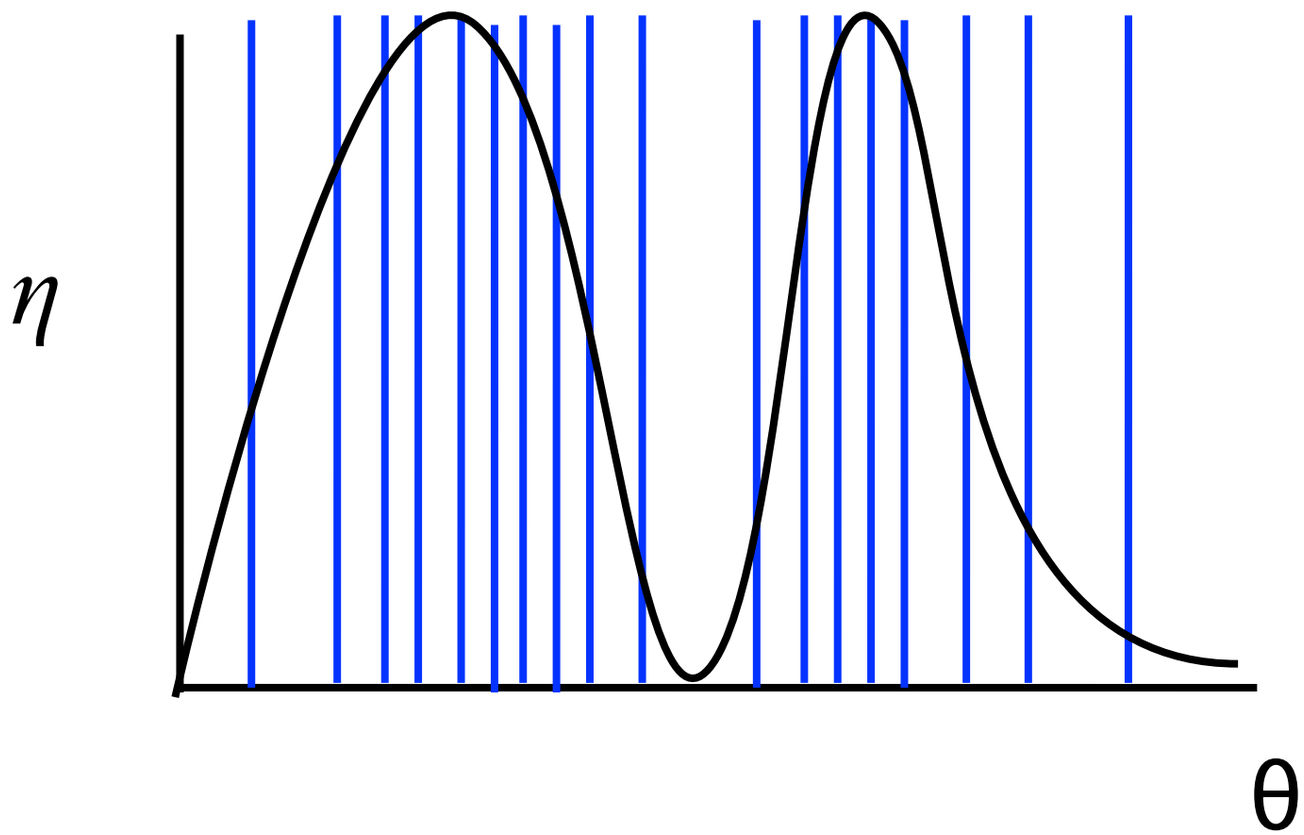
smooth by averaging

Average over initial data



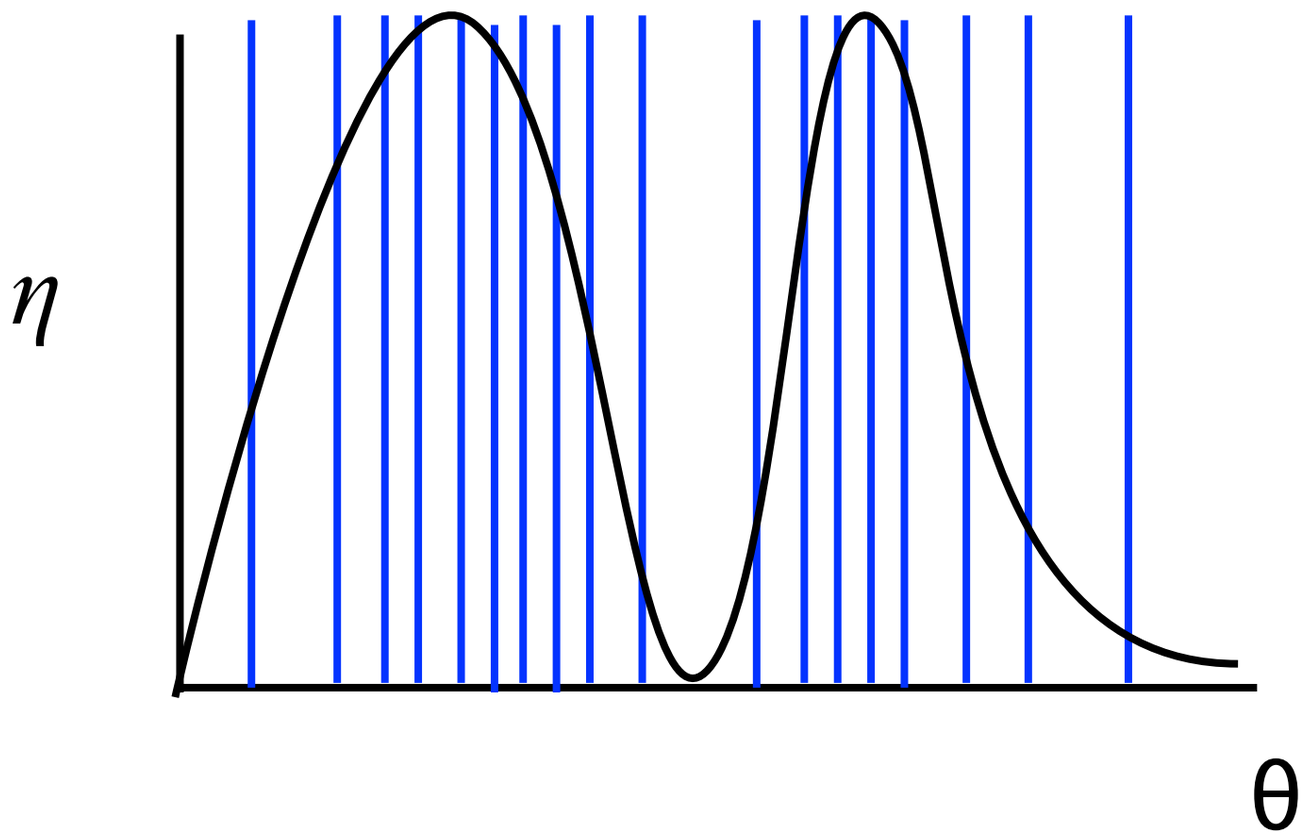
smooth by averaging

Average over initial data



smooth by averaging

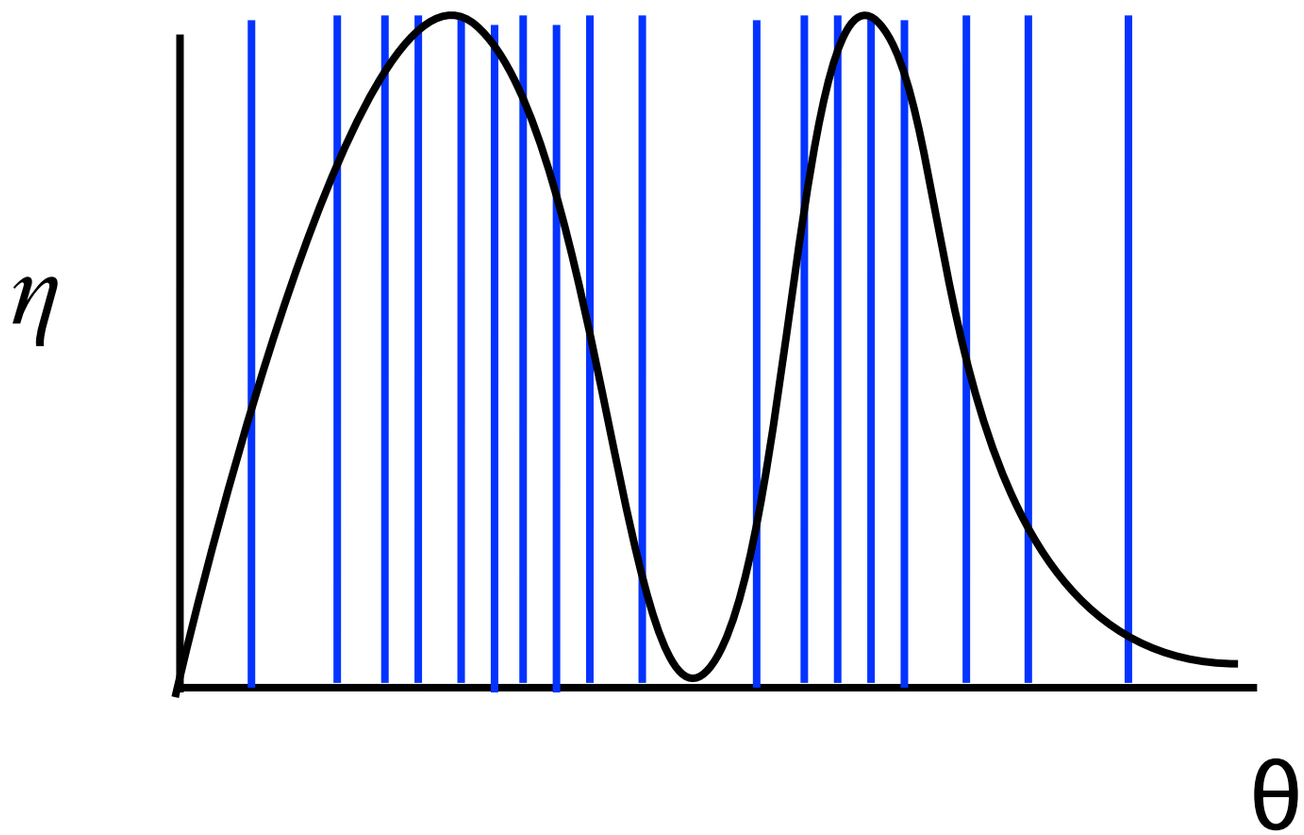
Average over initial data



smooth by averaging

$$\rho(\theta, t) = \langle \eta(\theta, t) \rangle$$

Average over initial data



smooth by averaging

$$\rho(\theta, t) = \langle \eta(\theta, t) \rangle$$

$$u_0 = \langle u \rangle$$

Average over initial data

$$\dot{u}(t) = -\beta u(t) + \beta \left[I(t)\eta + \alpha u \eta \right]$$

Average over initial data

$$\left\langle \; \dot{u}(t) = -\beta u(t) + \beta \left[I(t)\eta + \alpha u \eta \right] \; \right\rangle$$

Average over initial data

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$$\partial_t \eta + \partial_\theta \left[I(t)\eta + \alpha u \eta \right] = 0$$

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$$\partial_t \rho + \partial_\theta \left[I(t)\rho + \alpha \langle u\eta \rangle \right] = 0$$

$$\left(\partial_t u(t) + \beta u(t) - \beta \left[I(t)\eta + \alpha u\eta \right] \right) = 0$$

Average over initial data

$$\dot{u}_0(t) = -\beta u_0(t) + \beta \left[I(t)\rho + \alpha \langle u\eta \rangle \right]$$

$$\partial_t \rho + \partial_\theta \left[I(t)\rho + \alpha \langle u\eta \rangle \right] = 0$$

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$$\vdots$$

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$$\vdots$$

BBGKY moment hierarchy

Average over initial data

$$\dot{u}_0(t) = -\beta u_0(t) + \beta \left[I(t)\rho + \alpha \langle u\eta \rangle \right]$$

$$\partial_t \rho + \partial_\theta \left[I(t)\rho + \alpha \langle u\eta \rangle \right] = 0$$

$$\left\langle \; \eta \left(\partial_t u(t) + \beta u(t) - \beta \left[I(t)\eta + \alpha \underbrace{u\eta}_{\langle \eta u \eta \rangle} \right] \right) = 0 \; \right\rangle$$

$$\vdots$$

BBGKY moment hierarchy

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$$\dot{u}_0(t) = -\beta u_0(t) + \beta \left[I(t)\rho + \alpha \langle u\eta \rangle \right]$$

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$$\left\langle \; \eta \left(\partial_t u(t) + \beta u(t) - \beta \left[I(t)\eta + \alpha \underbrace{u\eta}_{\langle \eta u \eta \rangle} \right] \right) = 0 \; \right\rangle$$

$$\vdots$$

BBGKY moment hierarchy

Mean field theory

$$\langle u\eta \rangle = u_0 \rho + \frac{1}{N} C_{uv}$$

Mean field theory

$$\langle u\eta \rangle = u_0 \rho + \cancel{\frac{1}{N} C_{uv}}$$

Mean field theory

$$\langle u\eta \rangle = u_0\rho + \cancel{\frac{1}{N}C_{uv}}$$

Ignore correlations

Mean field theory

$$\dot{u}_0(t) = -\beta u_0(t) + \beta \nu(t)$$

$$\nu(t) = (I(t) + \alpha u_0(t))\rho(\pi, t)$$

$$\partial_t \rho + \partial_\theta \left[(I(t) + \alpha u_0(t))\rho \right] = 0$$

Mean field theory

$$\dot{u}_0(t) = -\beta u_0(t) + \beta \nu(t)$$

$$\nu(t) = (I(t) + \alpha u_0(t))\rho(\pi, t)$$

$$\partial_t \rho + \partial_\theta \left[(I(t) + \alpha u_0(t))\rho \right] = 0$$

Previous work went straight to mean field theory

e.g. Desai and Zwanzig, 1978; Strogatz and Mirollo, 1990; Treves 1993; Abbott and Van Vreeswijk, 1993; ...

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

Density functional

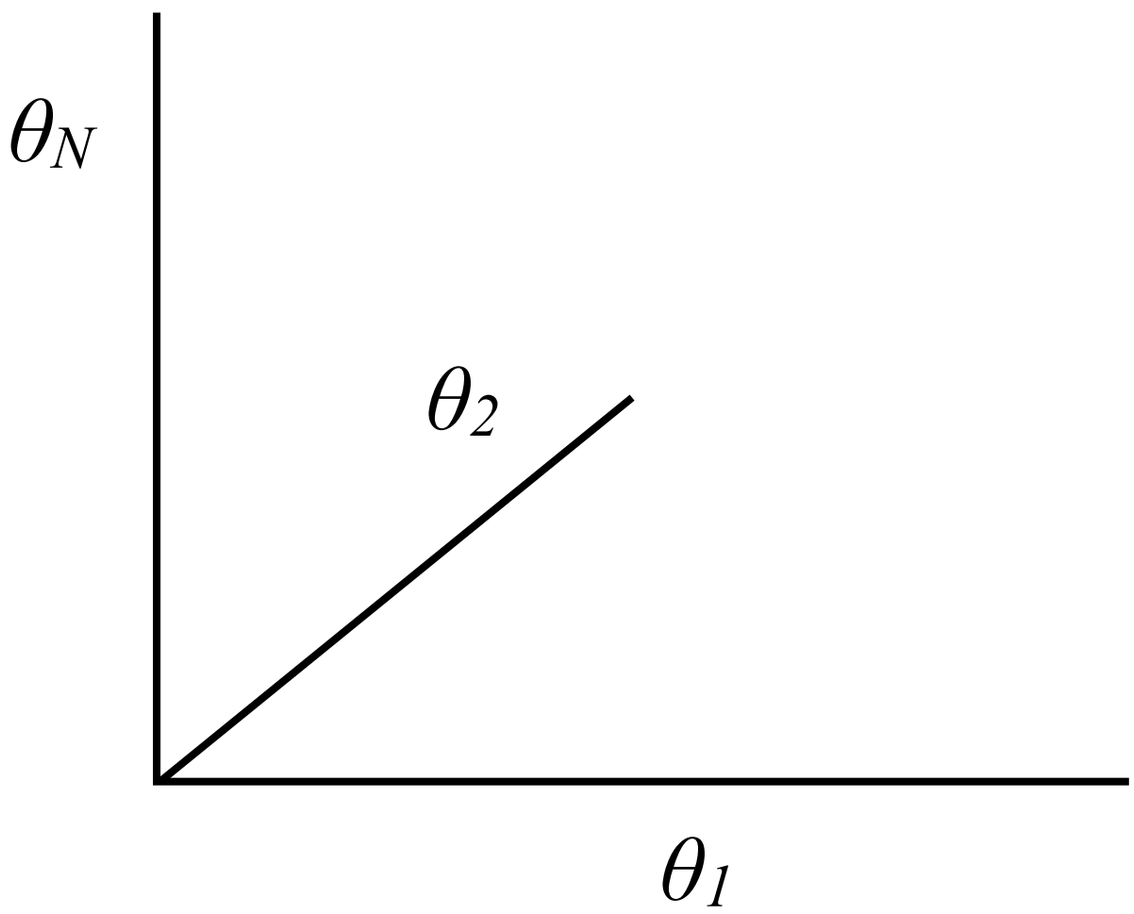
e.g. Buice and Chow, PRE, 76.031118, 2007

Liouville

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

Liouville



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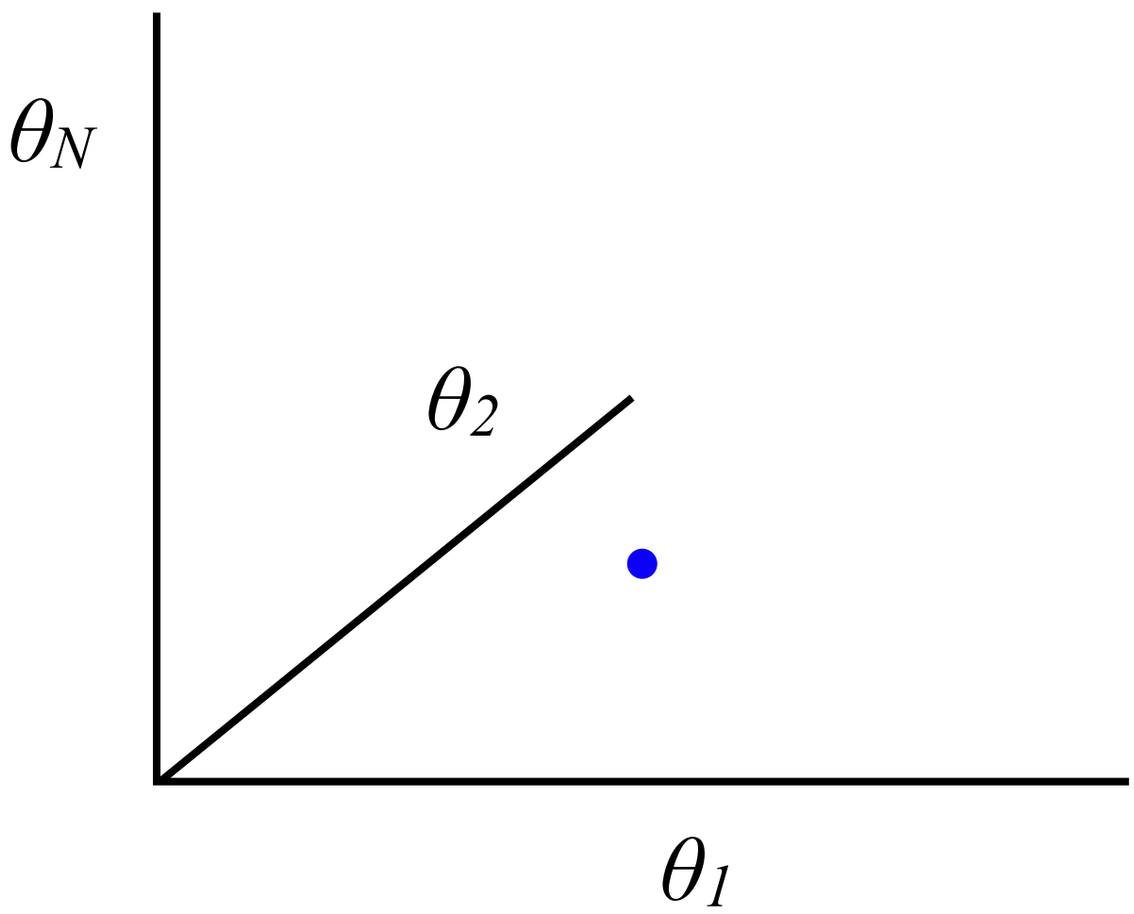

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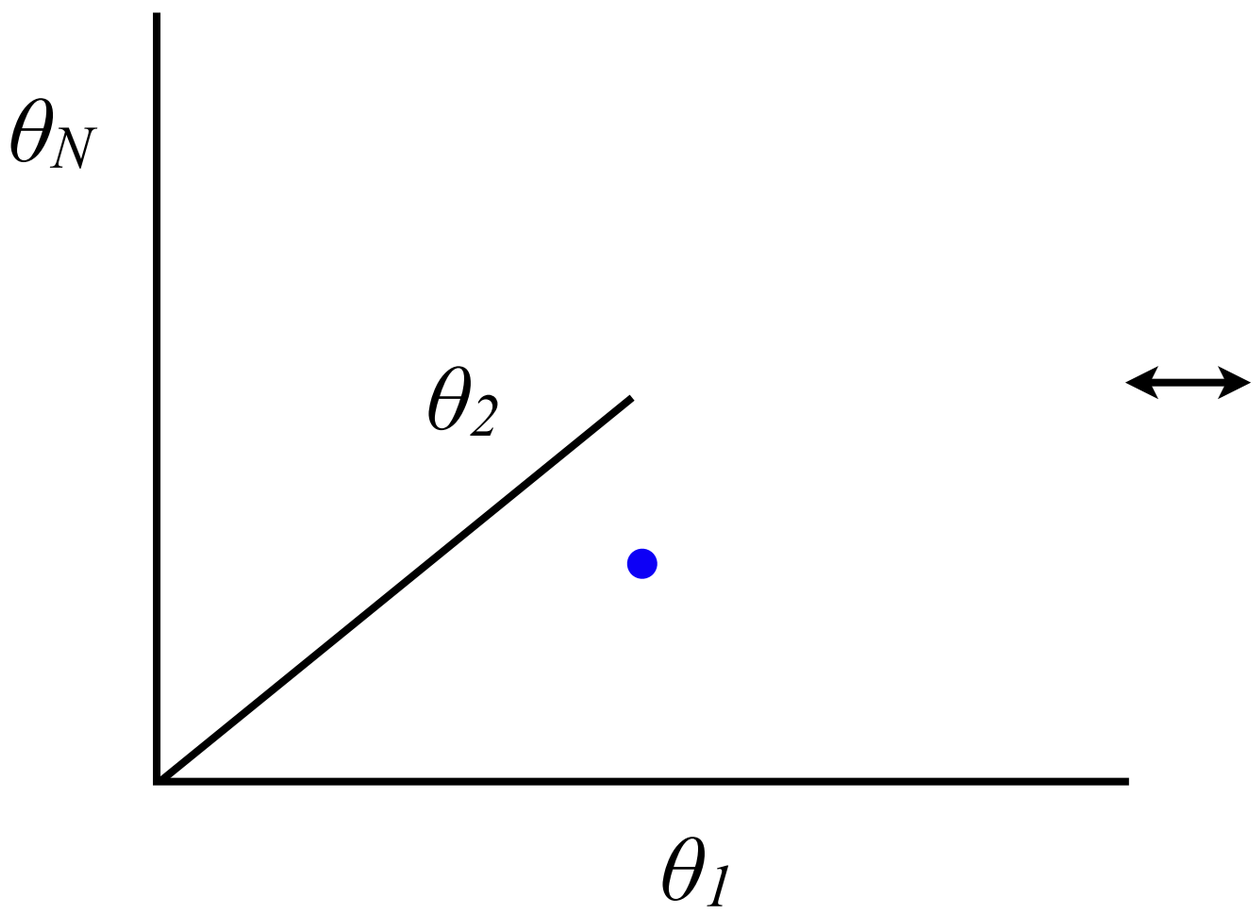

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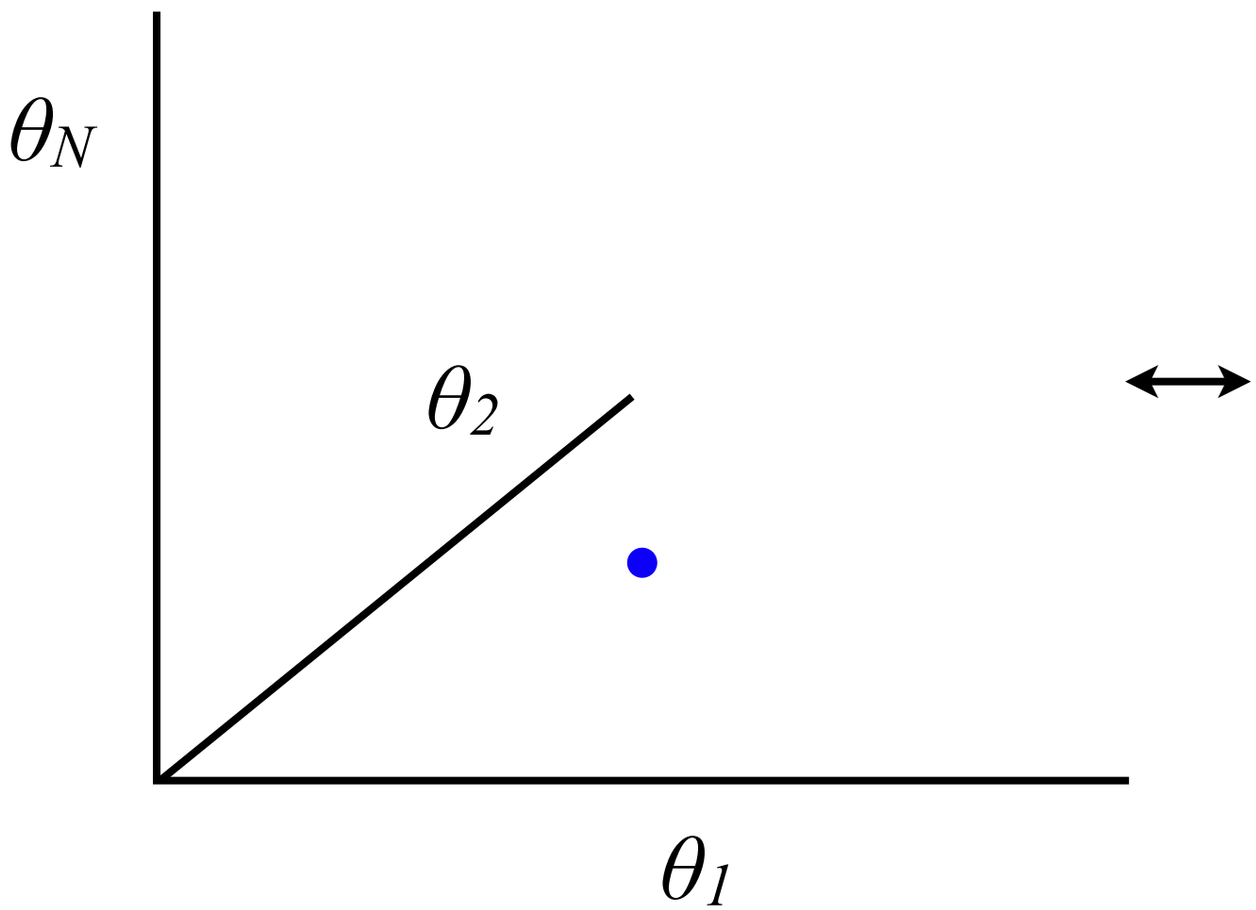
Klimontovich



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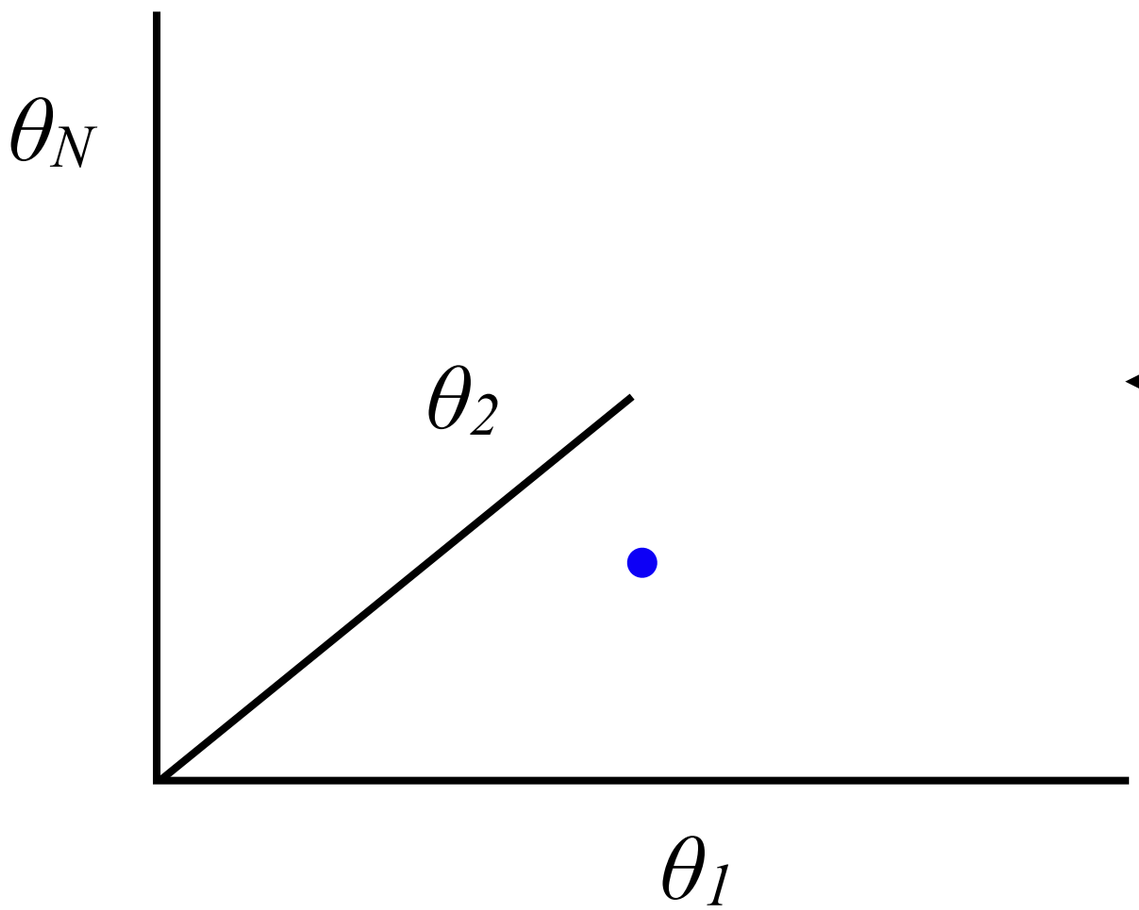
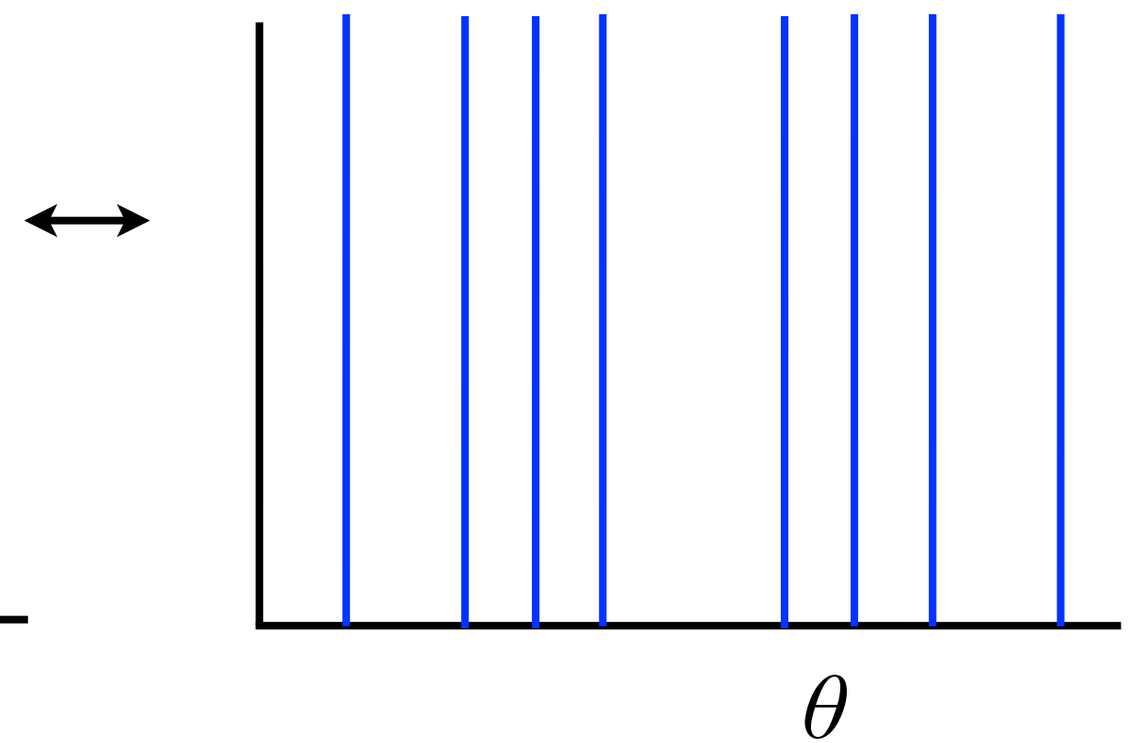


Klimontovich



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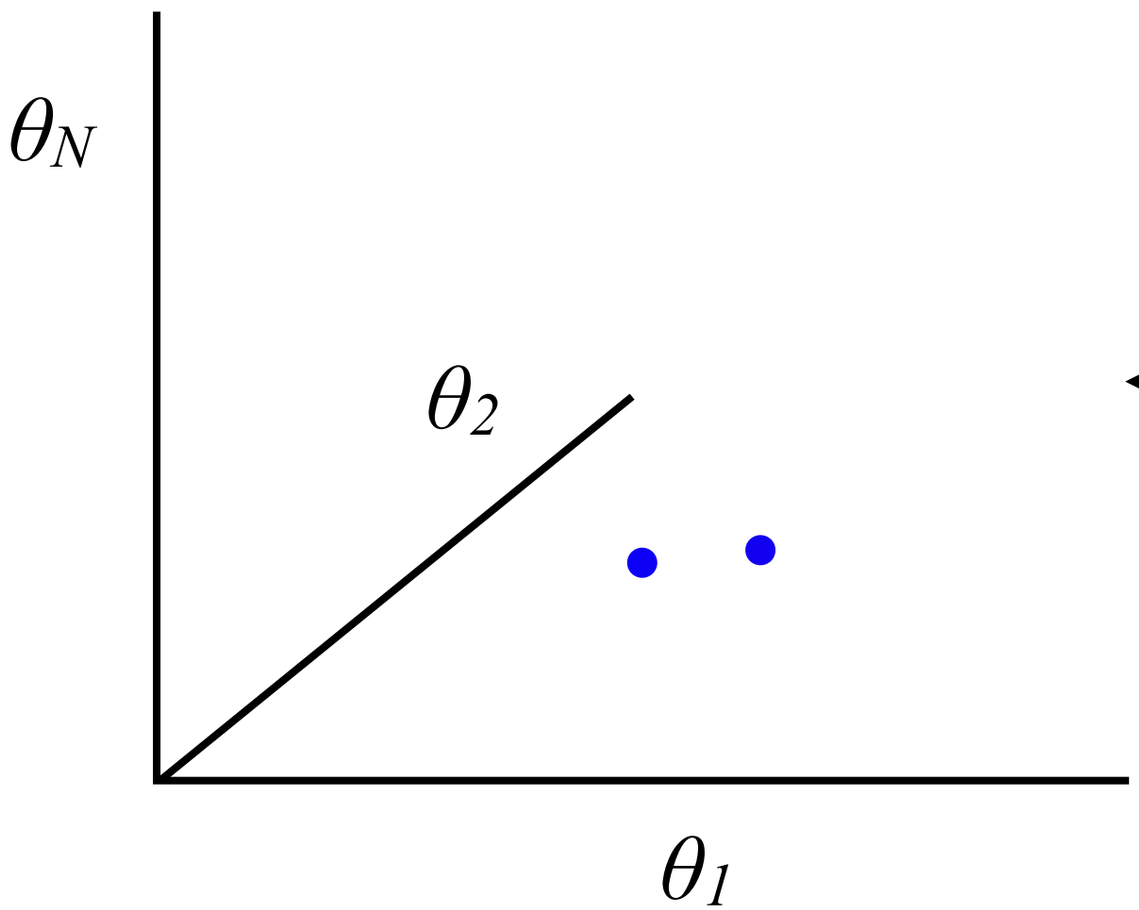

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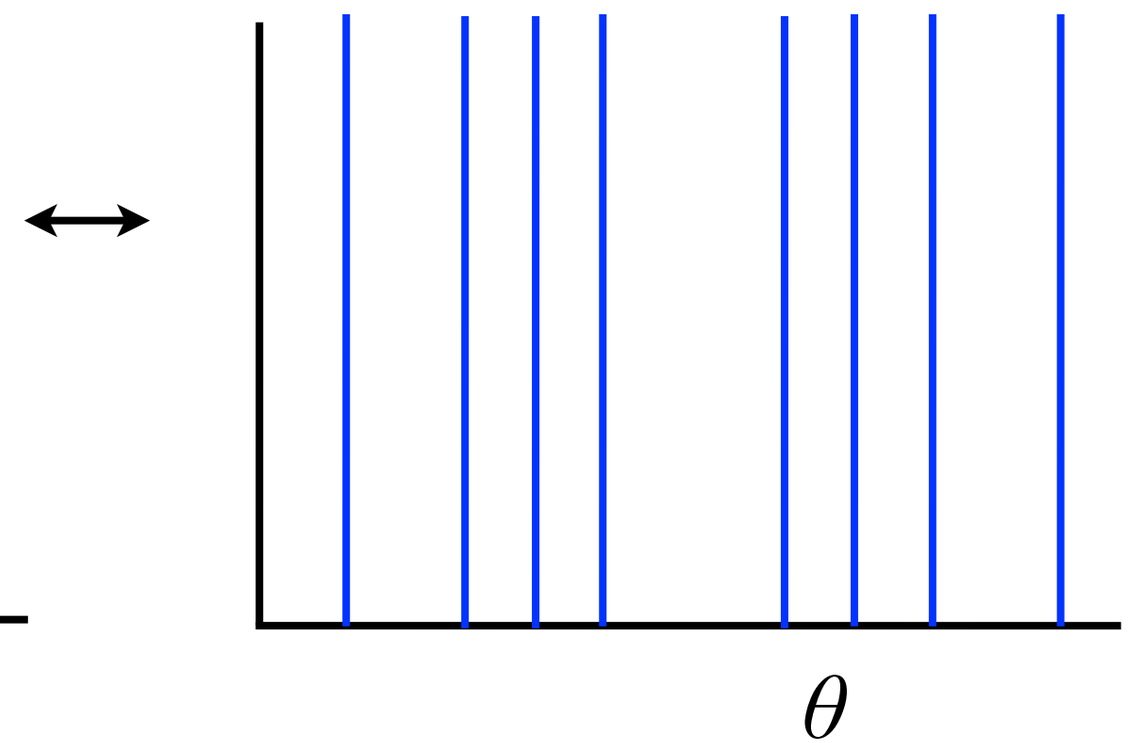

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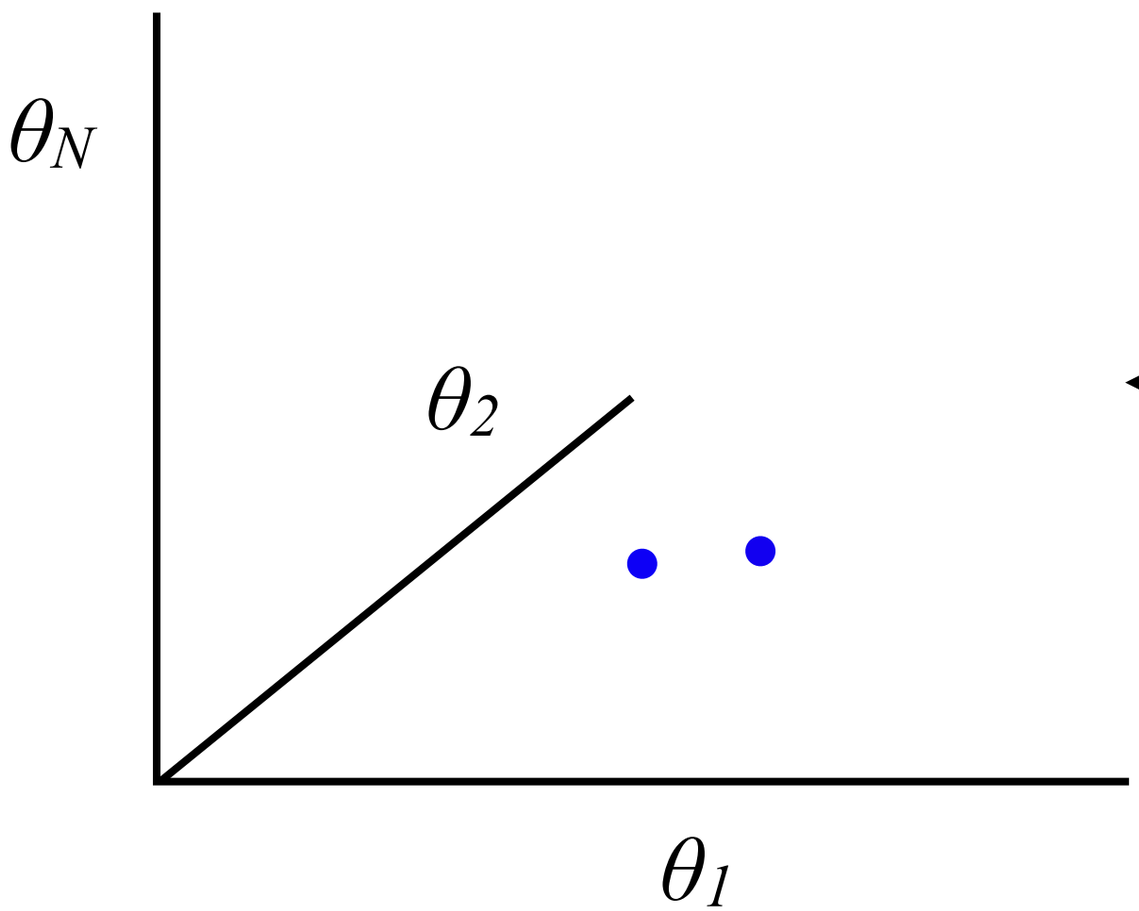

$\longleftrightarrow$

Klimontovich

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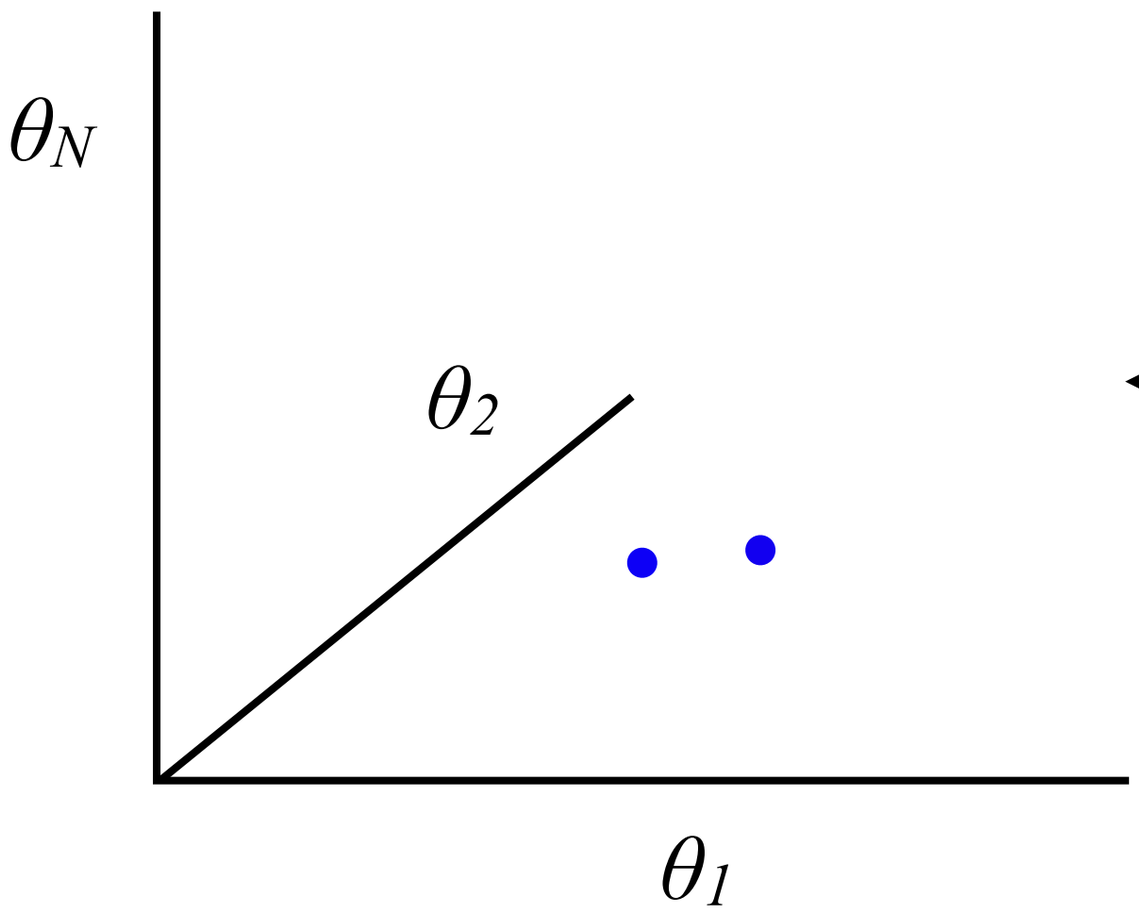
Density functional

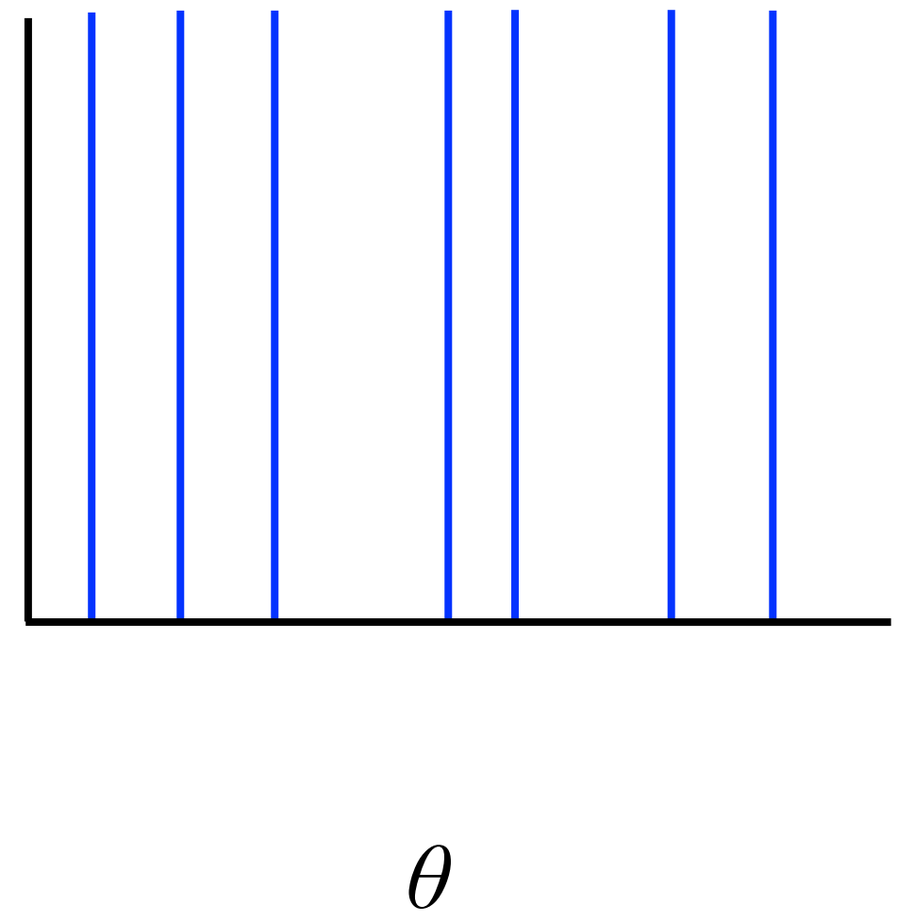
e.g. Buice and Chow, PRE, 76.031118, 2007

Liouville



$\longleftrightarrow$

Klimontovich



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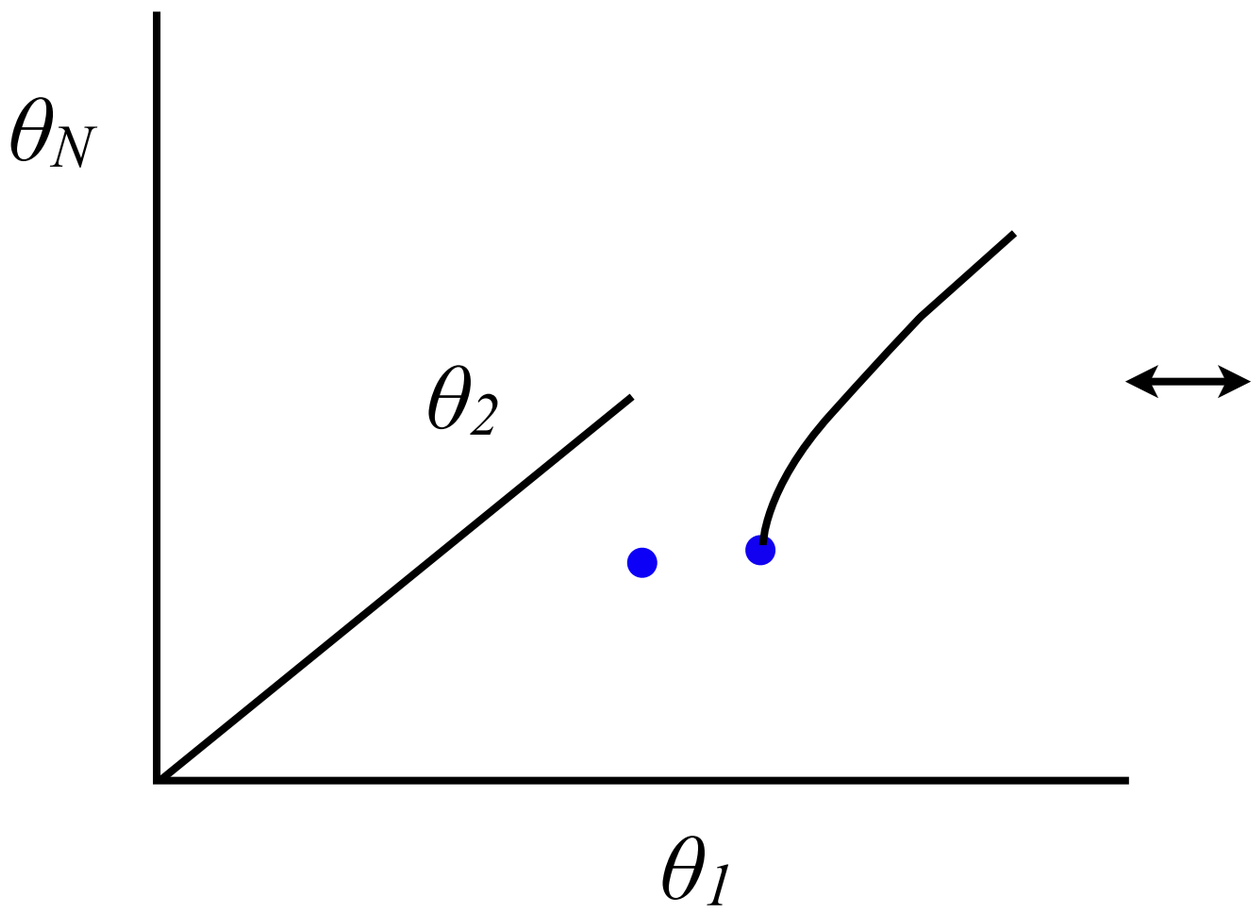
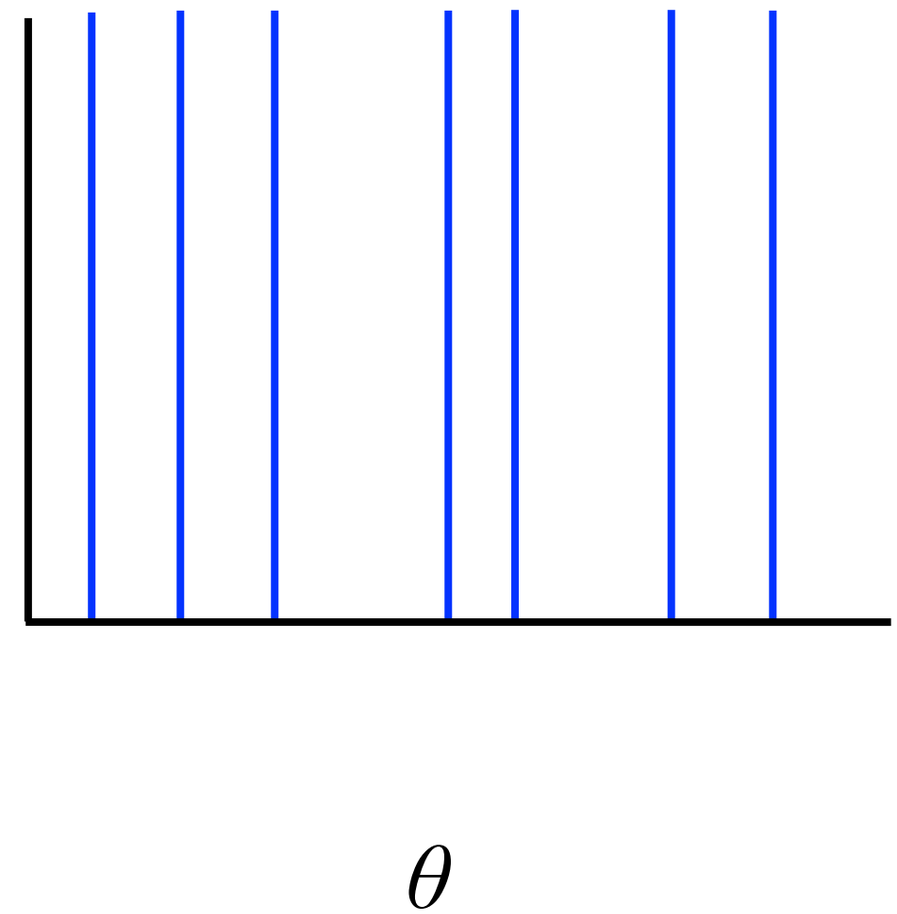


Klimontovich



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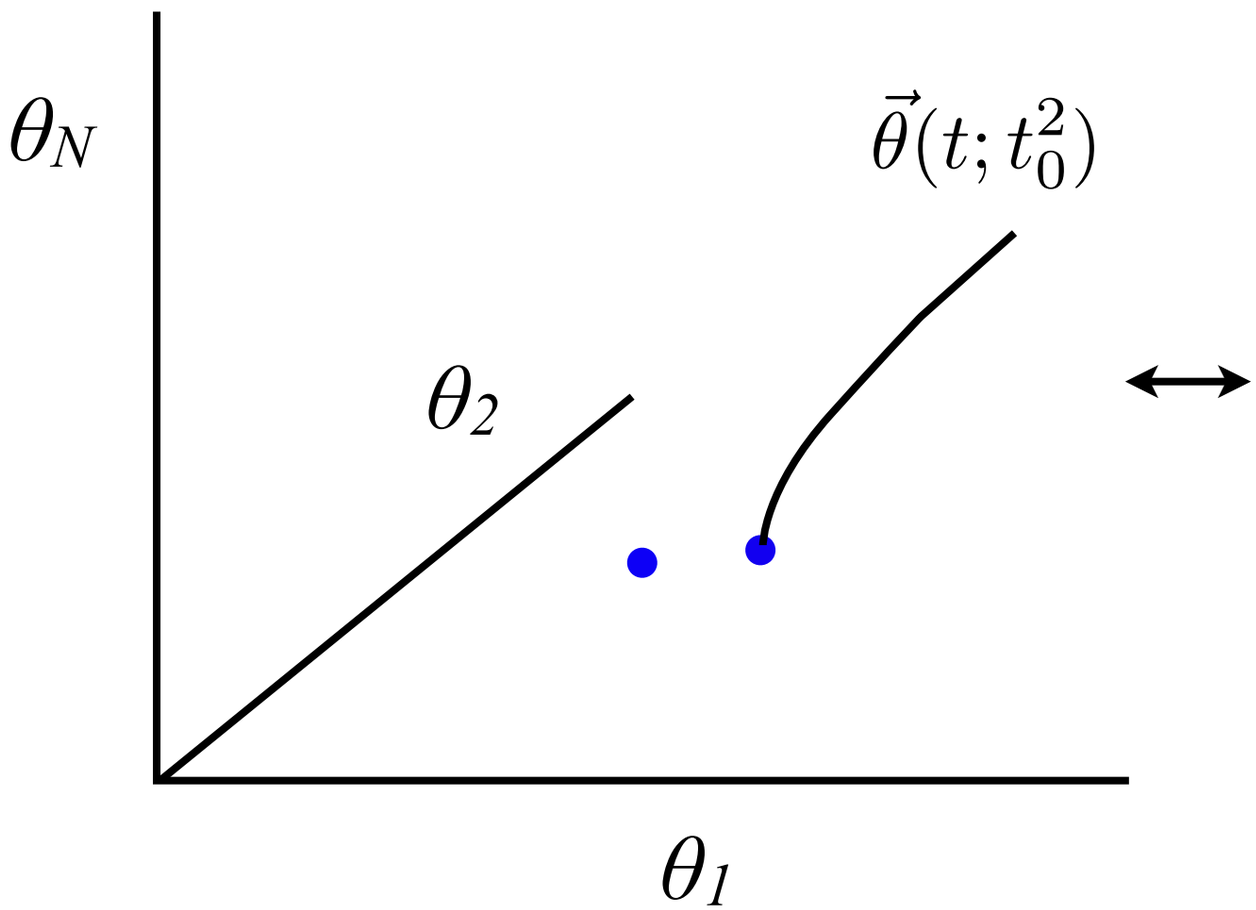

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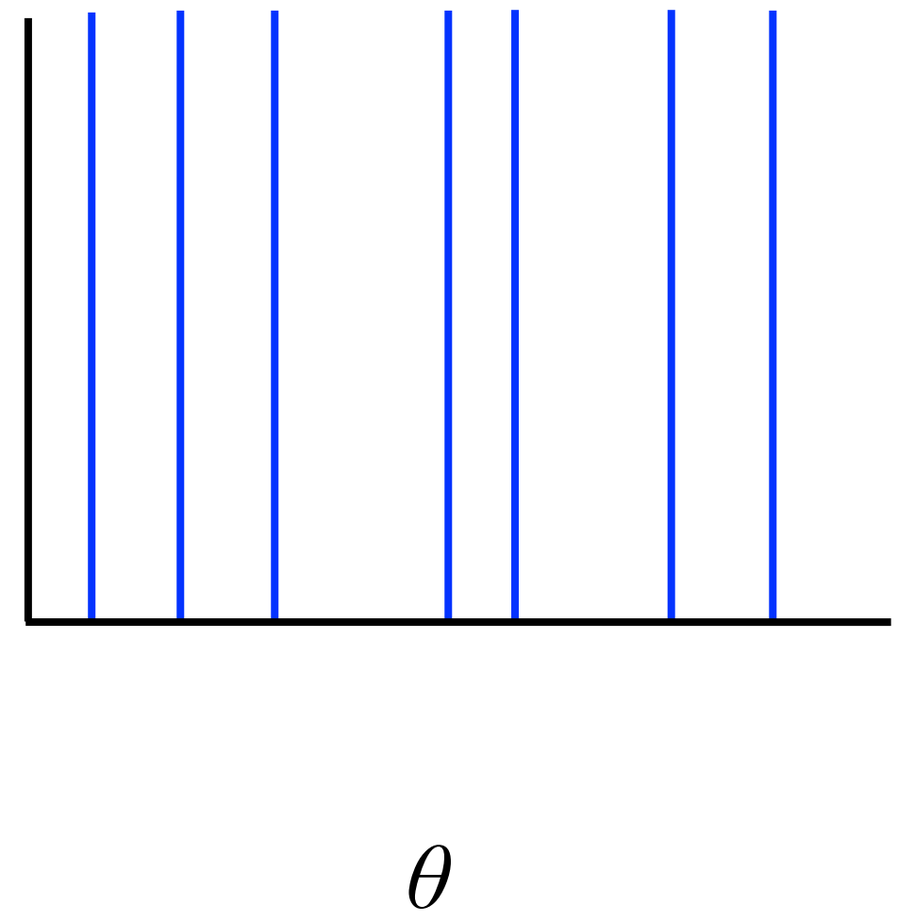

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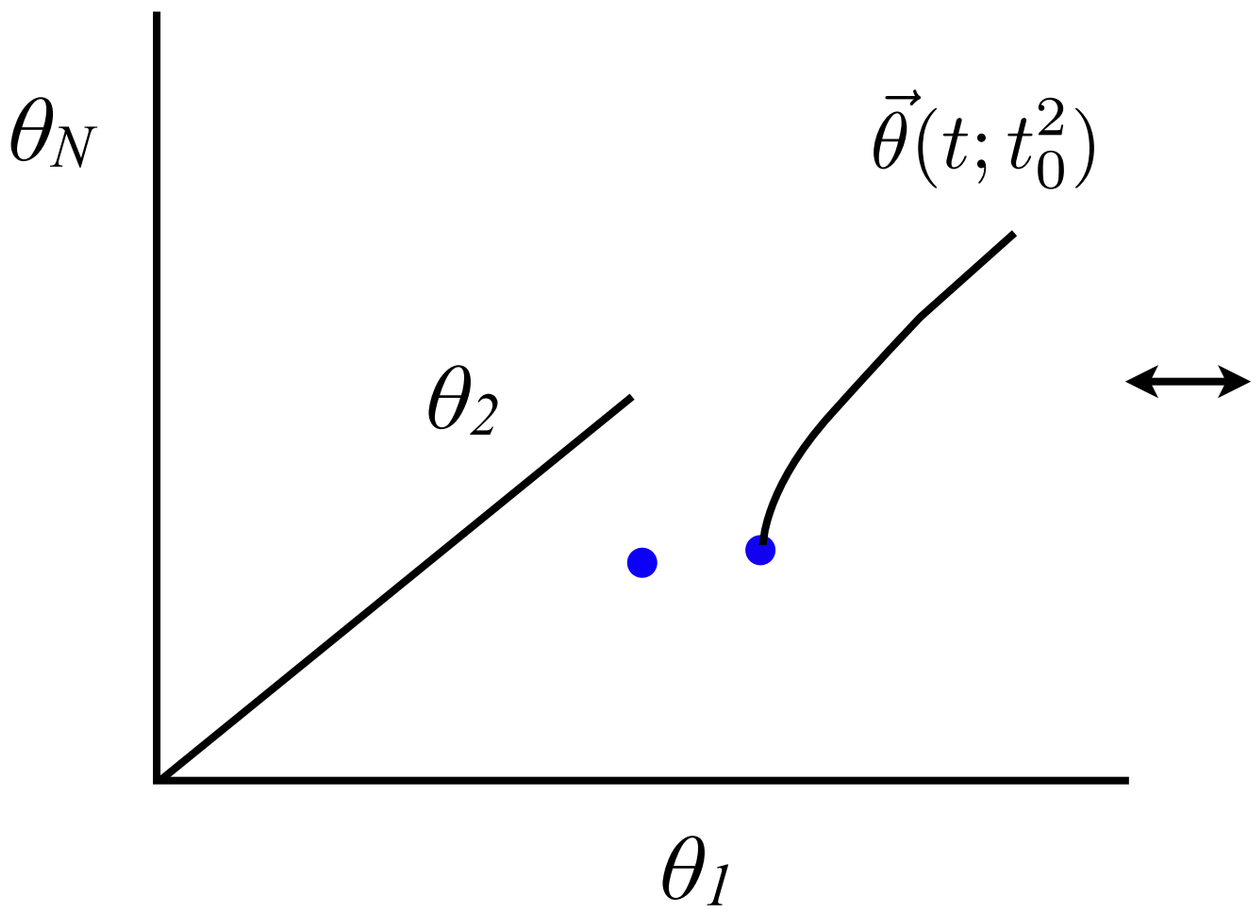

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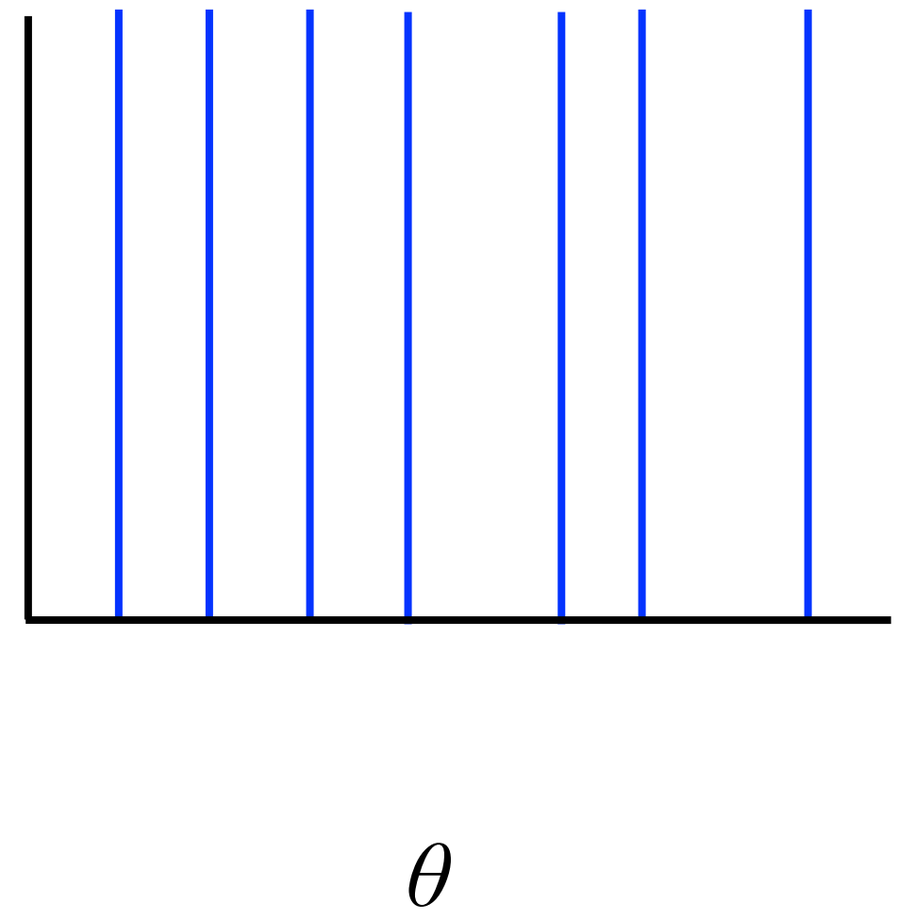

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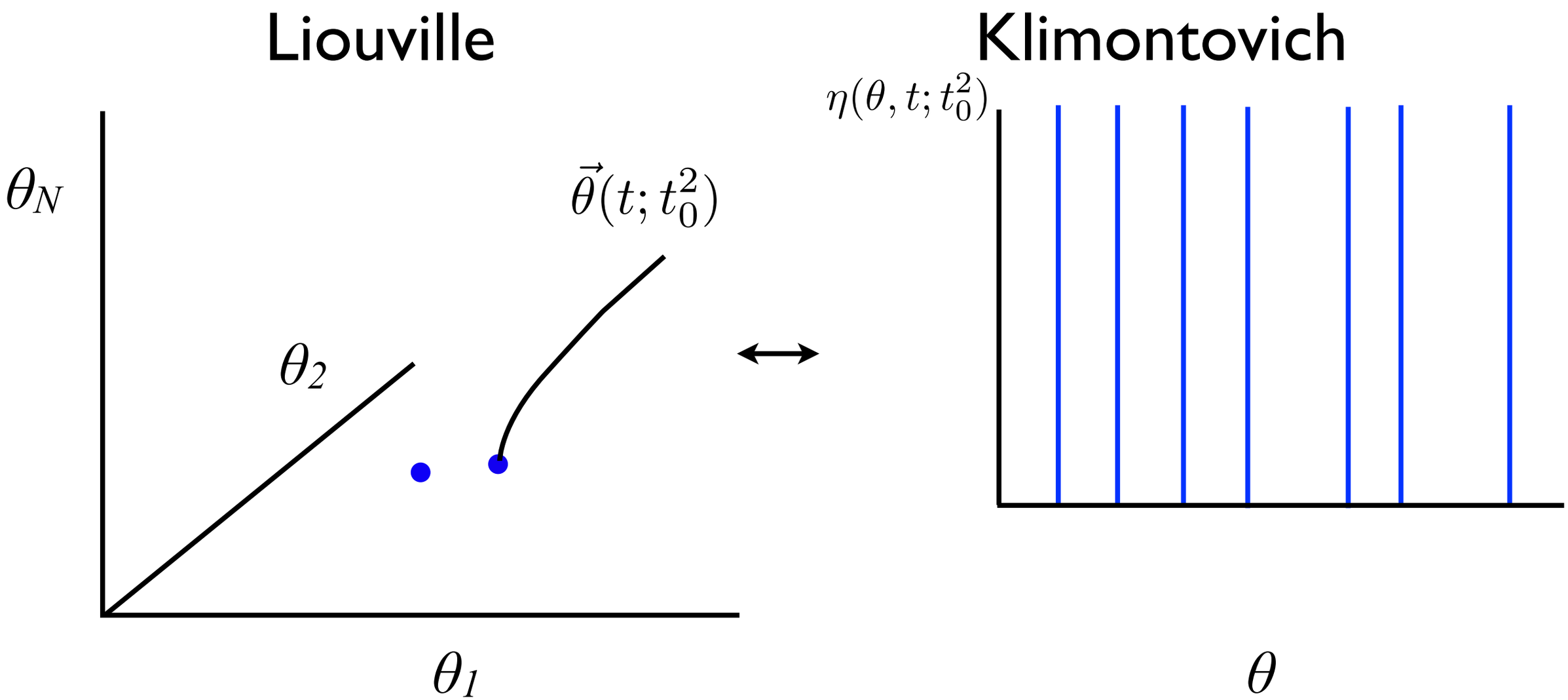

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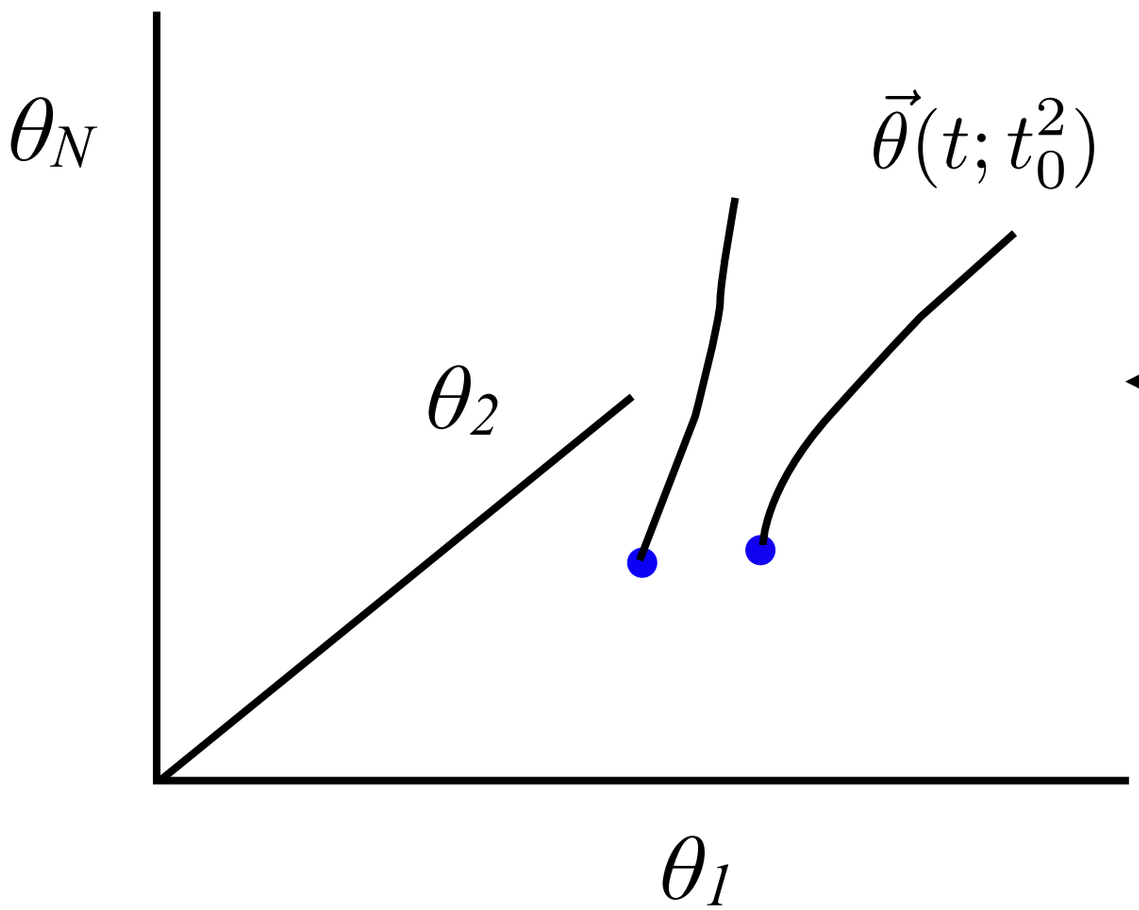

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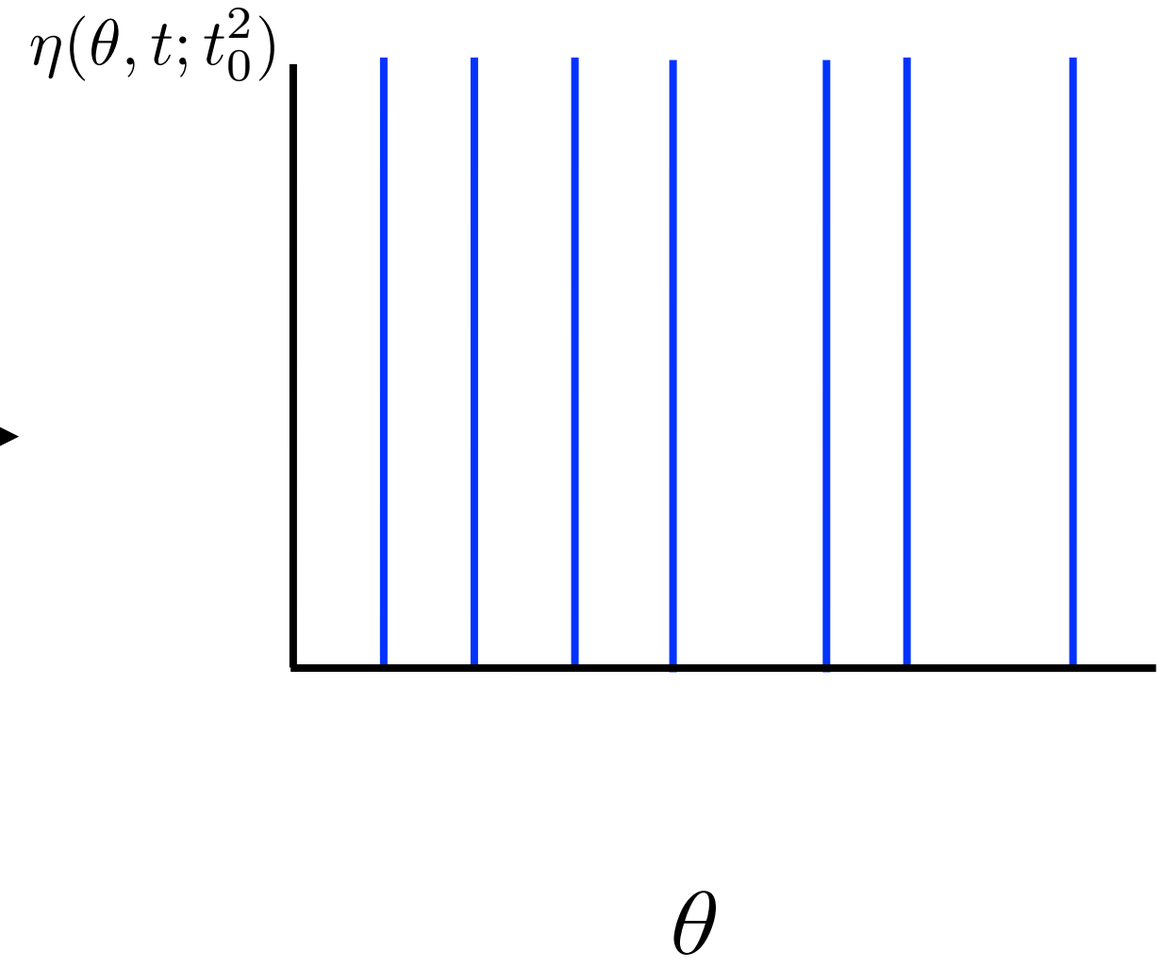

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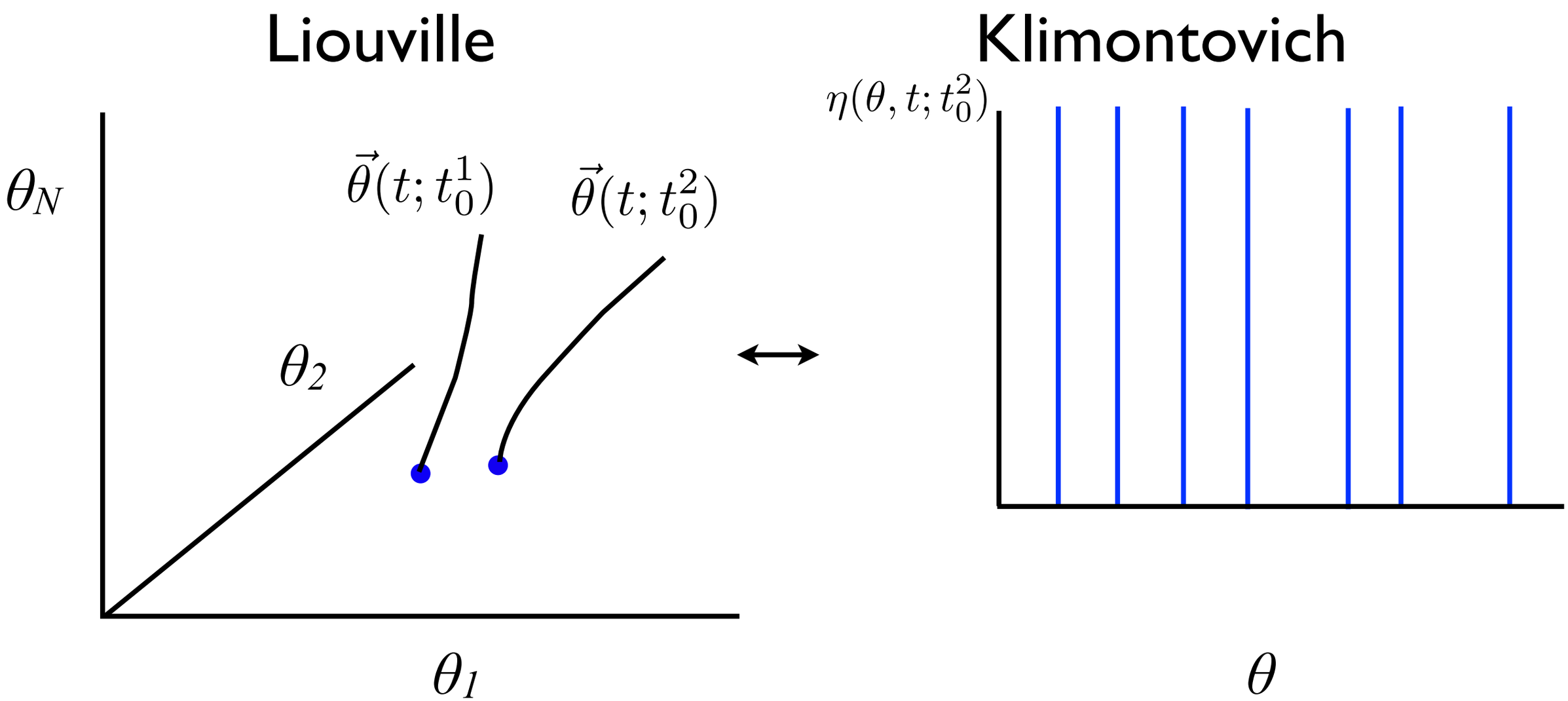

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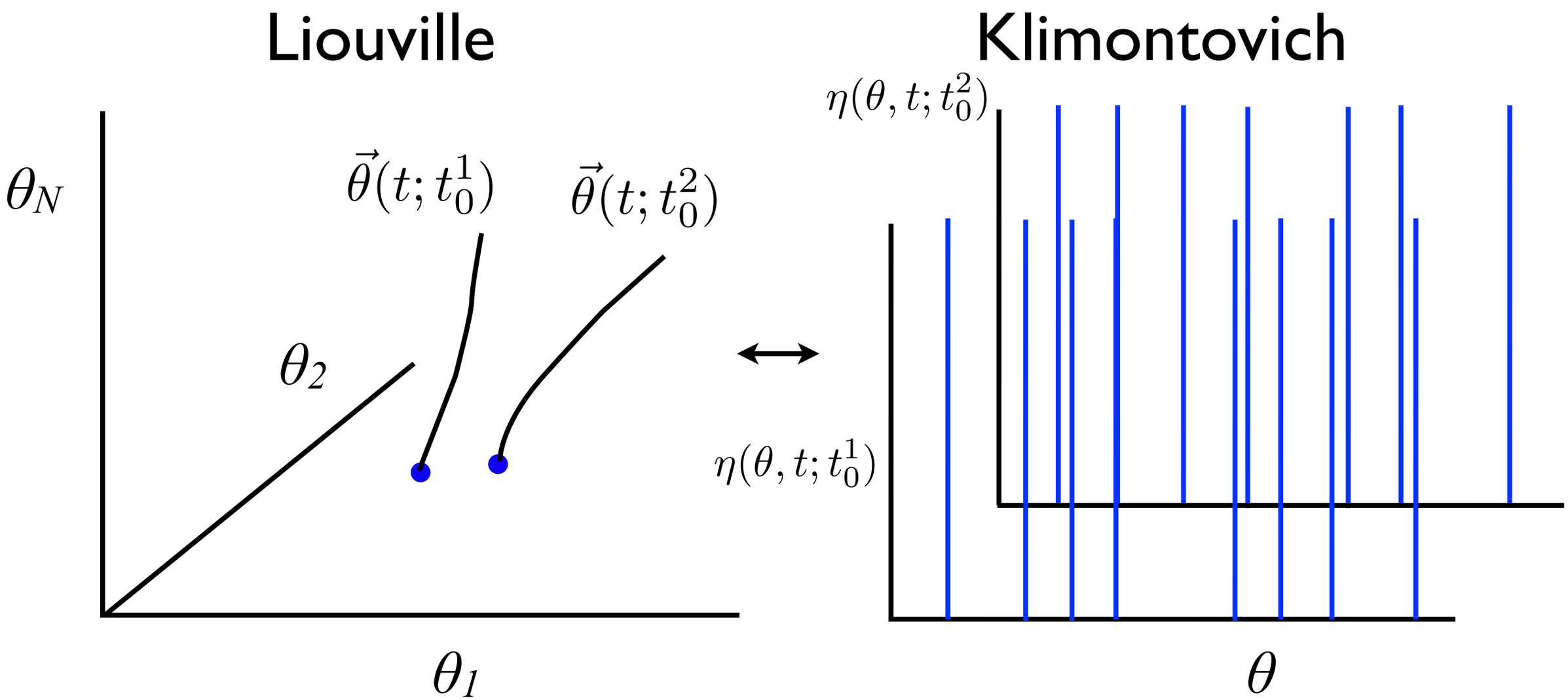

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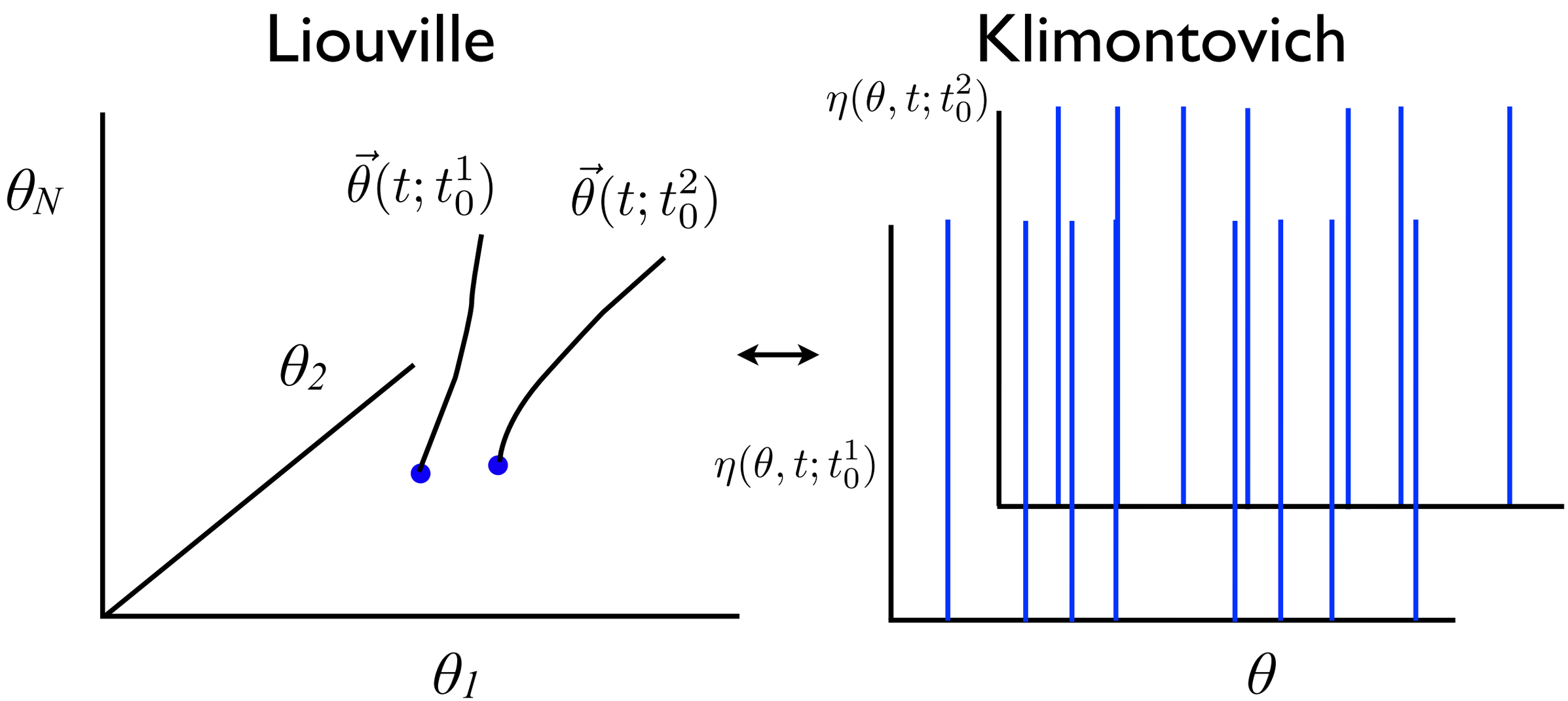
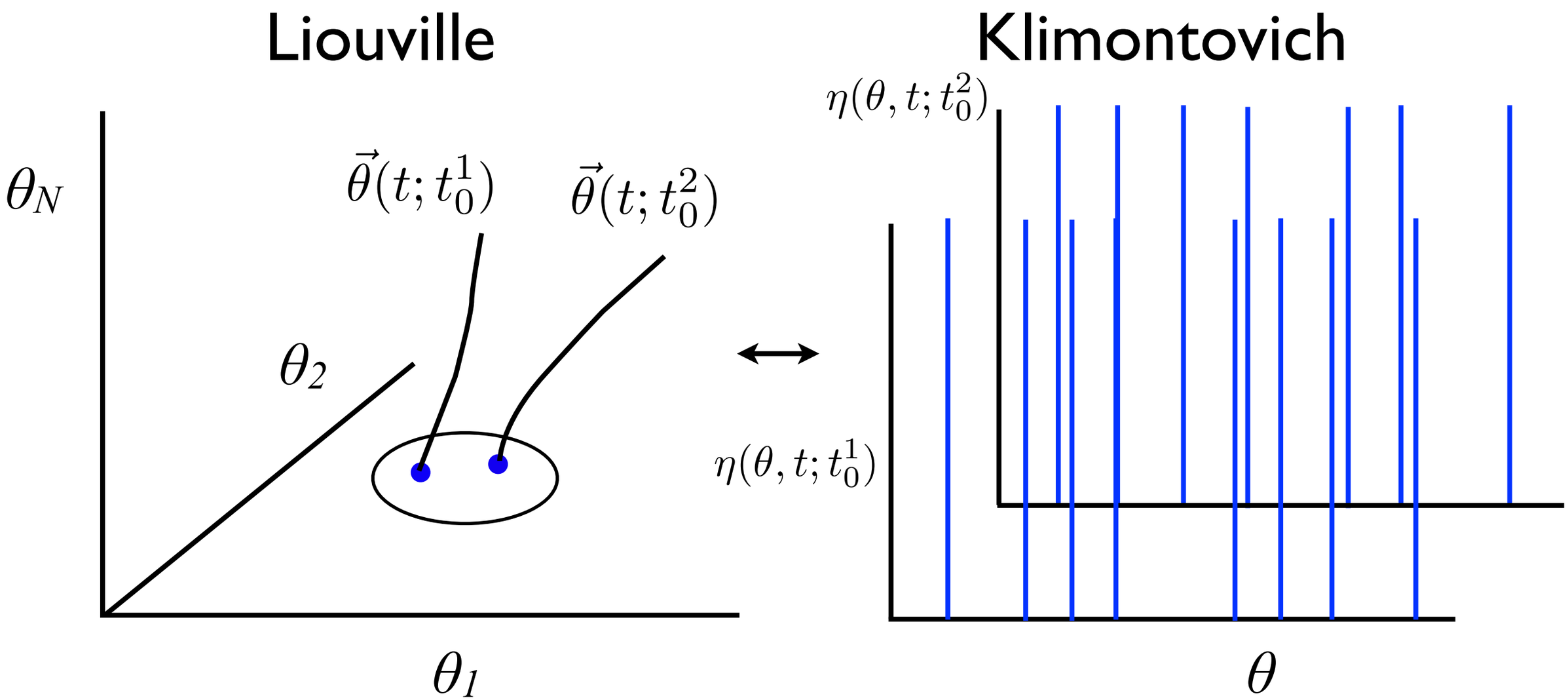


Ensemble of initial data

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007



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Density functional

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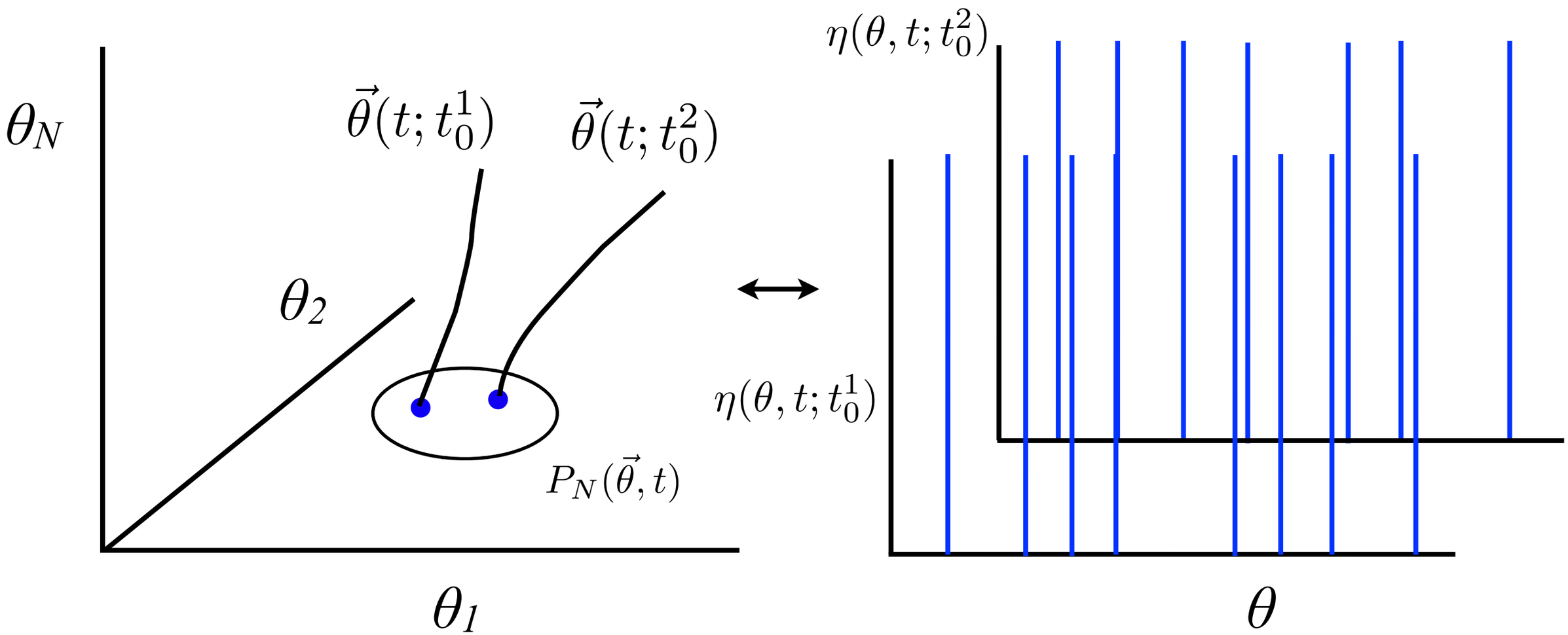
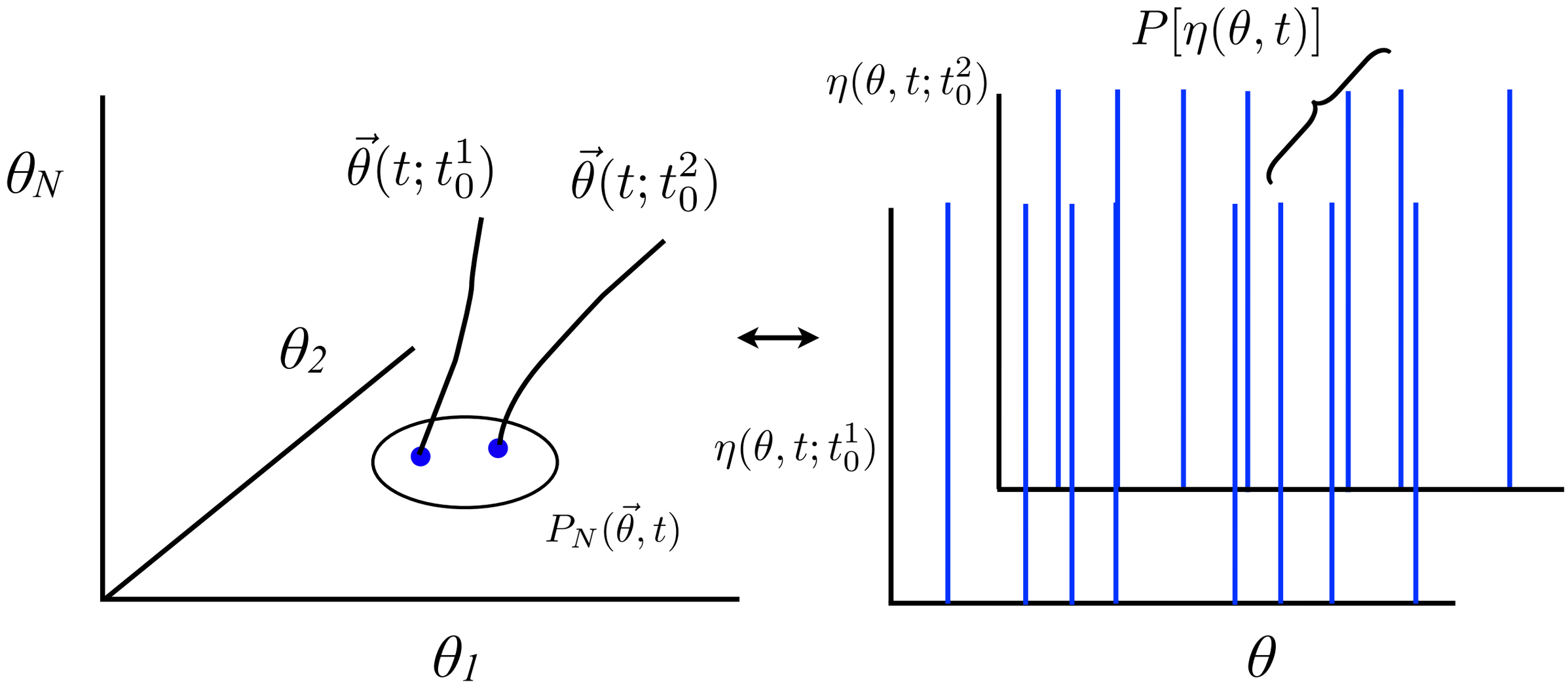


Ensemble of initial data $\Rightarrow$ Ensemble of systems

Density functional

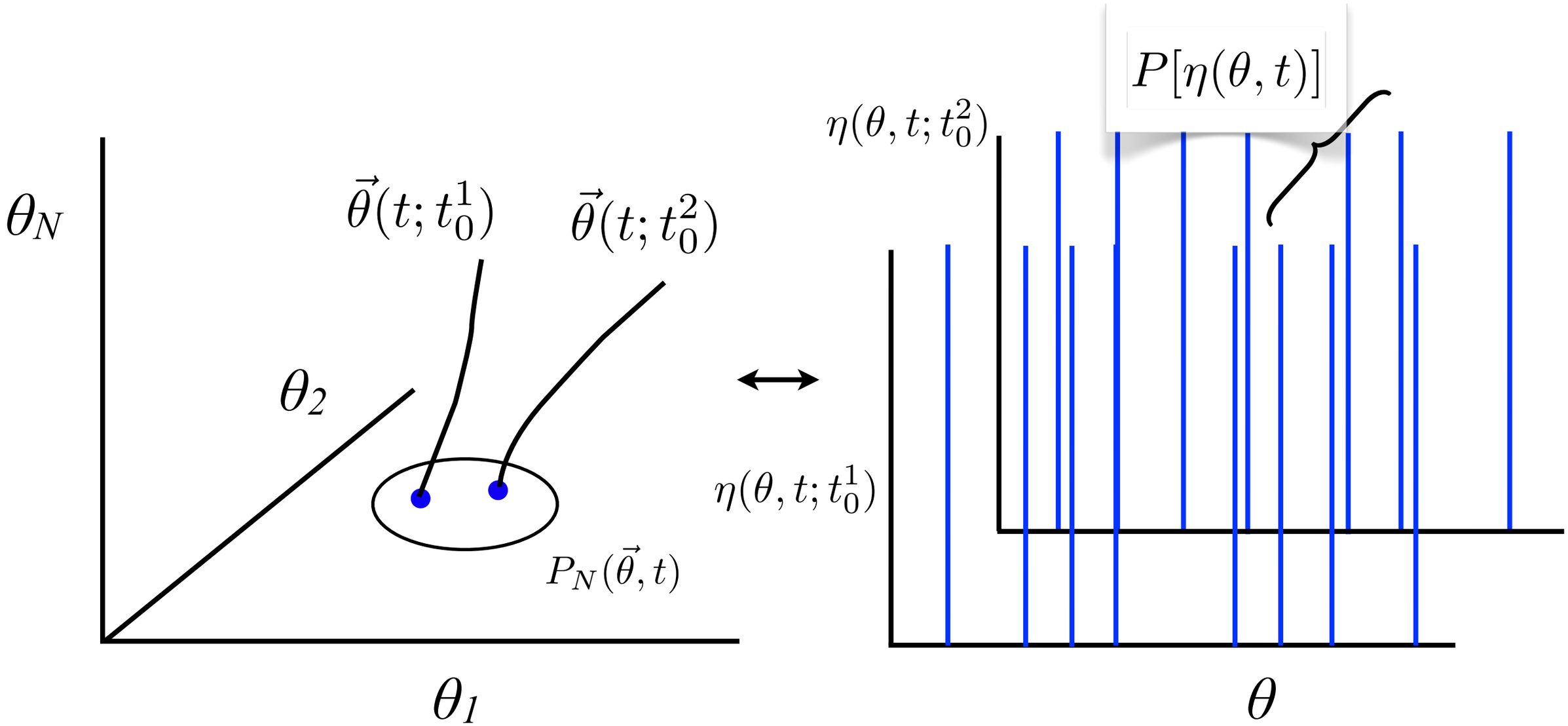
e.g. Buice and Chow, PRE, 76.031118, 2007



Ensemble of initial data

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007



Ensemble of initial data $\Rightarrow$ Density of densities

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

$$\partial_t \eta + \partial_\theta \left[(I(t) + \alpha u(t)) \eta \right] = 0$$

$$\dot{u} + \beta u = \beta (I + \alpha u) \eta(\pi, t)$$

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

$$\partial_t \eta + \partial_\theta \left[(I(t) + \alpha u(t)) \eta \right] = 0$$

$$\dot{u} + \beta u = \beta (I + \alpha u) \eta(\pi, t)$$

$$\eta(\theta, t_0) = \eta_0(\theta)$$

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

$$\partial_t \eta + \partial_\theta \left[(I(t) + \alpha u(t)) \eta \right] = 0$$

$$\dot{u} + \beta u = \beta (I + \alpha u) \eta(\pi, t)$$

$$\eta(\theta, t_0) = \eta_0(\theta) \quad u(t_0) = u_0$$

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

$$\left.\begin{aligned} &\partial_t \eta + \partial_\theta \left[(I(t) + \alpha u(t))\eta \right] = 0 \\ &\dot{u} + \beta u = \beta (I + \alpha u)\eta(\pi, t) \\ &\eta(\theta, t_0) = \eta_0(\theta) \quad u(t_0) = u_0 \end{aligned}\right\} \quad \mathcal{L}(u, \eta) = 0$$

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

$$\left.\begin{aligned} &\partial_t \eta + \partial_\theta \left[(I(t) + \alpha u(t))\eta \right] = 0 \\ &\dot{u} + \beta u = \beta (I + \alpha u)\eta(\pi, t) \\ &\eta(\theta, t_0) = \eta_0(\theta) \quad u(t_0) = u_0 \end{aligned}\right\} \quad \mathcal{L}(u, \eta) = 0$$

$$P[u, \eta] \propto \delta[\mathcal{L}(u, v) = 0]$$

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

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$$P[u, \eta] \propto \delta[\mathcal{L}(u, v) = 0]$$

Density of the density

$$\delta(x) = \int e^{ikx} dk$$

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$$P[u,\eta] = \delta[\mathcal{L}(u,v) = 0] \propto \int \mathcal{D}\tilde{u}\mathcal{D}\tilde{\eta}\, e^{-S[u,\tilde{u},\eta,\tilde{\eta}]}$$

$$\delta(x) = \int e^{ikx} dk$$

Action

$$P[u,\eta] = \delta[\mathcal{L}(u,v) = 0] \propto \int \mathcal{D}\tilde{u}\mathcal{D}\tilde{\eta}\, e^{-S[u,\tilde{u},\eta,\tilde{\eta}]}$$

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Path or functional integral

$$\delta(x) = \int e^{ikx} dk$$

Action

$$P[u,\eta] = \delta[\mathcal{L}(u,v) = 0] \propto \int \mathcal{D}\tilde{u}\mathcal{D}\tilde{\eta}\, e^{-S[u,\tilde{u},\eta,\tilde{\eta}]}$$

Path or functional integral

$$S[u,\tilde{u},\eta,\tilde{\eta}] = N \int dt d\theta\, \tilde{\eta}(\theta,t)\left(\partial_t \eta + \partial_\theta[(I+\alpha u)\eta]\right)$$

$$\delta(x) = \int e^{ikx} dk$$

Action

$$P[u,\eta] = \delta[\mathcal{L}(u,v) = 0] \propto \int \mathcal{D}\tilde{u}\mathcal{D}\tilde{\eta}\, e^{-S[u,\tilde{u},\eta,\tilde{\eta}]}$$

Path or functional integral

$$S[u,\tilde{u},\eta,\tilde{\eta}] = N \int dt d\theta\, \tilde{\eta}(\theta,t)\left(\partial_t \eta + \partial_\theta[(I+\alpha u)\eta]\right)$$

$$+ \int dt\, \tilde{u}\left(\dot{u} + \beta u - \beta[I+\alpha u]\eta(\pi,t)\right)$$

$$\delta(x) = \int e^{ikx} dk$$

Action

$$P[u,\eta] = \delta[\mathcal{L}(u,v) = 0] \propto \int \mathcal{D}\tilde{u}\mathcal{D}\tilde{\eta}\, e^{-S[u,\tilde{u},\eta,\tilde{\eta}]}$$

Path or functional integral

$$S[u,\tilde{u},\eta,\tilde{\eta}] = N \int dt d\theta\, \tilde{\eta}(\theta,t)\left(\partial_t \eta + \partial_\theta[(I+\alpha u)\eta]\right)$$

$$+ \int dt\, \tilde{u}\left(\dot{u} + \beta u - \beta[I+\alpha u]\eta(\pi,t)\right) \quad - \ln Z_0$$

e.g.

e.g. $$\langle v^2 \rangle = \int \mathcal{D}\tilde{v}\mathcal{D}v\, v^2 e^{-N\int dx \tilde{v}\Delta^{-1}v + a\tilde{v}^3 + b\tilde{v}v^2}$$

e.g. $\langle v^2 \rangle = \int \mathcal{D}\tilde{v}\mathcal{D}v\, v^2 e^{-N\int dx \tilde{v}\Delta^{-1}v + a\tilde{v}^3 + b\tilde{v}v^2}$

Steepest descent expansion in *1/N*

e.g. $\langle v^2 \rangle = \int \mathcal{D}\tilde{v}\mathcal{D}v\, v^2 e^{-N\int dx \tilde{v}\Delta^{-1}v + a\tilde{v}^3 + b\tilde{v}v^2}$

Steepest descent expansion in *1/N*

$$= \int \mathcal{D}\tilde{v}\mathcal{D}v\, e^{-N\int \tilde{v}\Delta^{-1}v} v^2 \int dx [1 + a\tilde{v}^3 + b\tilde{v}v^2 + \cdots]$$

e.g. $\langle v^2 \rangle = \int \mathcal{D}\tilde{v}\mathcal{D}v\, v^2 e^{-N\int dx \tilde{v}\Delta^{-1}v + a\tilde{v}^3 + b\tilde{v}v^2}$

Steepest descent expansion in *1/N*

$$= \int \mathcal{D}\tilde{v}\mathcal{D}v\, e^{-N\int \tilde{v}\Delta^{-1}v} v^2 \int dx [1 + a\tilde{v}^3 + b\tilde{v}v^2 + \cdots]$$

Moments in terms of combinations of Δ's

Linear Response

$$
\begin{aligned}
\Delta_u^u(t;t') &= \langle \delta u(t)\tilde{u}(t')\rangle \\
\Delta_u^\psi(t;\theta',t') &= \langle \delta u(t)\tilde{\psi}(\theta',t')\rangle \\
\Delta_\psi^u(\theta,t;t') &= \langle \psi(\theta,\vec{\alpha},t)\tilde{u}(t')\rangle \\
\Delta_\psi^\psi(\theta,t;\theta',t') &= \langle \psi(\theta,t)\tilde{\psi}(\theta',t')\rangle
\end{aligned}
$$

$$
\begin{aligned}
\psi(\theta,t) &= \eta(\theta,t)e^{-\tilde{\eta}(\theta,t)} - \rho(\theta,t) \\
\tilde{\psi}(\theta,t) &= e^{\tilde{\eta}(\theta,t)} - 1
\end{aligned}
$$

Linear Response

$$\left(\frac{d}{dt}+\beta\right)\Delta_u^u - \beta\rho(\pi,t)\Delta_u^u - \beta(I+\alpha\bar{u})\Delta_\psi^u = \delta(t-t')$$

$$\left(\frac{d}{dt}+\beta\right)\Delta_u^\psi - \beta\rho(\pi,t)\Delta_u^\psi - \beta(I+\alpha\bar{u})\Delta_\psi^\psi = 0$$

$$\partial_t\Delta_\psi^u + \partial_\theta\left[(I+\alpha\bar{u})\Delta_\psi^u\right] + \partial_\theta\rho\Delta_u^u = 0$$

$$\partial_t\Delta_\psi^\psi + \partial_\theta\left[(I+\alpha\bar{u})\Delta_\psi^\psi\right] + \partial_\theta\rho\Delta_u^\psi = \frac{1}{N}\delta(\theta-\theta')\delta(t-t')$$

$$\psi(\theta,t) = \eta(\theta,t)e^{-\tilde{\eta}(\theta,t)} - \rho(\theta,t)$$

$$\tilde{\psi}(\theta,t) = e^{\tilde{\eta}(\theta,t)} - 1$$

Linear Response

$\Delta_u^u =$

$\Delta_u^\psi =$

$\Delta_\psi^u =$

$\Delta_\psi^\psi =$

Steady state

$$\dot{u} = -\beta u + \beta(I + \alpha u)\rho(\pi, t)$$

$$\partial_t \rho = -\partial_\theta \left[(I(t) + \alpha u(t))\rho \right]$$

Steady state

$$\dot{u} = -\beta u + \beta(I + \alpha u)\rho(\pi, t) \ = 0$$

$$\partial_t \rho = -\partial_\theta \left[(I(t) + \alpha u(t))\rho \right]$$

Steady state

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Steady state

$$\dot{u} = -\beta u + \beta(I + \alpha u)\rho(\pi, t) \quad = 0$$

$$\partial_t \rho = -\partial_\theta \left[(I(t) + \alpha u(t))\rho \right] \qquad = 0$$

$$\bar{\rho} = \frac{1}{2\pi} \qquad \bar{u} = \frac{I}{2\pi}\left(1 - \frac{\alpha}{2\pi}\right)^{-1}$$

Steady state

$$\dot{u} = -\beta u + \beta(I + \alpha u)\rho(\pi, t) \quad = 0$$

$$\partial_t \rho = -\partial_\theta \left[(I(t) + \alpha u(t))\rho\right] \qquad = 0$$

$$\bar{\rho} = \frac{1}{2\pi} \qquad \bar{u} = \frac{I}{2\pi}\left(1 - \frac{\alpha}{2\pi}\right)^{-1}$$

$$\nu = (I + \alpha\bar{u})\bar{\rho} = \bar{u}$$

Drive Correlations

$$\langle \delta u(t) \delta u(t') \rangle$$

$$= \quad \beta \int dt'' \; (I + \alpha \bar{u}(t'')) \, \Delta_u^u(t, t'') \Delta_u^\psi(t', \pi, t'') \rho(\pi, \alpha, t'') + (t \leftrightarrow t')$$

$$- \quad \frac{N}{(2\pi)^2} \int d\theta \, \Delta_u^\psi(t, s) \int d\theta' \Delta_u^\psi(t', s')$$

Drive Correlations

$$\langle \delta u(t) \delta u(t') \rangle$$

$$= \quad \beta \int dt'' \; (I + \alpha \bar{u}(t'')) \, \Delta_u^u(t, t'') \Delta_u^\psi(t', \pi, t'') \rho(\pi, \alpha, t'') + (t \leftrightarrow t')$$

$$- \quad \frac{N}{(2\pi)^2} \int d\theta \, \Delta_u^\psi(t, s) \int d\theta' \Delta_u^\psi(t', s') + O\left(\frac{1}{N^2}\right)$$

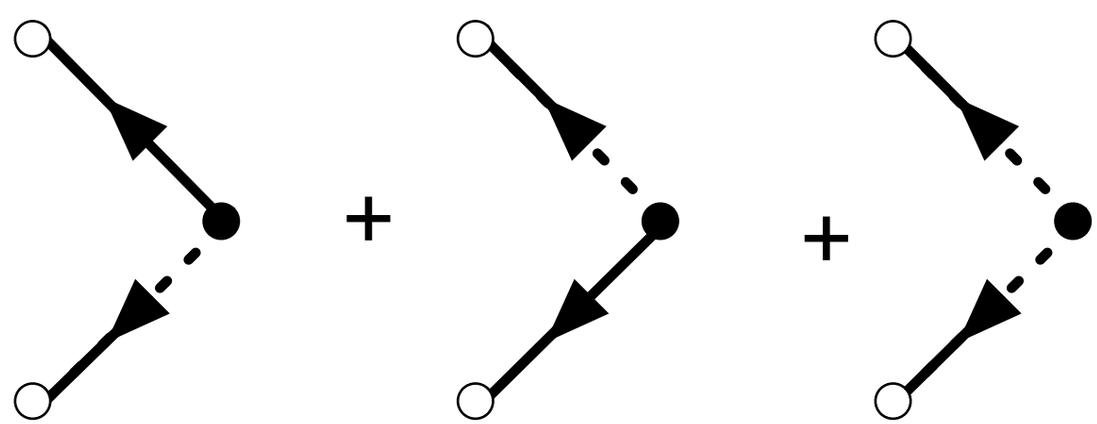
Drive Correlations

$$\langle \delta u(t) \delta u(t') \rangle$$

$$= \quad \beta \int dt'' \; (I + \alpha \bar{u}(t'')) \, \Delta_u^u(t, t'') \Delta_u^\psi(t', \pi, t'') \rho(\pi, \alpha, t'') + (t \leftrightarrow t')$$

$$- \quad \frac{N}{(2\pi)^2} \int d\theta \, \Delta_u^\psi(t, s) \int d\theta' \Delta_u^\psi(t', s') + O\left(\frac{1}{N^2}\right)$$

$=$



Input Correlations

$$
\begin{aligned}
\langle \delta u(t)^2 \rangle &= \frac{1}{N} \sum_{k=0}^{\infty} \left(1 - \frac{1}{2} \delta_{k,0} \right) \frac{\beta^2}{\pi \delta} \left(I + \alpha \bar{u}_0 \right) \\
&\quad \times e^{-\beta \delta \Delta t_k} \left[1 - e^{-2\beta\delta(t - t_0 - \Delta t_k)} \right] H(t - t_0 - \Delta t_k) \\
&\quad - \frac{1}{N} \bar{u}_0^2 \left(1 - e^{-\beta\delta(t - t_0)} \right)^2
\end{aligned}
$$

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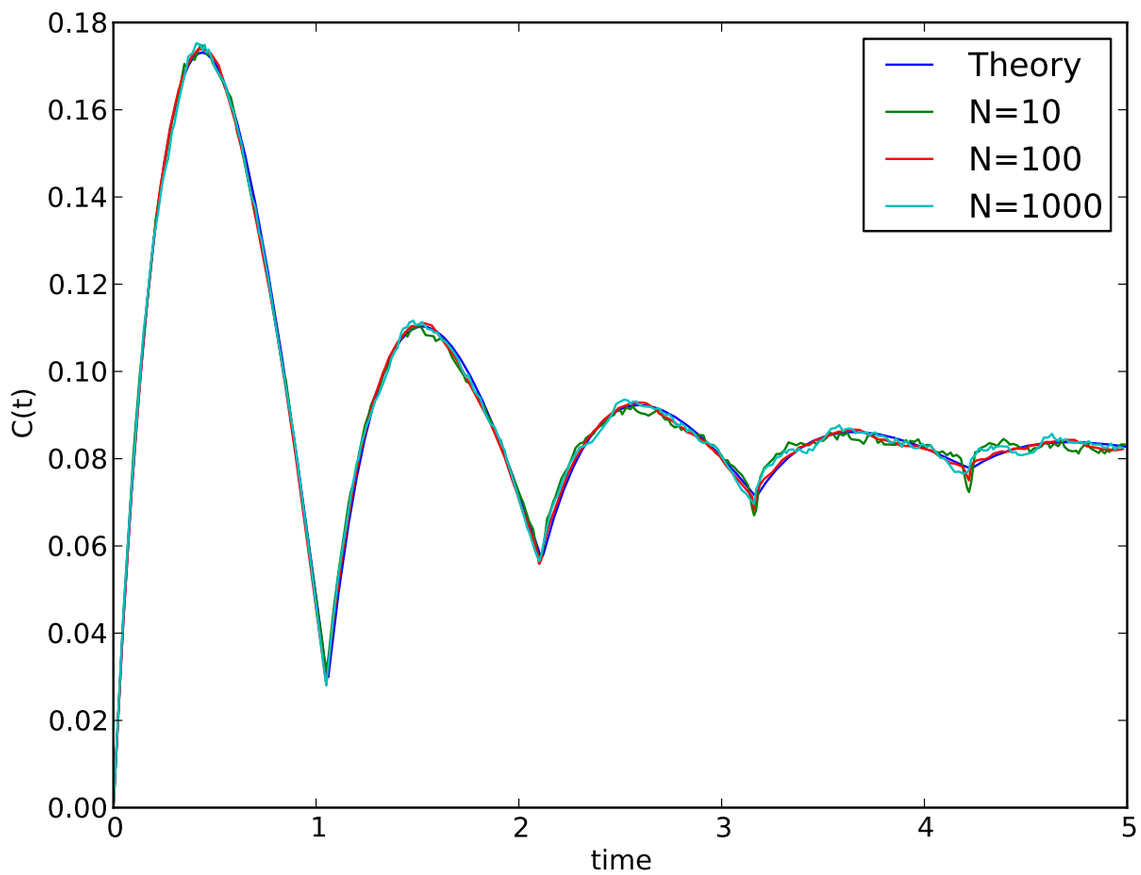
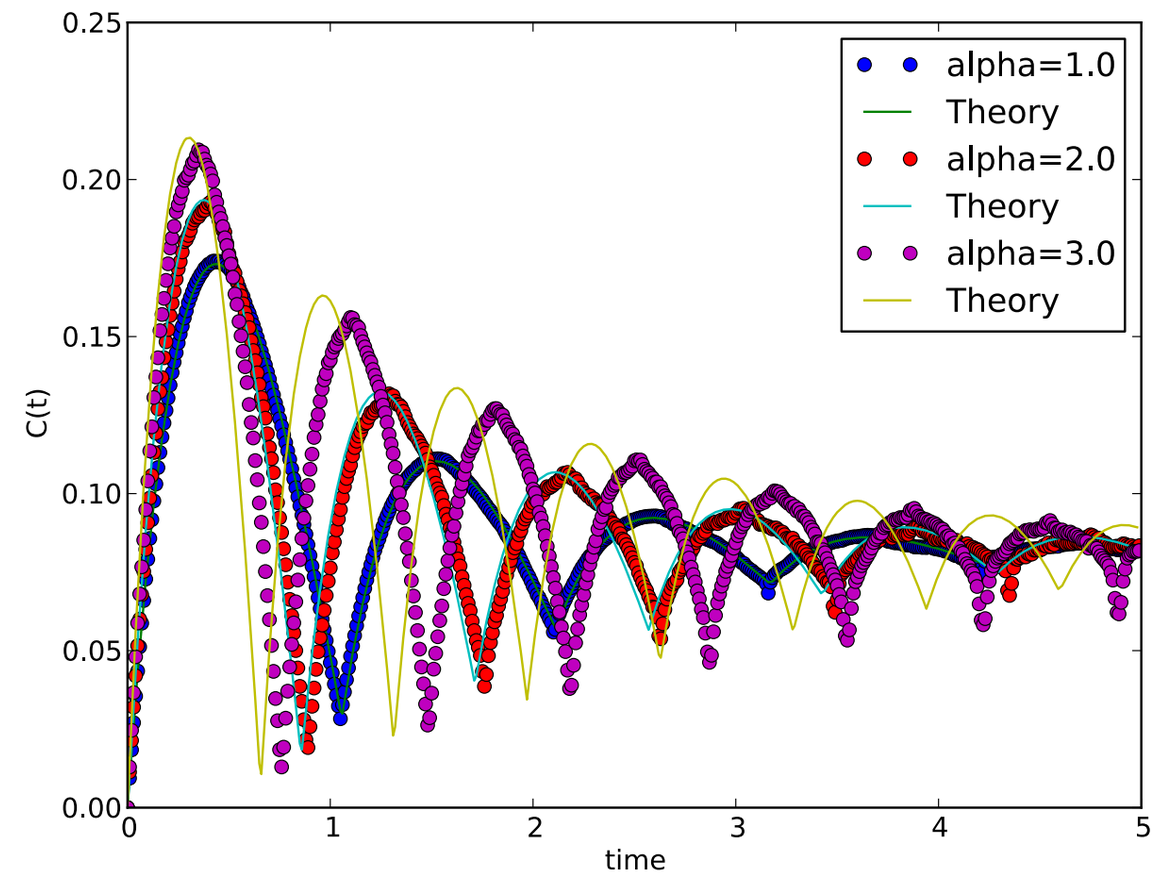
$$- \frac{1}{N} \bar{u}_0^2 \left(1 - e^{-\beta\delta(t - t_0)} \right)^2$$

Firing times of
a "fictitious neuron"

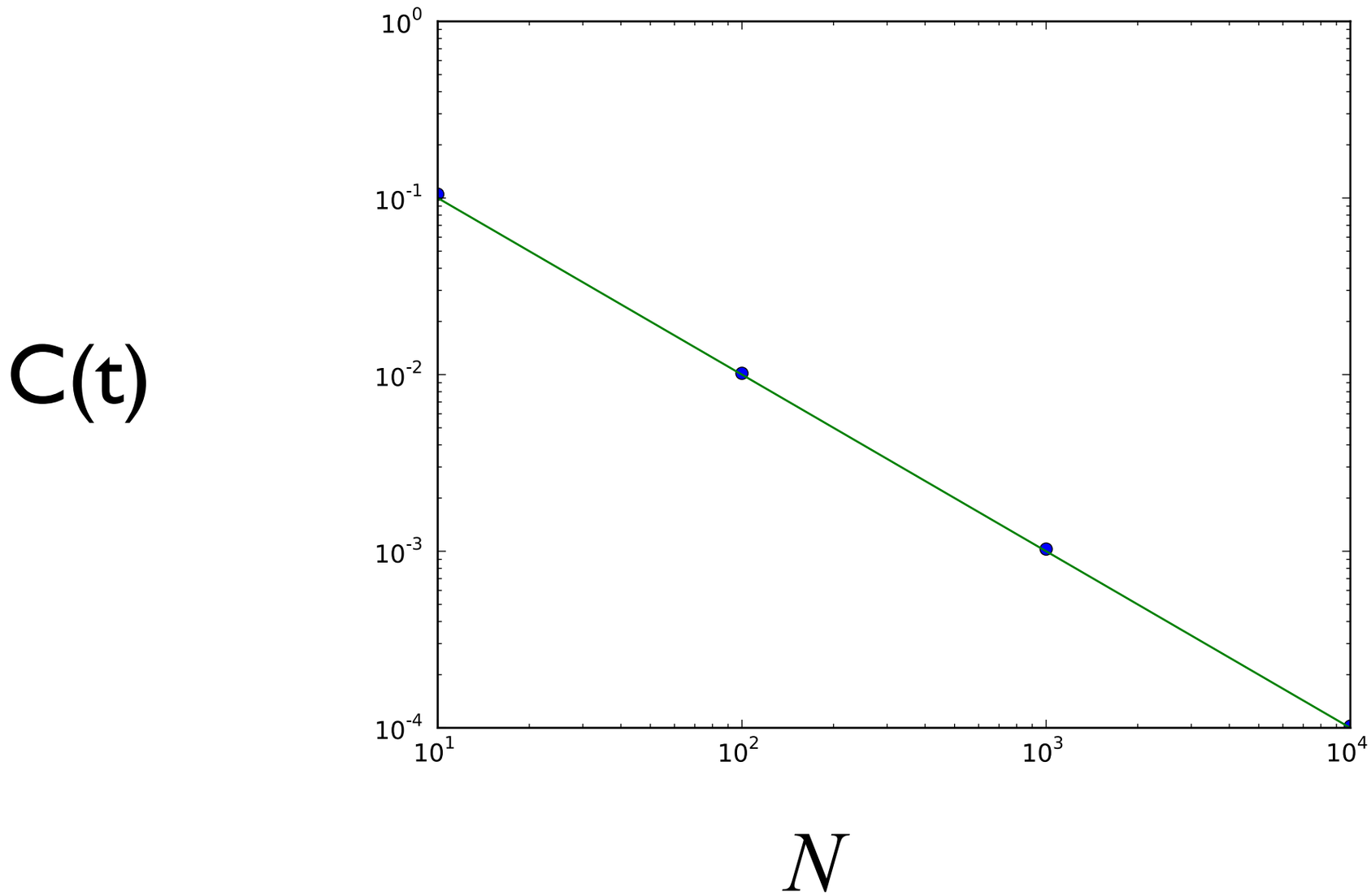
Transients



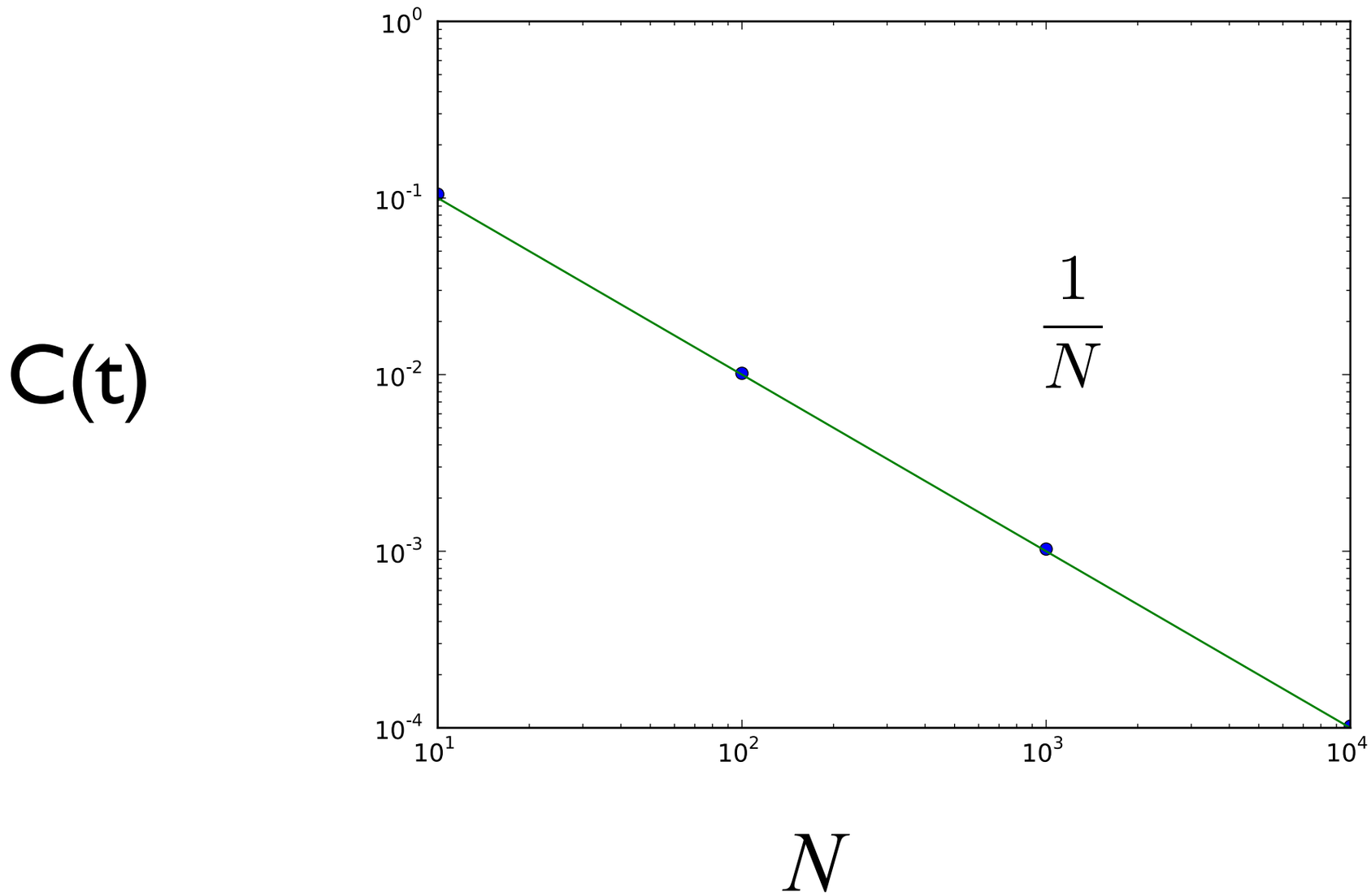


$$C(t) = \langle u(t)^2 \rangle - \langle u(t) \rangle^2 \propto \frac{1}{N}$$

Asymptotic state



Asymptotic state



Firing rate fluctuations

$$\nu(t) = (I(t) + \alpha u(t))\eta(\pi, t)$$

$$\langle \nu(t) \rangle = (I(t) + \alpha u(t))\bar{\rho} = \bar{u}$$

$$\langle (\nu(t) - \bar{u})\,(\nu(t') - \bar{u}) \rangle \;\; = (I + \alpha\bar{u})^2 \,\langle \eta(\pi, t)\eta(\pi, t') \rangle$$

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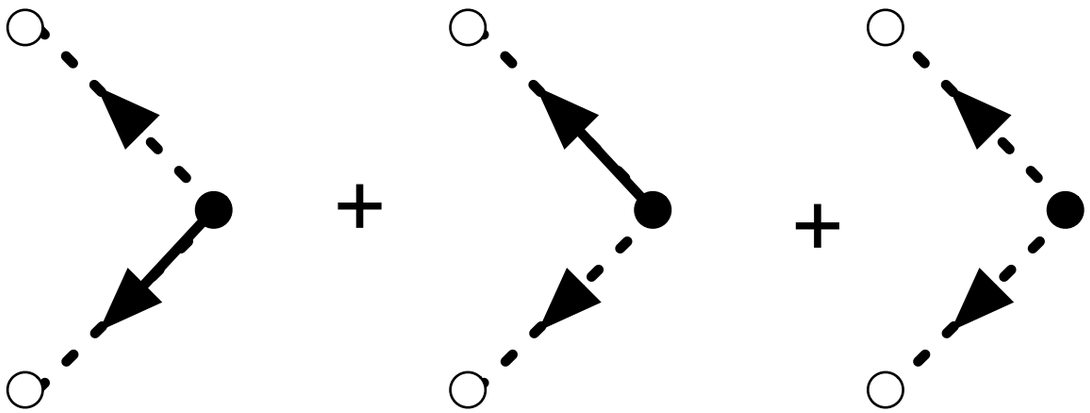

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$$=$$

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Poisson ansatz
e.g. Van V and S,
Brunel and Hakim

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Poisson ansatz
e.g. Van V and S,
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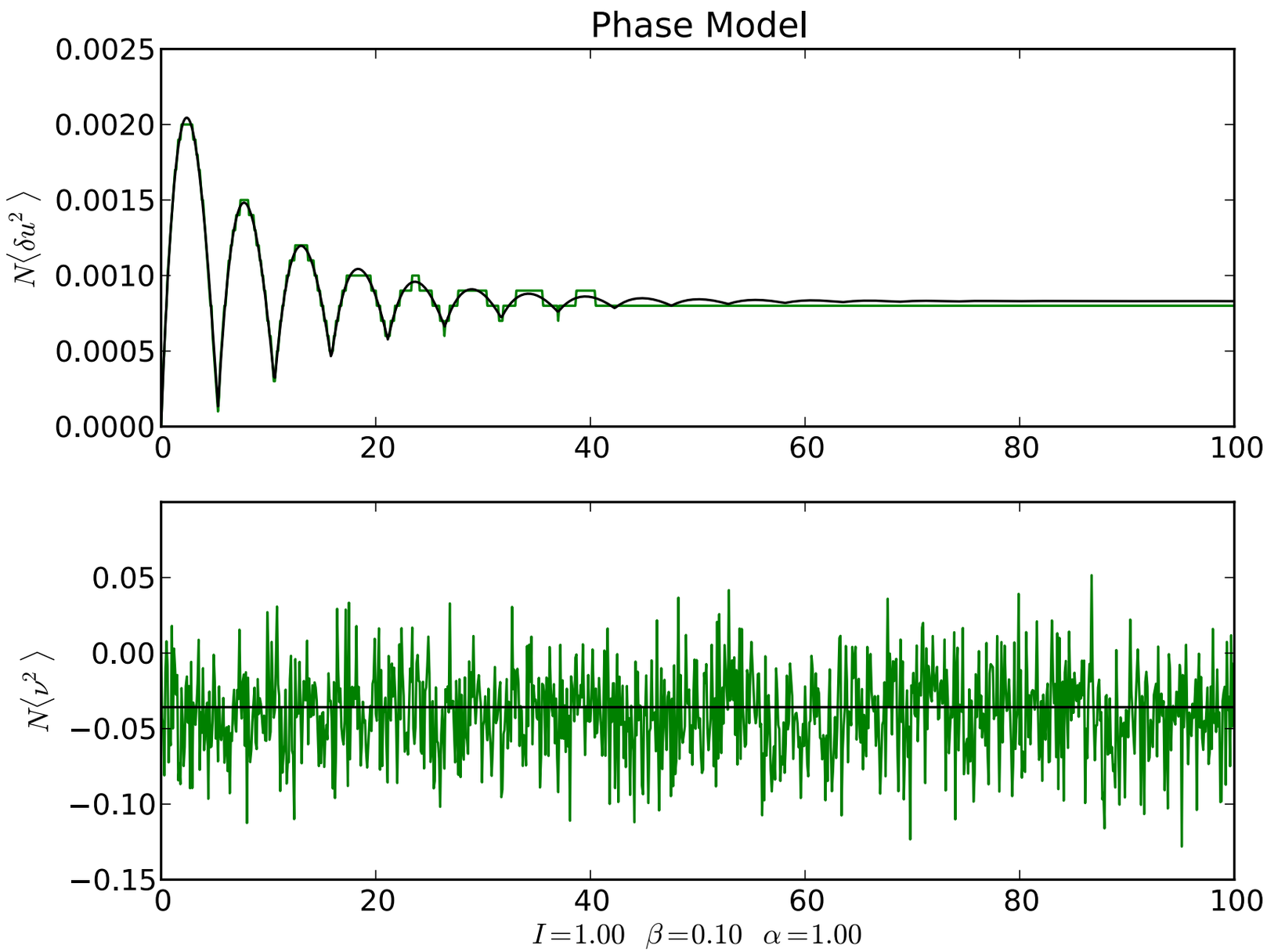
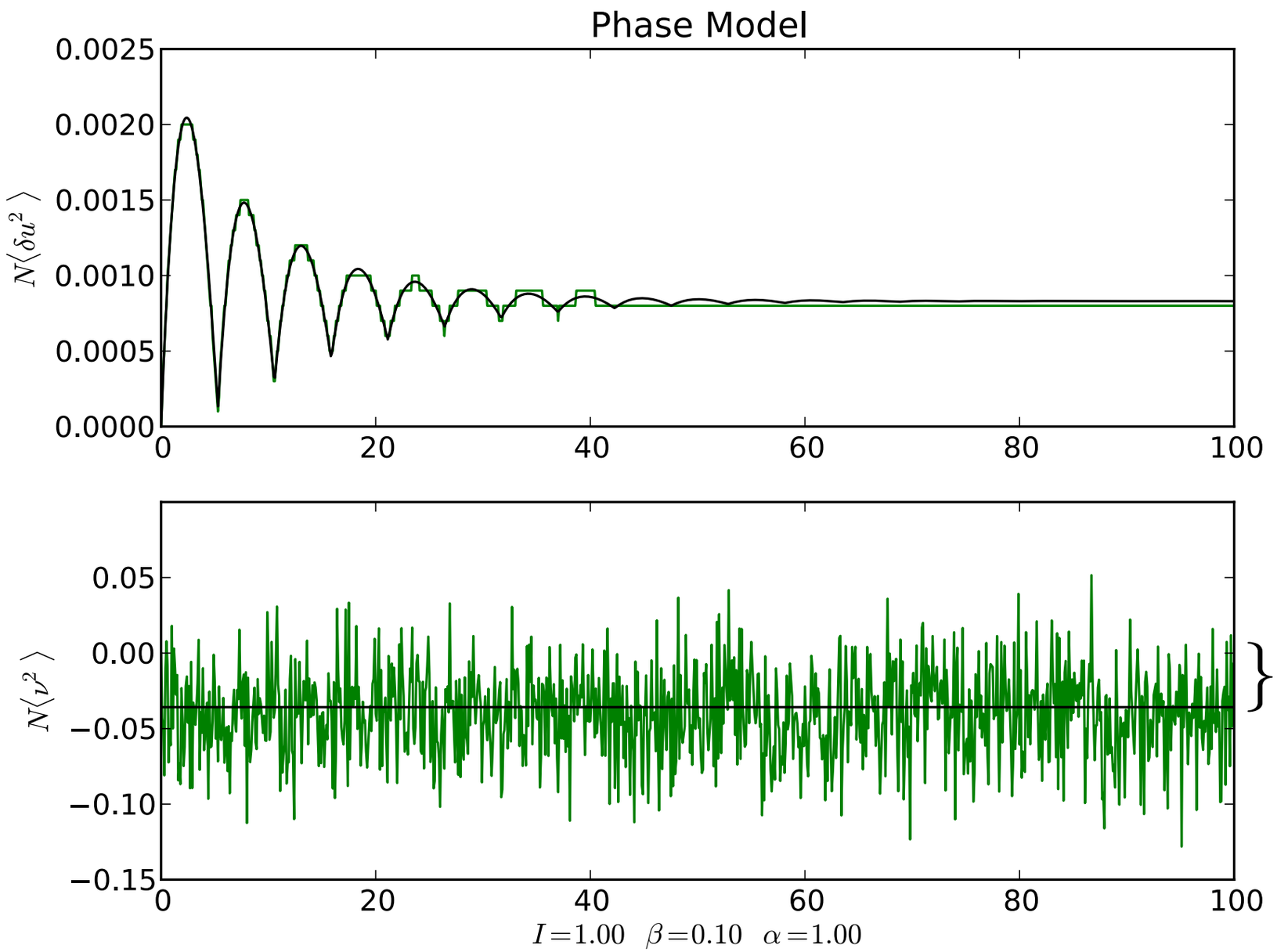
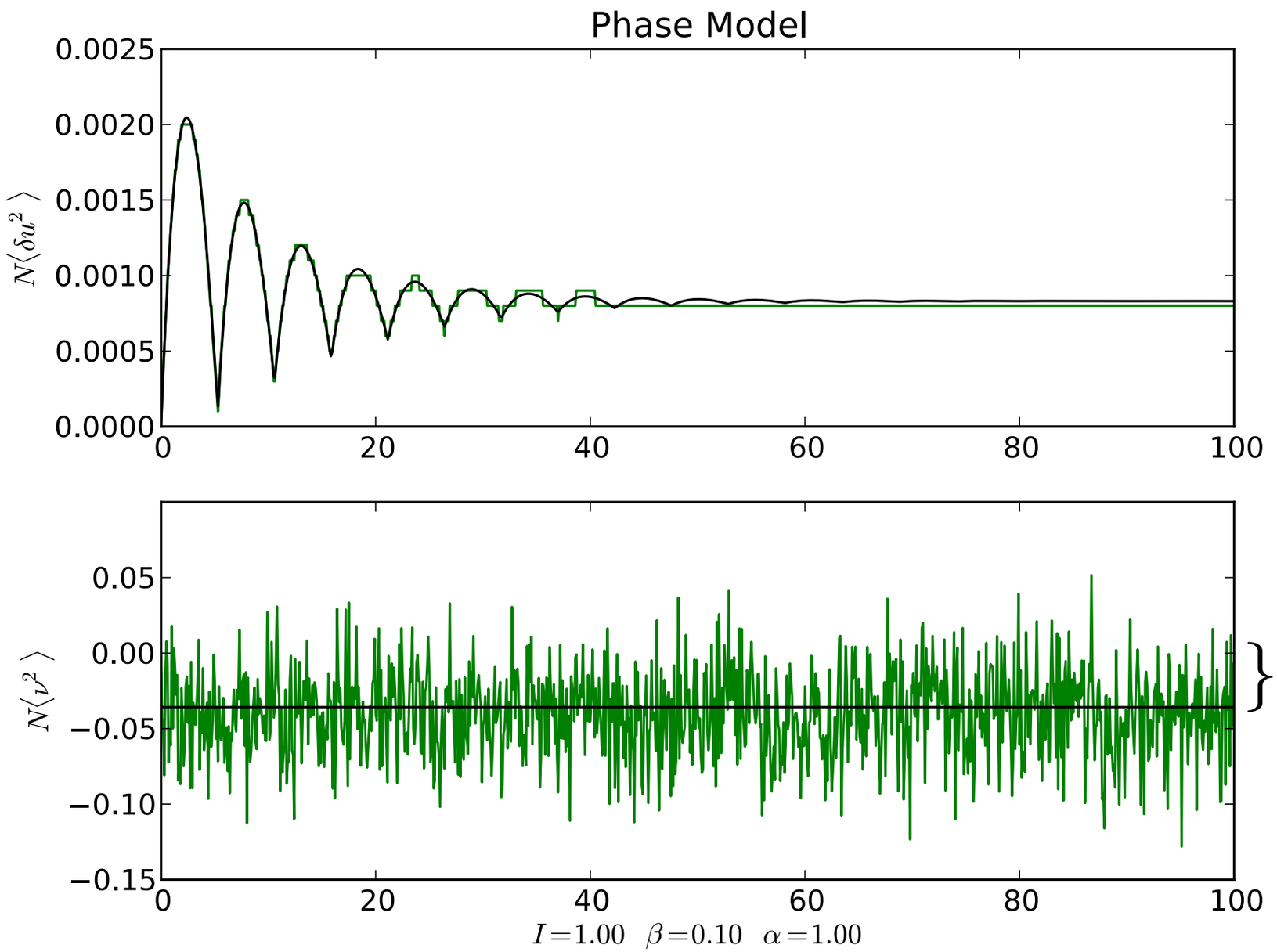
Initial state
sampling correction


Phase Model
0.0025
0.0020
0.0015
0.0010
0.0005
0.0000
$N\langle \delta u^2 \rangle$
0
20
40
60
80
100
0.05
0.00
−0.05
−0.10
−0.15
$N\langle v^2 \rangle$
0
20
40
60
80
100
$I=1.00 \quad \beta=0.10 \quad \alpha=1.00$


Phase Model
0.0025
0.0020
0.0015
0.0010
0.0005
0.0000
$N\langle \delta u^2 \rangle$
0
20
40
60
80
100
0.05
0.00
−0.05
−0.10
−0.15
$N\langle v^2 \rangle$
$I = 1.00 \quad \beta = 0.10 \quad \alpha = 1.00$



Deviation from Poisson

Theta Model

$$\dot{\theta}_i(t) = 1 - \cos\theta_i(t) + (I_i(t) + \alpha_i u(t))(1 + \cos\theta_i(t))$$

$$\dot{u}_i + \beta u_i = \frac{\beta}{N} \sum_j \delta(t - t_j^s)$$

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$$S = S[\tilde{u}(t), u(t)] + S[\tilde{\varphi}(\theta, t), \varphi(\theta, t)]$$

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$$S = S[\tilde{u}(t), u(t)] + S[\tilde{\varphi}(\theta, t), \varphi(\theta, t)]$$

$$\begin{aligned} S[\varphi, \tilde{\varphi}] &= N \int dt d\theta \, \tilde{\varphi}(\theta, t) \left[\partial_t \varphi(\theta, t) + \partial_\theta \left[1 - \cos\theta \right.\right. \\ &\quad \left.\left. + (1 + \cos\theta) \left\{I + \alpha u(t)\right\} \varphi(\theta, t)\right]\right] - \ln Z[\tilde{\varphi}_0(\theta, t_0)] \end{aligned}$$

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$$\begin{aligned} S[\tilde{u}(t), u(t)] &= \int_{t_0}^{t} ds \, \tilde{u}(s) \left(\frac{d}{ds} u(s) + \beta u(s) \right. \\ &\quad \left. - 2\beta \left\{\tilde{\varphi}(\pi, s)\varphi(\pi, s) + \varphi(\pi, s)\right\}\right) - \ln Z[\tilde{u}(t_0)] \end{aligned}$$

Steady state

$$\rho_0(\theta) = \frac{\sqrt{I + u_0}}{\pi(1 - \cos\theta + (I + \alpha u_0)(1 + \cos\theta))}$$

$$u_0 = \sqrt{I + \alpha u_0}$$

$$\nu = \frac{1}{\pi}\sqrt{I + \alpha u_0}$$

Firing rate fluctuations

$$\langle \nu(t) \rangle = 2\rho(\pi, t)$$

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Firing rate fluctuations

$$\langle \nu(t) \rangle = 2\rho(\pi, t)$$

Poisson ansatz

$$\langle \nu(t) \rangle = \int d\alpha d\Omega d\alpha' d\Omega' \langle \psi(x_\pi)\psi(x'_\pi) \rangle + \frac{1}{N dt} \langle \nu(t) \rangle$$

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Anomalous finite size effects

Firing rate fluctuations

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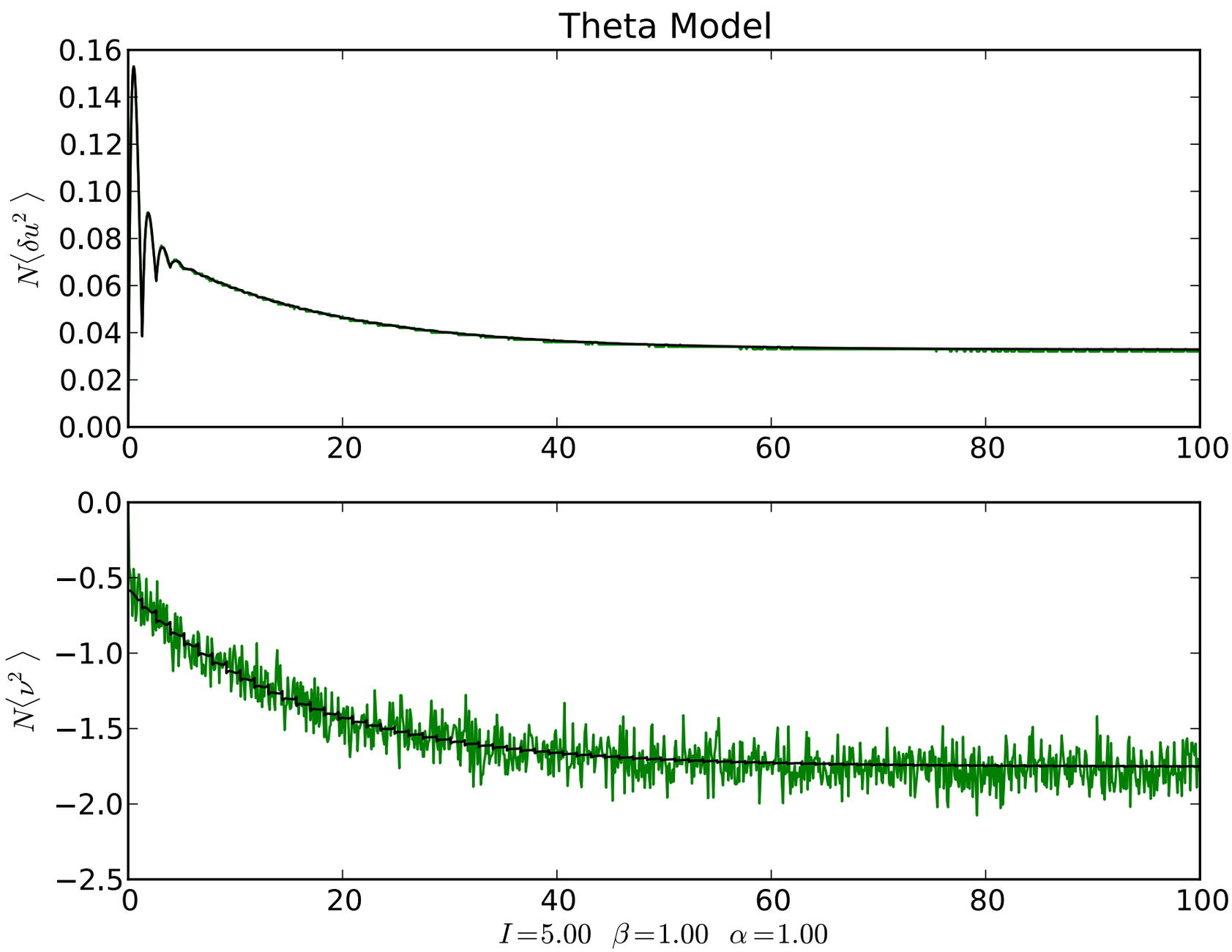
Poisson ansatz

$$\langle \nu(t) \rangle = \boxed{\int d\alpha d\Omega d\alpha' d\Omega' \langle \psi(x_\pi)\psi(x'_\pi) \rangle} + \frac{1}{N dt} \langle \nu(t) \rangle$$

Anomalous finite size effects

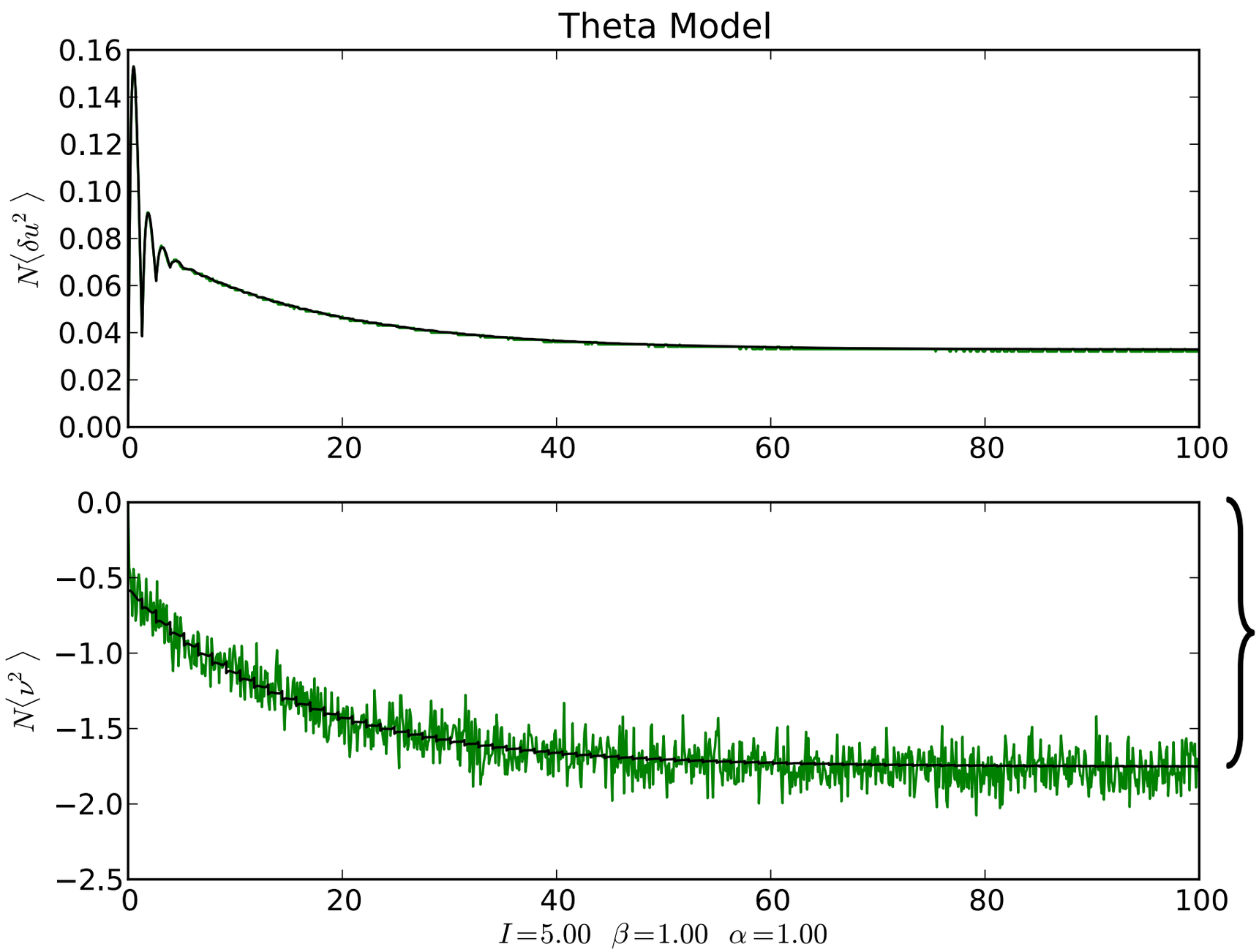
not in phase model

Simulations



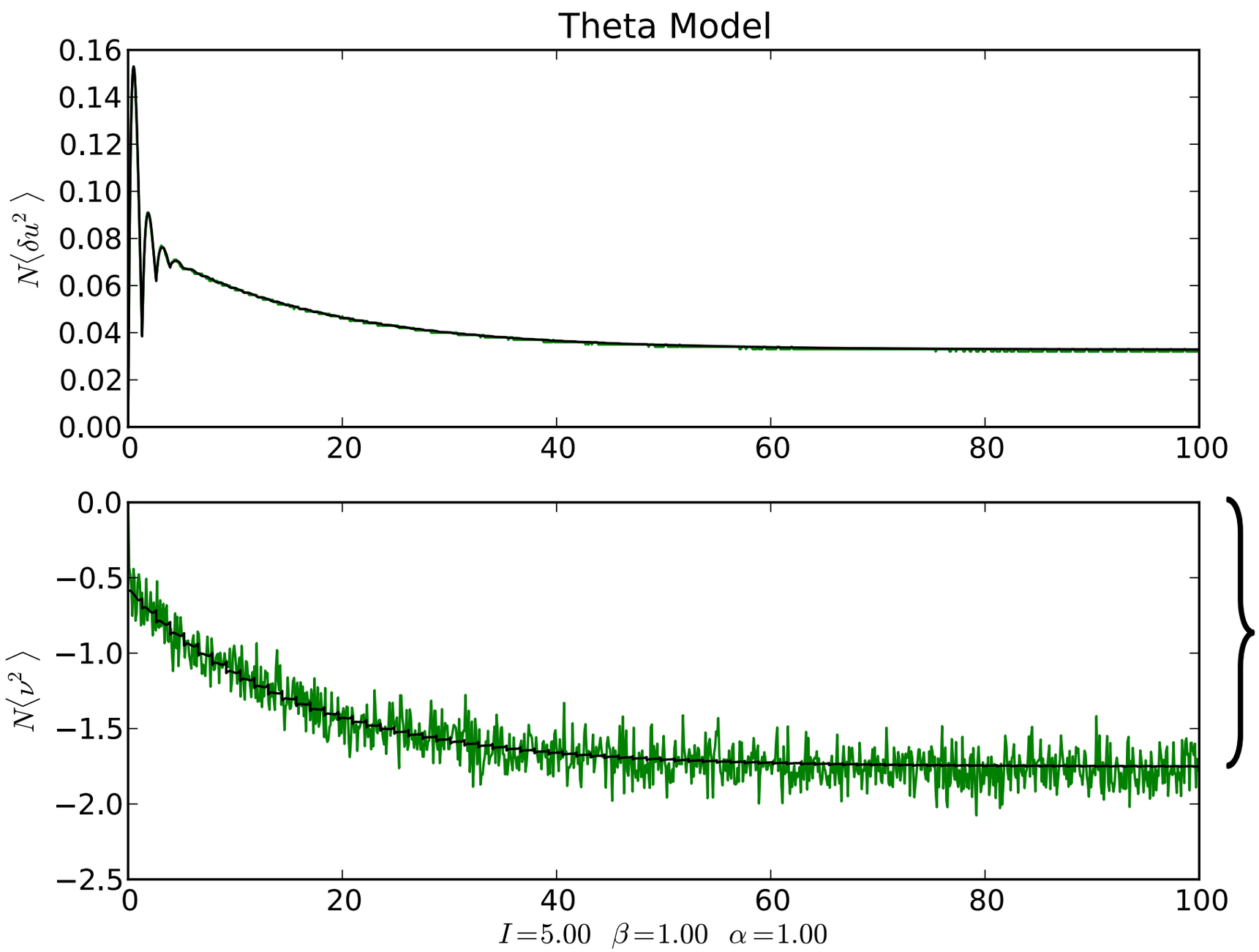
N=1000

Simulations



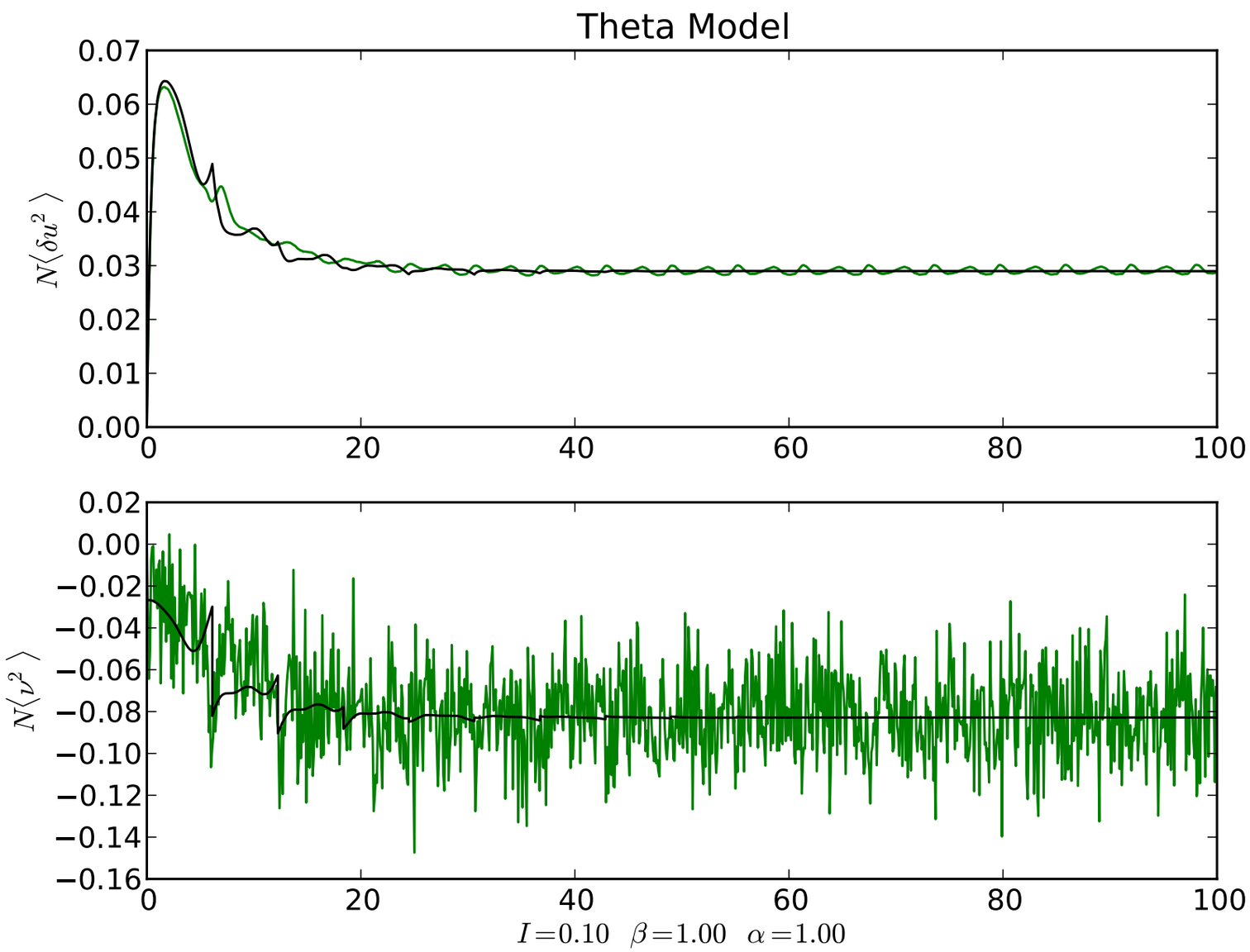
N=1000

Simulations

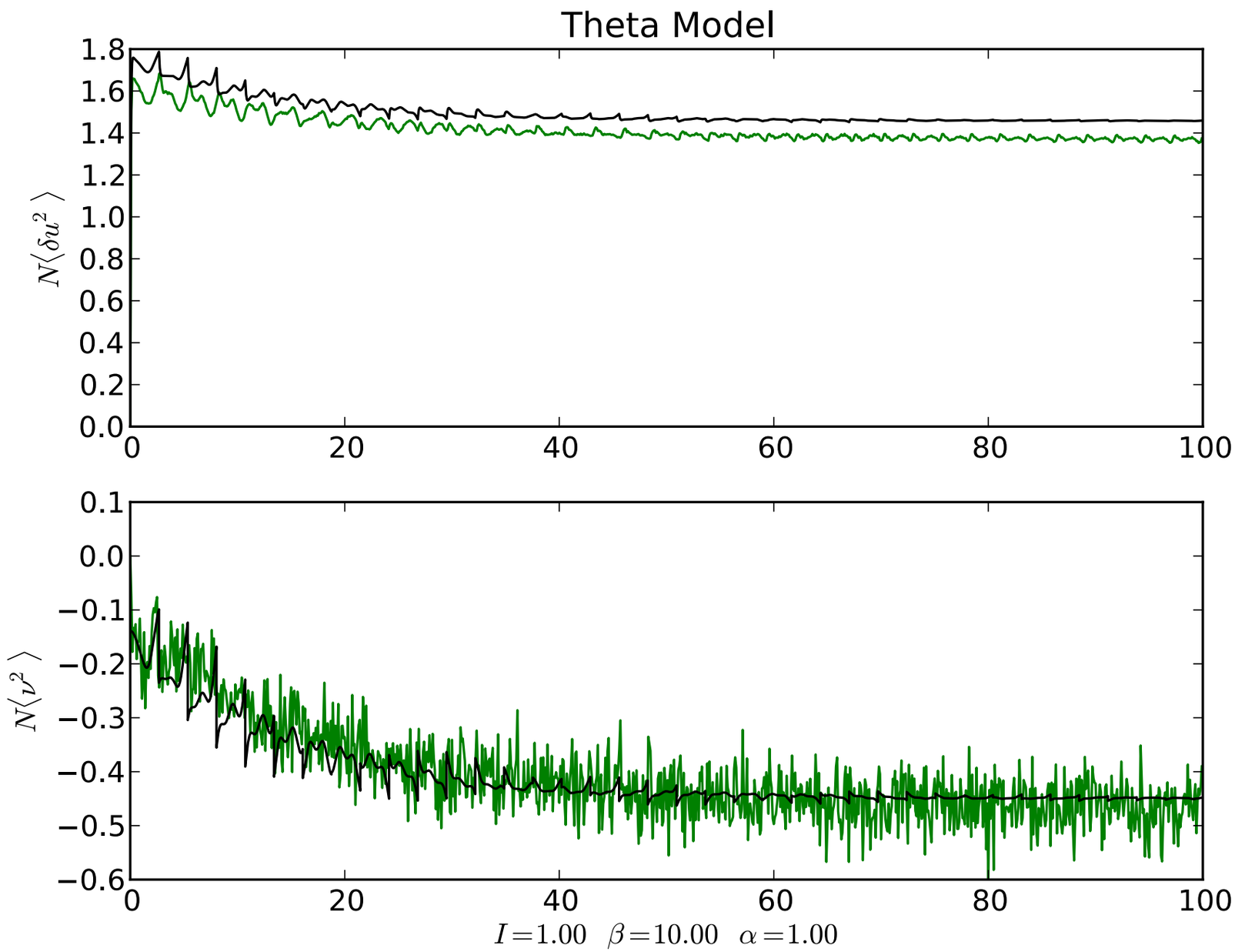


N=1000

deviation from Poisson


Theta Model
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
$N\langle \delta u^2 \rangle$
0
20
40
60
80
100
0.02
0.00
−0.02
−0.04
−0.06
−0.08
−0.10
−0.12
−0.14
−0.16
$N\langle \nu^2 \rangle$
0
20
40
60
80
100
$I = 0.10 \quad \beta = 1.00 \quad \alpha = 1.00$

N = 10


Theta Model
$N\langle \delta u^2 \rangle$
$N\langle v^2 \rangle$
$I=1.00 \quad \beta=10.00 \quad \alpha=1.00$

N=10

Slides on sciencehouse.wordpress.com