Neural networks: some results about

(i) Spike train statistics ; (ii) Linear response theory.

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Spike train statistics and Gibbs distributions

Characterizing spike trains statistics

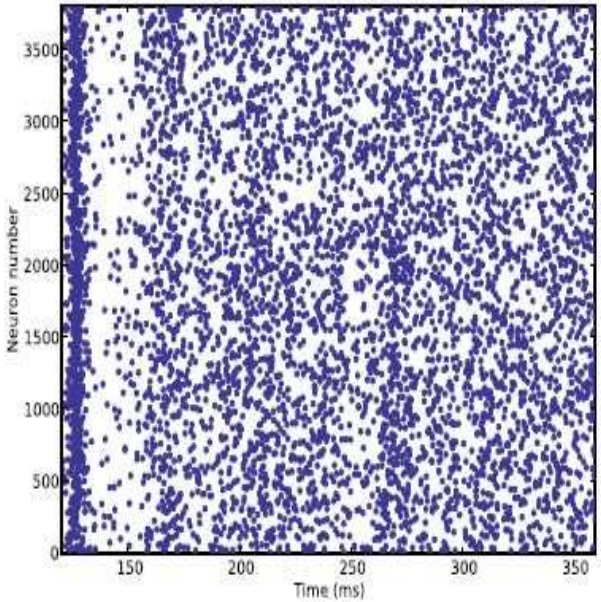

Figure: Raster plot/spike train.

Characterizing spike trains statistics

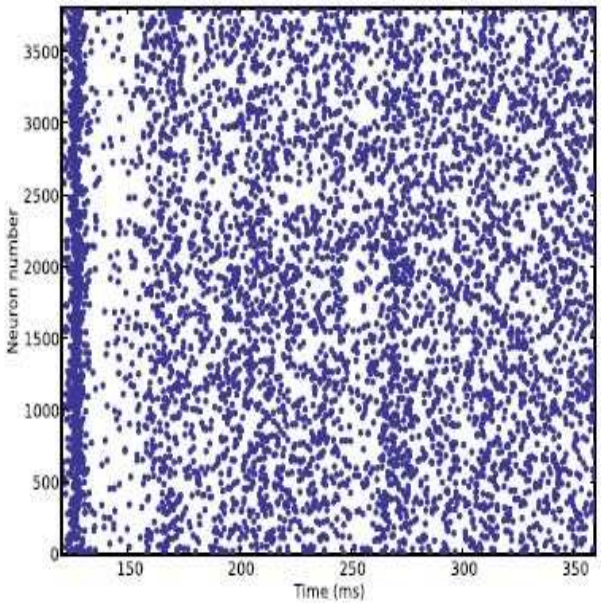

Figure: Raster plot/spike train.

Assume that spike trains statistics is distributed according to an hidden probability μ.

Characterizing spike trains statistics

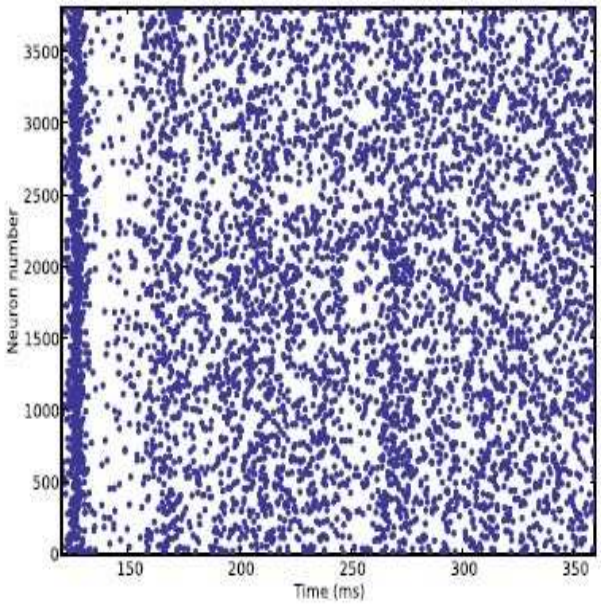

Figure: Raster plot/spike train.

Assume that spike trains statistics is distributed according to an hidden probability μ.

Can one have a reasonable idea of what μ is in a neural network model ?

Raster plot

"spiking state"

$$\omega_k(n) = \begin{cases} 1 & \text{if} \quad \exists t \in]n-1, n] \quad \text{such that} \quad V_k(t) \geq \theta; \\ 0 & \text{otherwise.} \end{cases}$$

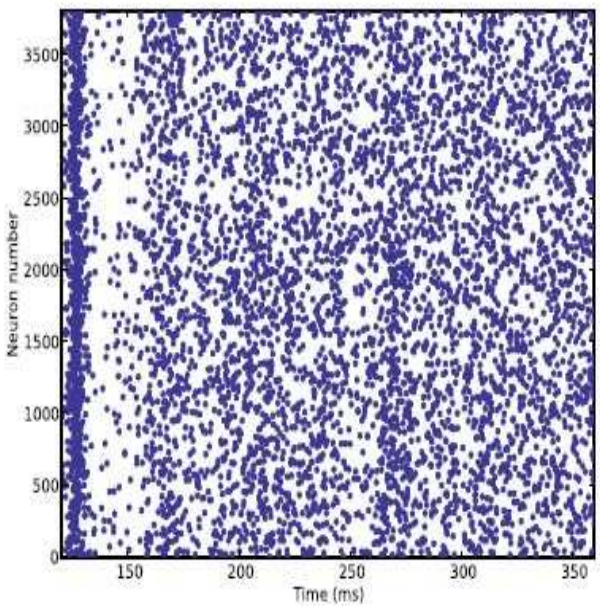

Figure: Raster plot.

Spike pattern

$$\omega(n) = (\omega_k(n))_{k=1}^{N}$$

Spike block

$$\omega_m^n = \{\, \omega(m)\, \omega(m+1) \ldots \omega(n) \,\}$$

Raster plot

$$\omega \stackrel{\text{def}}{=} \omega_{-\infty}^{+\infty}$$

The Generalized Integrate and Fire Model Rudolph-Destexhe,2006

$$C_k \frac{dV_k}{dt} = -g_{L,k}\left(V_k - E_L\right) - \sum_{j=1}^{N} g_{kj}(t,\omega)\left(V_k - E_j\right) + I_k(t),$$

$$g_{kj}(t,\omega) = G_{kj}\alpha_{kj}\left(t,\omega\right)$$

$$\alpha_{kj}\left(t,\omega\right) = \sum_{r<t} \alpha_{kj}\left(t - r\right)\omega_j(r)$$

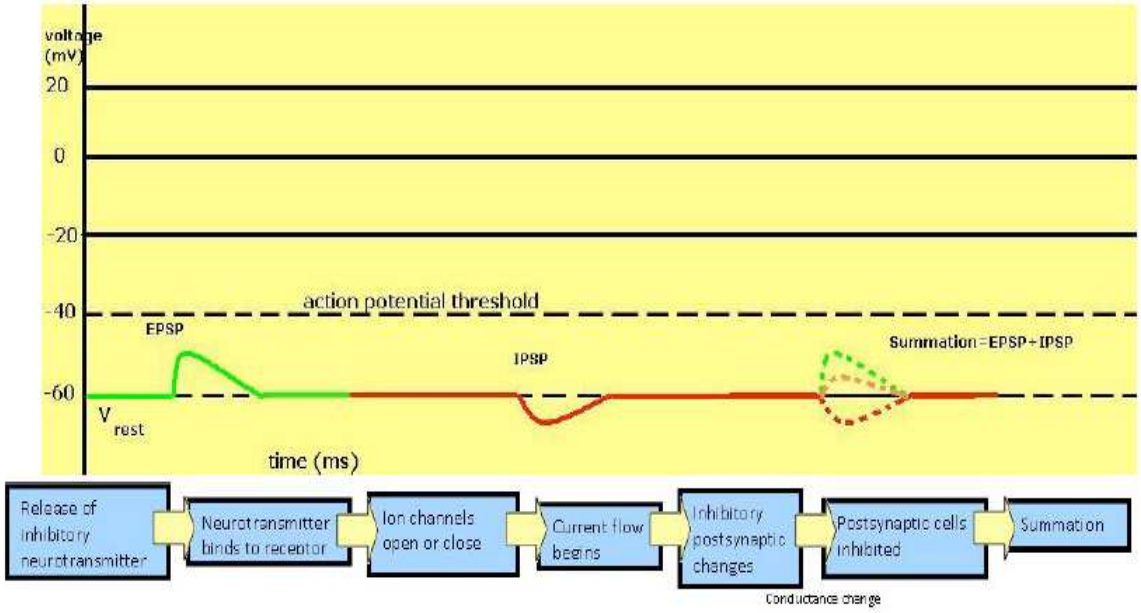

Figure: PostSynaptic Potential. From F. Grammont, Lecture in Les Houches, 2009.

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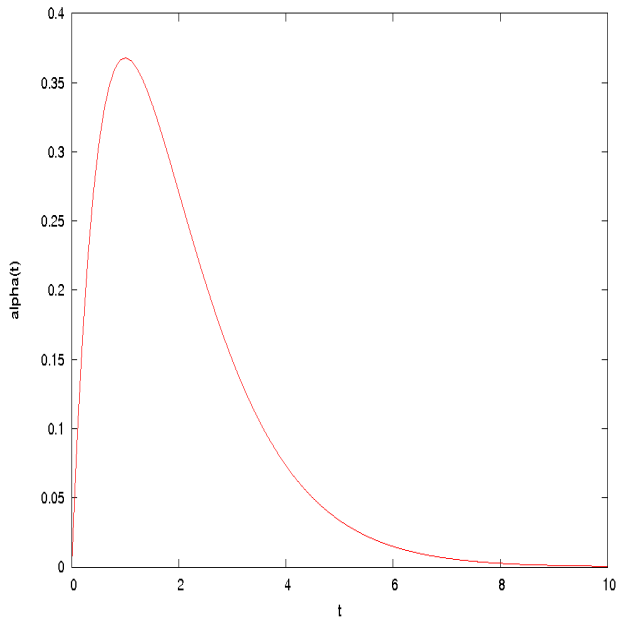
Synaptic response

$$\alpha_{kj}\left(t\right) = \frac{t}{\tau_{kj}} e^{-\frac{t}{\tau_{kj}}} H(t)$$



The Generalized Integrate and Fire Model Rudolph-Destexhe,2006

$$C_k \frac{dV_k}{dt} = -g_{L,k}\left(V_k - E_L\right) - \sum_{j=1}^{N} g_{kj}(t,\omega)\left(V_k - E_j\right) + i_k^{(ext)}(t) + \sigma_B \xi_k(t),$$

$$g_{kj}(t,\omega) = G_{kj}\alpha_{kj}\left(t,\omega\right)$$

Canonical equations

$$C_k \frac{dV_k}{dt} + g_k\left(t,\omega\right) V_k = i_k(t,\omega),$$

$$i_k(t,\omega) = g_{L,k}\, E_L + \sum_{j=1}^{N} W_{kj}\, \alpha_{kj}\left(t,\omega\right) + i_k^{(ext)}(t) + \sigma_B \xi_k(t),$$

$$W_{kj} = E_j\, G_{kj}$$

Flow given a raster

$$C_k \frac{dV_k}{dt} + g_k(t,\omega)\, V_k = i_k(t,\omega),$$

$$\Gamma_k(t_1, t_2, \omega) = e^{-\frac{1}{C_k}\int_{t_1}^{t_2} g_k(u,\omega)\, du}.$$

$$V_k(t,\omega) = \Gamma_k(s,t,\omega)\, V_k(s) + \frac{1}{C_k}\int_s^t \Gamma_k(t_1,t,\omega)\, i_k(t_1,\omega)\, dt_1.$$

Last reset time

If $V_k(t) \geq \theta$, neuron k fires.

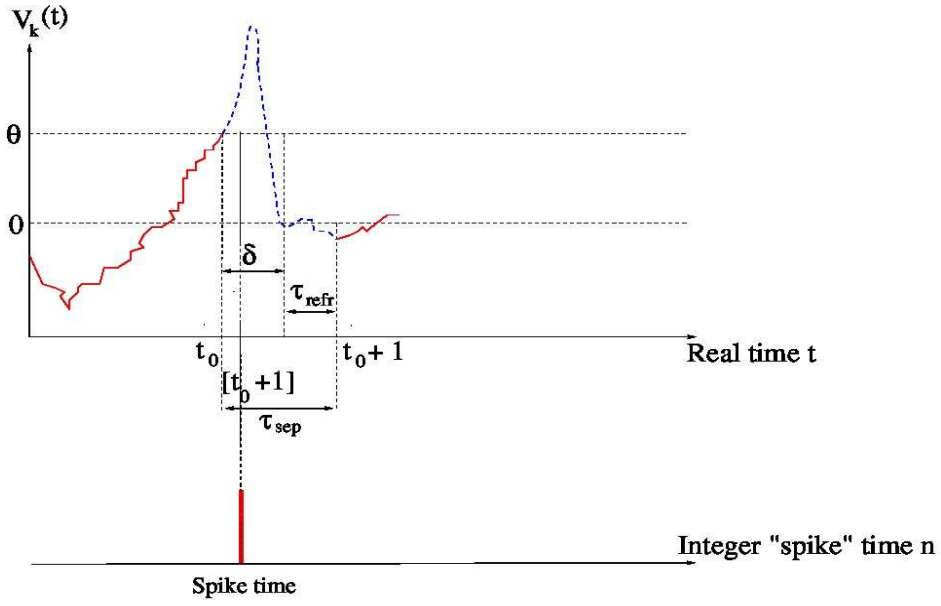


Figure: From Cessac, J. Math. Neuro. 2011.

Delayed reset to a random value V_{reset}.

Spikes are registered at integer times (in a time unit that can be arbitrary small).

$$V_k(t,\omega) = \Gamma_k(\tau_k(t,\omega), t, \omega)\, V_{reset} + \frac{1}{C_k} \int_{\tau_k(t,\omega)}^{t} \Gamma_k(t_1, t, \omega)\, i_k(t_1, \omega)\, dt_1.$$

Explicit form of the membrane potential given a raster

$$V_k(t,\omega) = \Gamma_k(\tau_k(t,\omega), t, \omega)\, V_{reset} \; + \; \frac{1}{C_k} \int_{\tau_k(t,\omega)}^{t} \Gamma_k(t_1, t, \omega)\, i_k(t_1, \omega)\, dt_1 .$$

$$i_k(t,\omega) = g_{L,k}\, E_L + \sum_{j=1}^{N} W_{kj}\, \alpha_{kj}\,(t,\omega) \; + \; i_k^{(ext)}(t) \; + \; \sigma_B \xi_k(t).$$

$$V_k(t,\omega) = V_k^{(det)}(t,\omega) + V_k^{(noise)}(t,\omega).$$

Deterministic part

$$V_k^{(det)}(t,\omega) = V_k^{(syn)}(t,\omega) + V_k^{(ext)}(t,\omega)$$

Synaptic contribution

$$V_k^{(syn)}(t,\omega) = \frac{1}{C_k}\sum_{j=1}^{N} W_{kj} \int_{\tau_k(t,\omega)}^{t} \Gamma_k(t_1,t,\omega)\alpha_{kj}(t_1,\omega)\, dt_1,$$

External + leak contribution

$$V_k^{(ext)}(t,\omega) = \frac{E_L}{\tau_{L,k}} \int_{\tau_k(t,\omega)}^{t} \Gamma_k(t_1,t,\omega)dt_1 + \frac{1}{C_k}\int_{\tau_k(t,\omega)}^{t} i_k^{(ext)}(t_1)\Gamma_k(t_1,t,\omega)dt_1,$$

where $\tau_{L,k} \stackrel{\text{def}}{=} \frac{C_k}{g_{L,k}}$.

Stochastic part

$$V_k^{(noise)}(\tau_k(t,\omega),t,\omega) = \Gamma_k(\tau_k(t,\omega),t,\omega)V_{reset} + V_k^{(B)}(\tau_k(t,\omega),t,\omega)$$

with

$$V_k^{(B)}(t,\omega) = \frac{\sigma_B}{C_k}\int_{\tau_k(t,\omega)}^{t} \Gamma_k(t_1,t,\omega)dB_k(t_1).$$

Gaussian process with mean zero and variance:

$$\sigma_k^2(t,\omega) = \Gamma_k^2(\tau_k(t,\omega),t,\omega)\,\sigma_R^2 + \left(\frac{\sigma_B}{C_k}\right)^2 \int_{\tau_k(t,\omega)}^{t} \Gamma_k^2(t_1,t,\omega)\,dt_1.$$

Conditional probability

Proposition

The probability of $\omega(n)$ conditionally to $\omega_{-\infty}^{n-1}$ is given by:

$$P_n\left[\omega(n)\,\middle|\,\omega_{-\infty}^{n-1}\right] = \prod_{k=1}^{N} P_n\left[\omega_k(n)\,\middle|\,\omega_{-\infty}^{n-1}\right],$$

with $P_n\left[\omega_k(n)\,\middle|\,\omega_{-\infty}^{n-1}\right] =$

$$\omega_k(n)\,\pi\left(X_k(n-1,\omega)\right) + \left(1-\omega_k(n)\right)\left(1-\pi\left(X_k(n-1,\omega)\right)\right),$$

where

$$X_k(n-1,\omega) = \frac{\theta - V_k^{(det)}(n-1,\omega)}{\sigma_k(n-1,\omega)},$$

and

$$\pi(x) = \frac{1}{\sqrt{2\pi}}\int_x^{+\infty} e^{-\frac{u^2}{2}}\,du.$$

Gibbs potential

Set:

$$\phi(n,\omega) = \sum_{k=1}^{N} \phi_k(n,\omega)$$

$\phi_k(n,\omega) =$

$$\omega_k(n)\log\pi\left(X_k(n-1,\omega)\right) + \left(1-\omega_k(n)\right)\log\left(1-\pi\left(X_k(n-1,\omega)\right)\right),$$

so that

$$P_n\left[\omega(n)\,\middle|\,\omega_{-\infty}^{n-1}\right] = e^{\phi(n,\omega)}.$$

Then:

Conditional probability of blocks given the past

$$P_n\left[\omega_m^n\,\middle|\,\omega_{-\infty}^{m-1}\right] = e^{\sum_{l=m}^{n}\phi(l,\omega)}.$$

Gibbs measure

Theorem, Cessac 2011, J. Math. Neuro.

For each choice of parameters the gIF model has a unique Gibbs distribution with potential ϕ.

Explicit Gibbs potential.

Explicit dependence in parameters.

Holds for a time-dependent stimulus (non stationarity).

Markovian approximations.

The Gibbs potential has infinite range (non Markovian).

Markovian approximations with memory depth D approaches the exact statistics with a Kullback-Leibler divergence converging exponentially fast to 0 as $D \to \infty$.

Markovian approximations.

Polynomial expansion.

$$\phi^{(D)}(\omega_{n-D}^{n}) = \sum_{l=0}^{L} \lambda_l(n)\, \phi_l(\omega_{n-D}^{n}),$$

where:

$$\phi_l(\omega_{n-D}^{n}) = \omega_{i_1}(t_1) \ldots \omega_{i_n}(t_n), \quad i_l \in \{1, \ldots, N\},\ t_l \in \{n-D, \ldots, n\}.$$

The maximal entropy principle

- Consider the stationary case.

$$\phi^{(D)}(\omega^0_{-D}) = \sum_{l=0}^{L} \lambda_l \, \phi_l(\omega^0_{-D}),$$

- Variational principle and topological pressure.

$$P\left(\phi^{(D)}\right) = \sup_{\nu \in \mathcal{M}_{inv}} \left\{ h(\nu) + \sum_{l=0}^{L} \lambda_l \, \nu \left[\phi_l(\omega^0_{-D})\right] \right\}.$$

- Therefore, the monomials ϕ_l constitute a canonical basis for constraints while the λ_l's are conjugated parameters.
- The λ_l's depend explicitly on network parameters (synaptic weights, stimulus).

Statistical Models hierarchy

- Bernoulli $D = 0$. Memoryless. Neurons are independent.
- "Ising" (Schneidman et al, nature 2006) $D = 0$. Memoryless. Neurons are spatially correlated but time-independent.
- Polynomial expansion (Marre et al, 2009; Vasquez et al 2011)
- Infinite range

Linear response in chaotic neural networks.

(Cessac-Sepulchre, 2004, 2006)

A "firing rate" model.

$$u_i(t+1) = \sum_j J_{ij} f(g u_j(t)) + \theta_i \qquad i = 1..N$$

membrane potential of neuron i at time t+1.

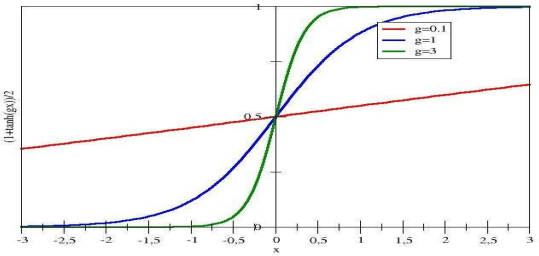
Synapses.

Fully connected network,

Quenched random synapses (uncorrelated).

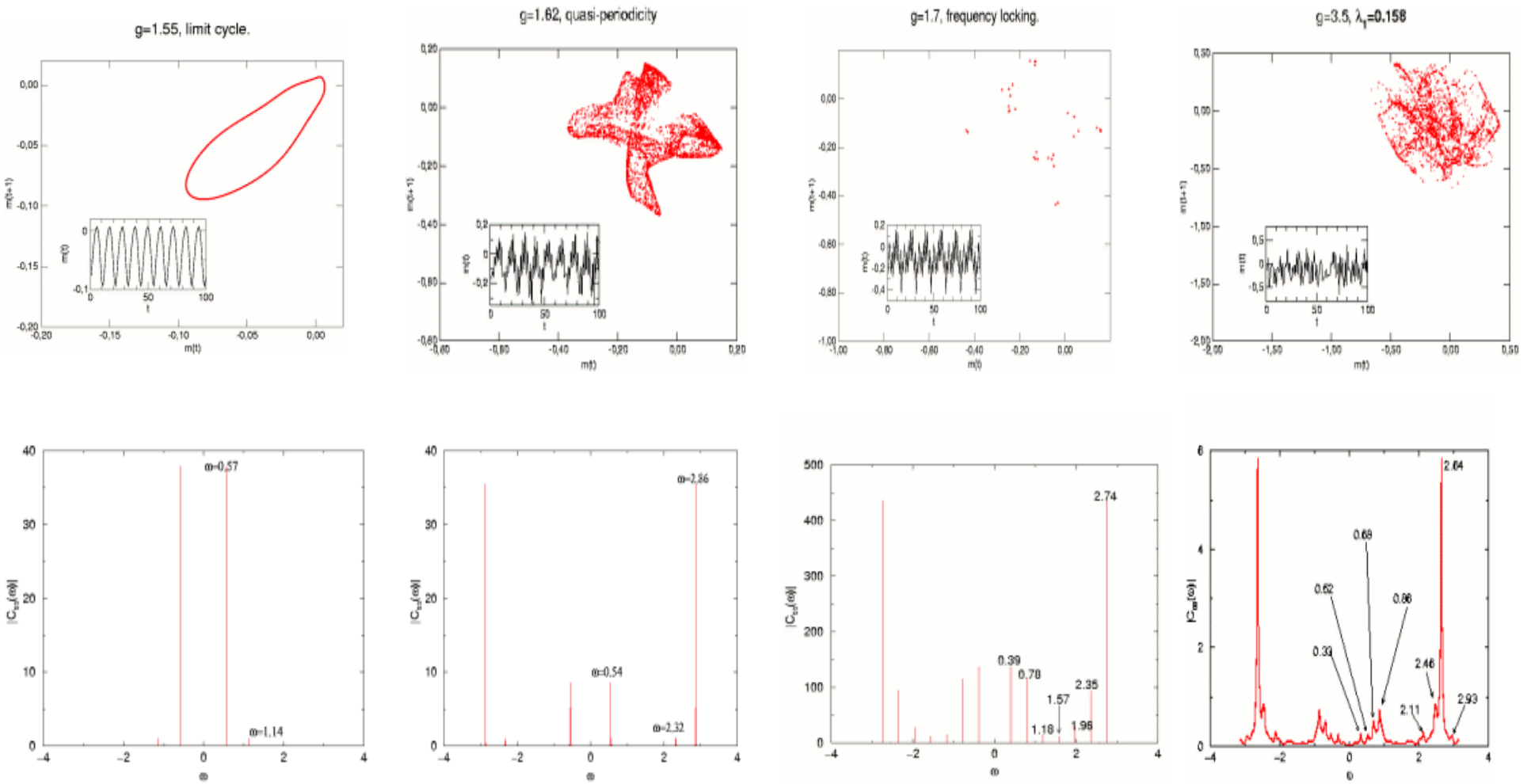
firing rate of neuron j at time t.

Stimulus/input.



Ruelle Takens Transition to chaos by quasi-periodicity (Doyon et al, 1994; Cessac, 1995).

Increasing g.



Signal transmission.

Signal transmission.

Superimpose a weak signal upon the background activity of some neuron

$$\tilde{u}_i(t+1) = \sum_{j=1}^{N} J_{ij} f(\tilde{u}_j(t)) + \xi_i[t, \tilde{\mathbf{u}}(t)]$$

Signal transmission.

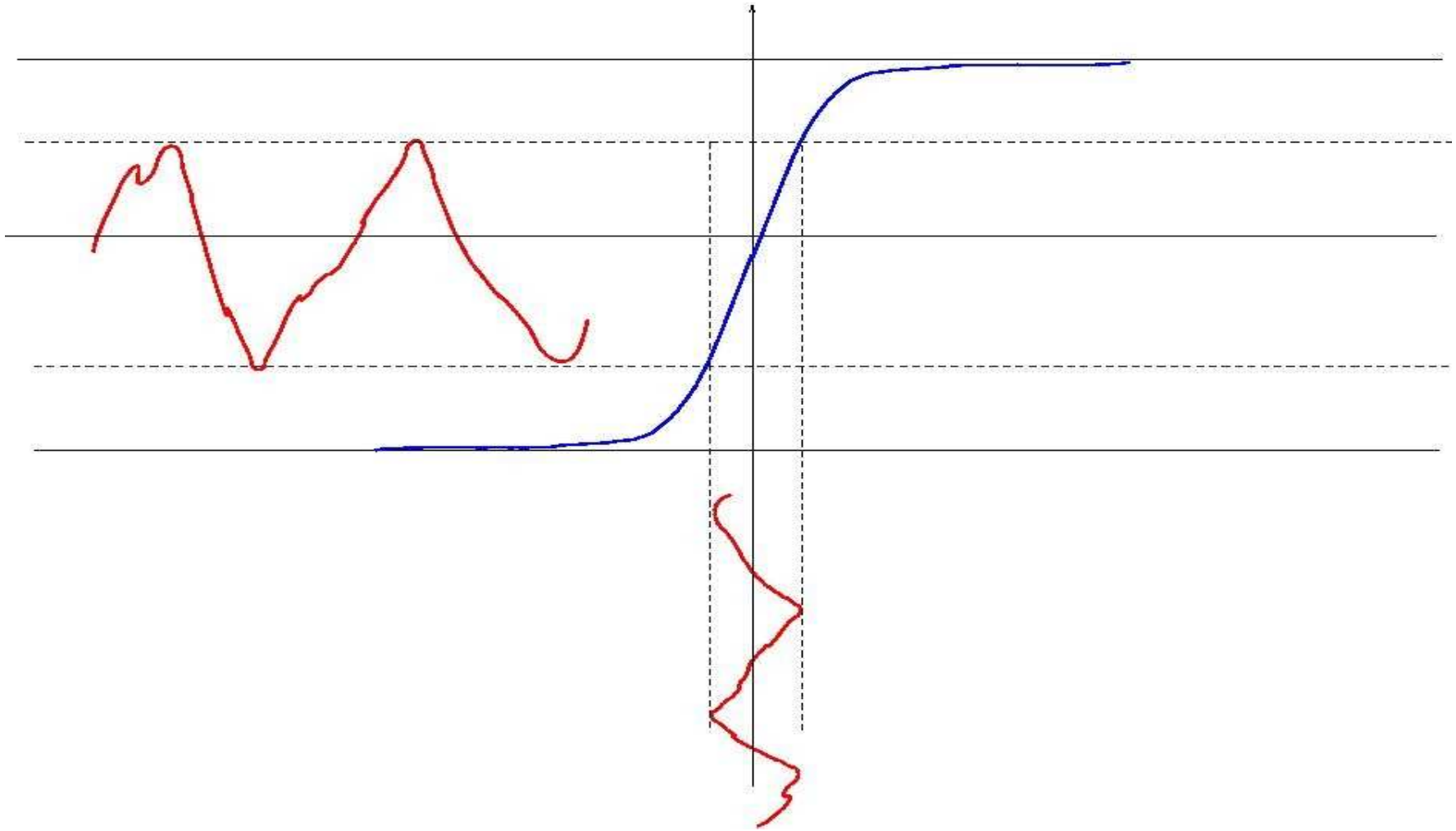
Superimpose a weak signal upon the background activity of some neuron

$$\tilde{u}_i(t+1) = \sum_{j=1}^{N} J_{ij} f(\tilde{u}_j(t)) + \xi_i[t, \tilde{\mathbf{u}}(t)]$$

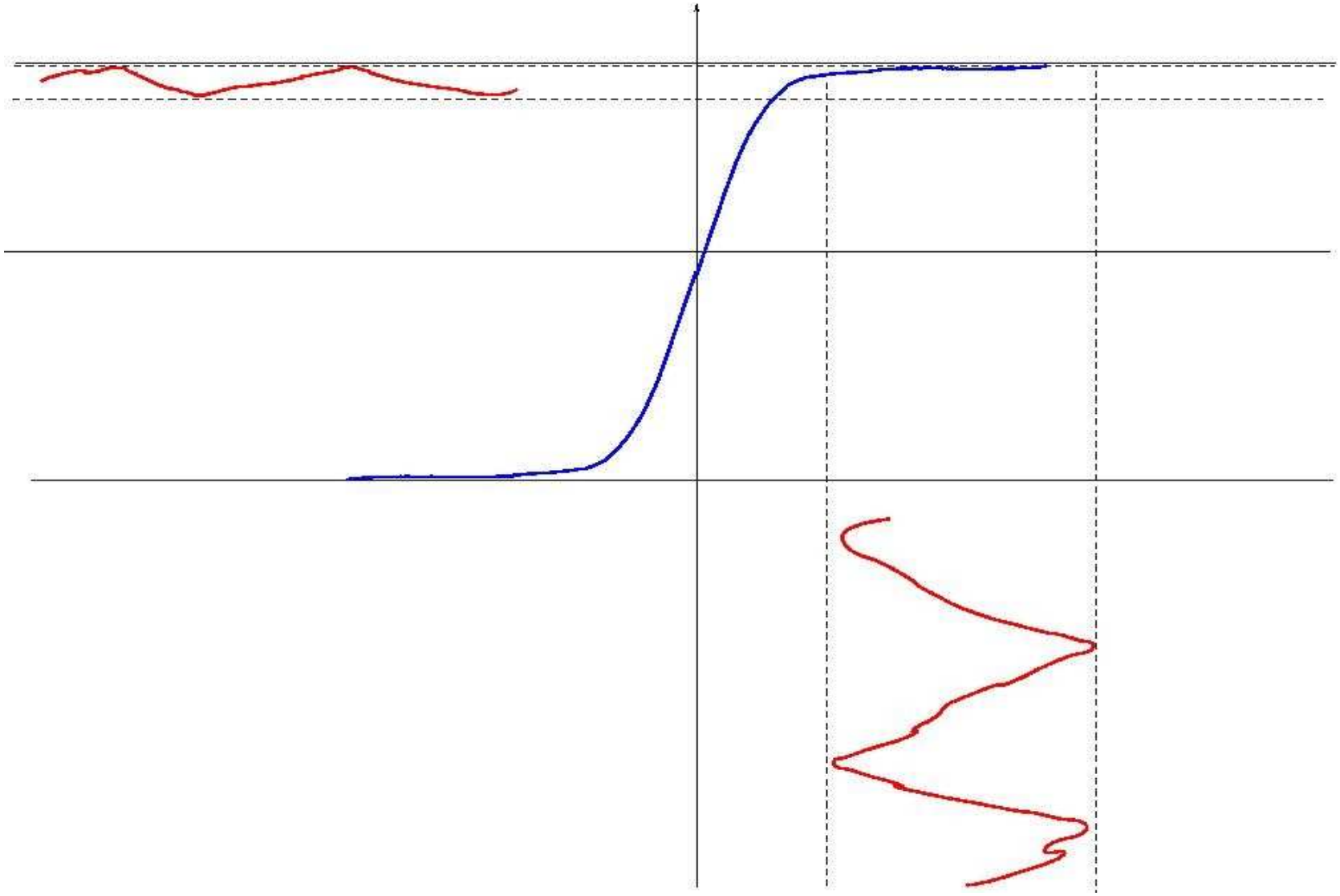
Global dynamics.

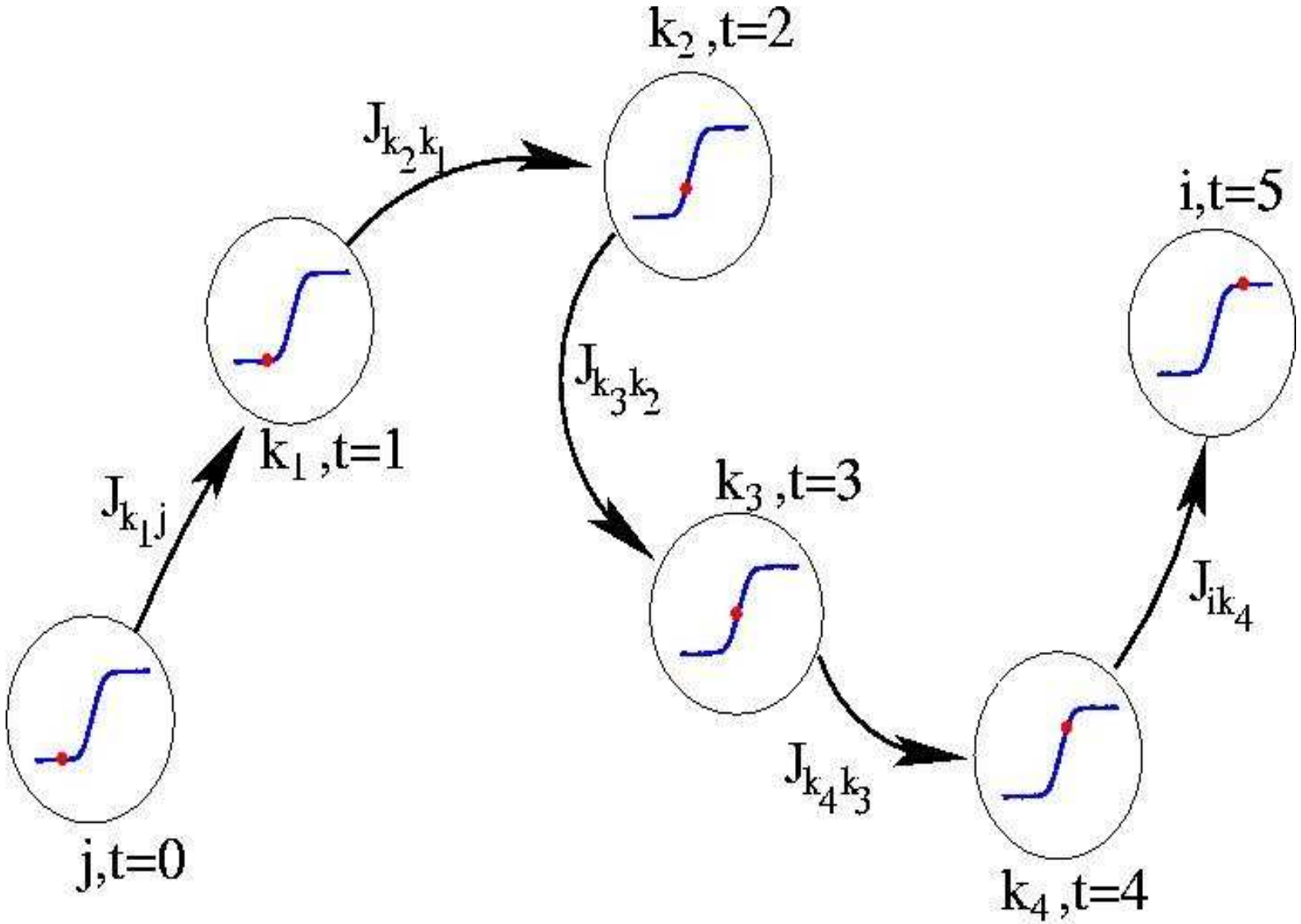
$$\tilde{\mathbf{u}}(t+1) = \mathbf{G}\left[\tilde{\mathbf{u}}(\mathrm{t})\right] + \xi\left[\mathrm{t}, \tilde{\mathbf{u}}(\mathrm{t})\right] = \tilde{\mathbf{G}}\left[\tilde{\mathbf{u}}(\mathrm{t})\right]$$

Amplification



Saturation




k_2 ,t=2
$J_{k_2k_1}$
i,t=5
$J_{k_3k_2}$
k_1 ,t=1
k_3 ,t=3
J_{k_1j}
J_{ik_4}
$J_{k_4k_3}$
j,t=0
k_4 ,t=4

Complex interplay between topology and dynamics.

How can we characterize / measure the effects of this interplay ?

How can we measure the action of a weak signal superimposed upon a chaotic background ?

“Butterfly effect !

SRB measure.

Let μ be an initial state (a.c. probability measure on Ω).

The SRB measure is given by: $$\rho \overset{w}{=} \lim_{t \to +\infty} \mathrm{G}^{*\mathrm{t}} \mu$$

It is a “natural” probability measure, carried by the attractor.

- *Ergodic in a strong sense:*

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \Phi \left[\mathbf{G}^{\mathbf{t}}(\mathbf{X}) \right] \overset{\mu \; a.s.}{=} \int_{\Omega} \Phi(\mathbf{X}) \rho(\mathbf{dX})$$

- *Gibbs state and equilibrium state.*
- *Robust to noise.*

Out of equilibrium SRB state

(Ruelle, 99).

$$\rho_t = \rho + \delta_t \rho = \lim_{n \to +\infty} \tilde{\mathbf{G}}_t \ldots \tilde{\mathbf{G}}_{t-n} \mu$$

$$\Phi\left[\tilde{\mathbf{u}}(t)\right] = \Phi\left[\mathbf{u}(\mathbf{t})\right] + \nabla_{\mathbf{u}(\mathbf{t})} \Phi . \delta \mathbf{u}(\mathbf{t}) + \mathrm{O}(\|\delta \mathbf{u}(\mathbf{t})\|^{\mathbf{2}})$$

$$\delta_t \rho\left[\Phi\right] = \sum_{\tau=-\infty}^{t-1} \int \rho(d\mathbf{u}) \mathbf{D}\mathbf{G}_{\mathbf{u}}^{\mathbf{t}-\tau-\mathbf{1}} \xi_\tau \left[\mathbf{G}^{-\mathbf{1}}(\mathbf{u})\right] . \nabla_{\mathbf{u}(\mathbf{t}-\tau-\mathbf{1})} \mathbf{\Phi} + \mathbf{NL}$$

Linear response theory

(Ruelle, 99).

$$\delta_t \rho[\Phi] = \sum_{\sigma} \left\langle \kappa_\sigma \xi_{t-\sigma-1} \circ \mathbf{G}^{-1} | \Phi \right\rangle$$

Set:

$$\mathbf{X} = \xi_0 \circ \mathbf{G}^{-1}$$

Then:

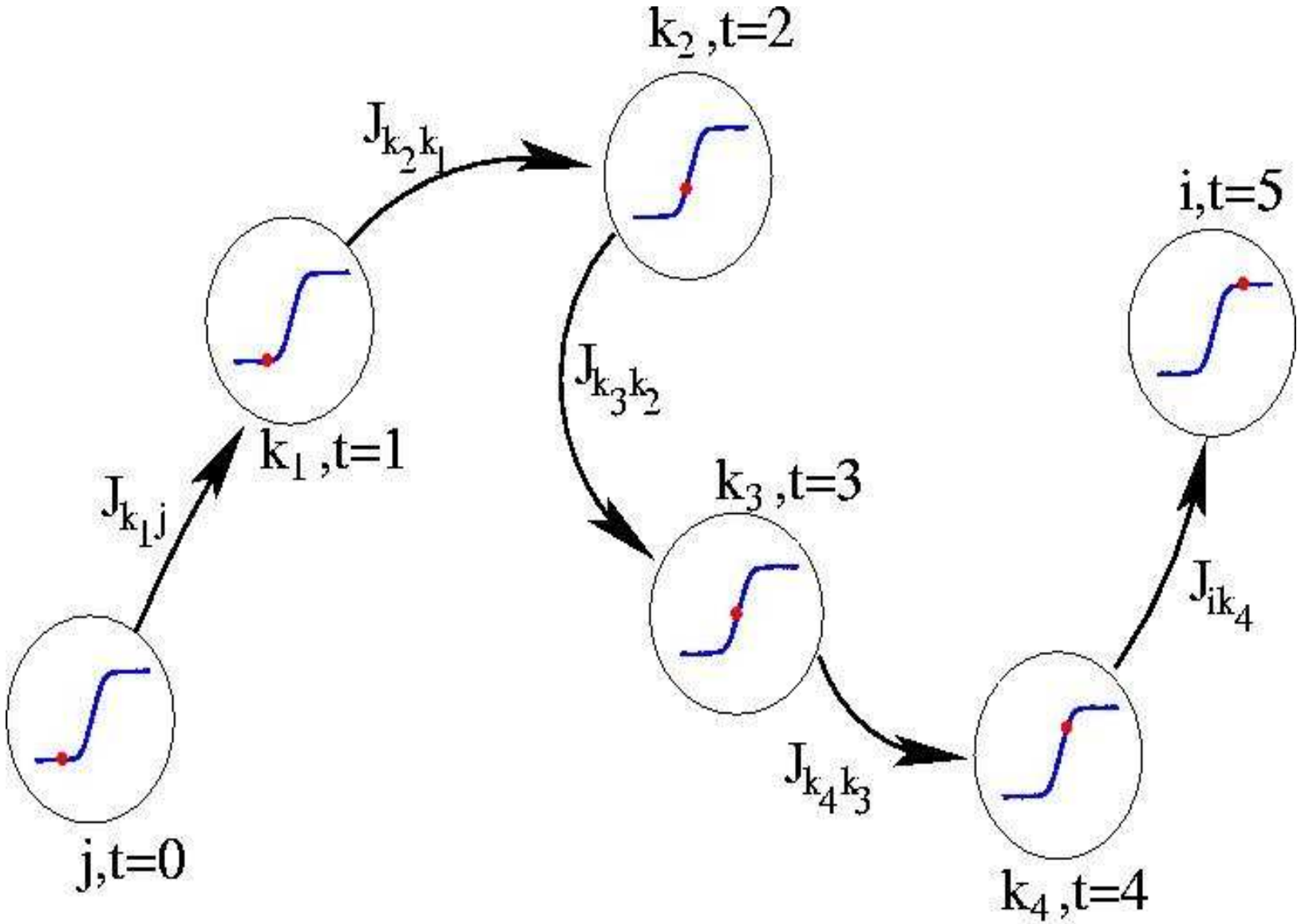
$$\begin{cases} \langle \kappa_\sigma \mathbf{X} | \Phi \rangle & = & \int \rho(d\mathbf{u}) \mathrm{D}\mathbf{G}_{\mathbf{u}}^{\sigma} \mathbf{X}(\mathbf{u}).\nabla_{\mathbf{u}(\sigma)} \Phi & \mathbf{if} & \sigma \geq 0 \\ \kappa_\sigma \mathbf{X} & = & 0 & if & \sigma < 0 \quad (causality). \end{cases}$$

Average effect on i of a perturbation applied at j (Cessac-Sepulchre, 2004).

$$\Phi(\mathbf{u}) = \mathbf{u_i}$$
$$\xi(\mathbf{t}) = \xi_\mathbf{j}(\mathbf{t})\mathbf{e_j}$$

$$\delta_t \rho[\Phi] = \sum_\sigma \int \rho(d\mathbf{u}) \mathbf{DG}^\sigma_\mathbf{u}(\mathbf{i},\mathbf{j}) \xi_\mathbf{j}(\mathbf{t}-\sigma-\mathbf{1})$$

$$\chi_{i,j}(\sigma) = \int \rho(d\mathbf{u}) \mathrm{DG}^\sigma_\mathbf{u}(\mathbf{i},\mathbf{j}) = \sum_{\gamma_{\mathrm{ij}}(\sigma)} \prod_{\mathrm{l}=1}^{\sigma} \mathrm{J}_{\mathrm{k_l k_{l-1}}} \left\langle \prod_{\mathrm{l}=1}^{\sigma} \mathrm{f}'(\mathrm{u}_{\mathrm{k_{l-1}}}(\mathrm{l}-1)) \right\rangle$$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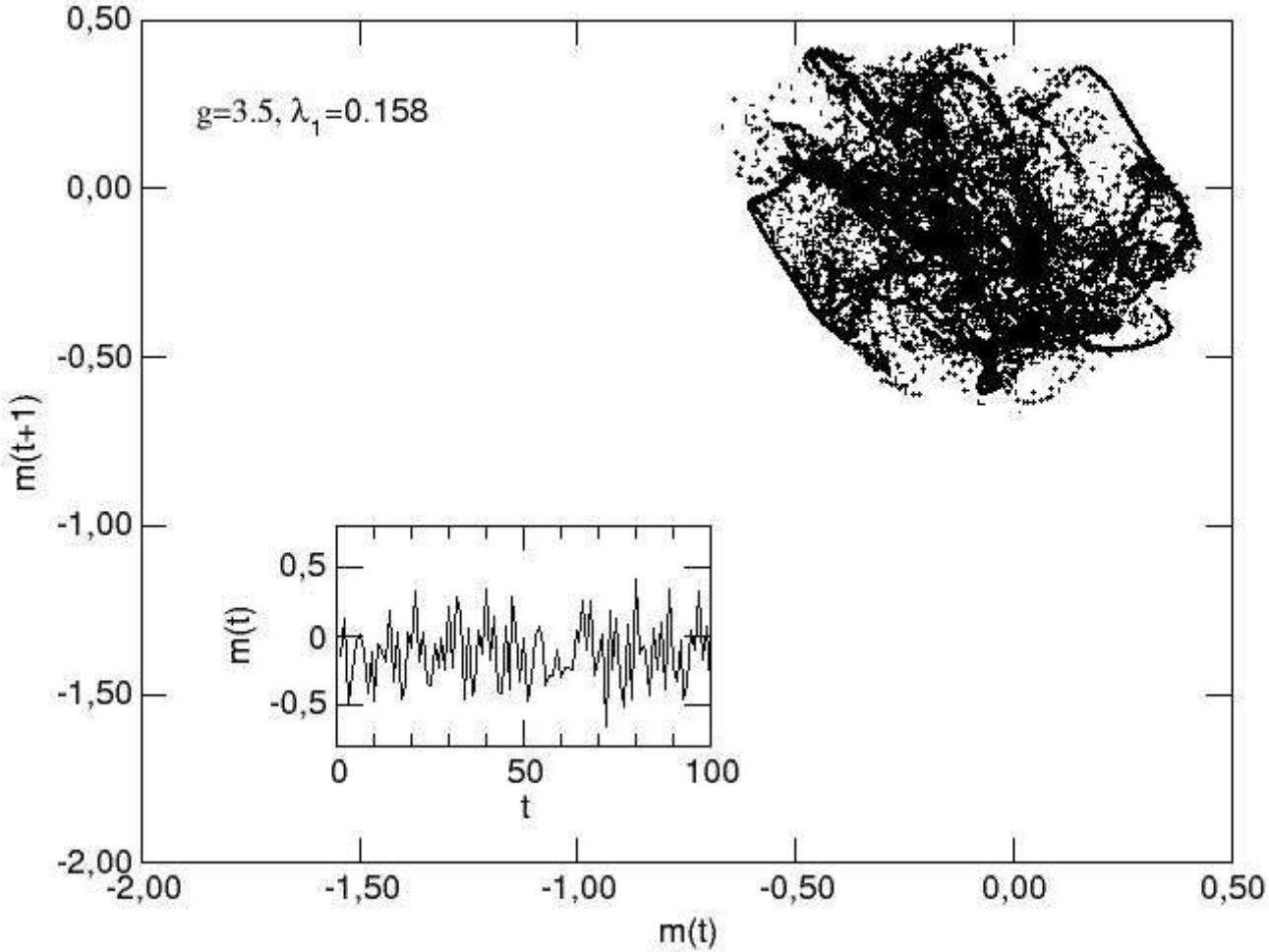

g=3.5, λ1=0.158
m(t+1)
m(t)
0,50
0,00
-0,50
-1,00
-1,50
-2,00
-2,00
-1,50
-1,00
-0,50
0,00
0,50
m(t)
t
0,5
0
-0,5
0
50
100

Unstable-Stable manifold decomposition *(Ruelle, 99).*

By locally projecting X on the stable and unstable direction one can write :

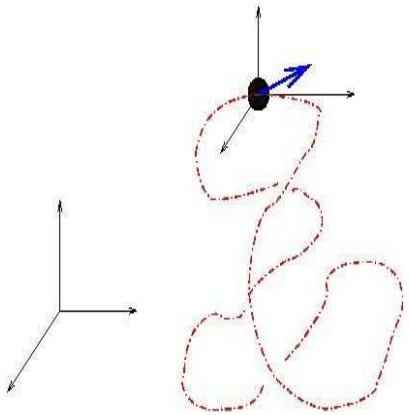
$$\langle \kappa_\sigma \mathbf{X} | \mathbf{\Phi} \rangle = \langle \kappa_\sigma^s \mathbf{X} | \mathbf{\Phi} \rangle + \langle \kappa_\sigma^u \mathbf{X} | \mathbf{\Phi} \rangle$$

where

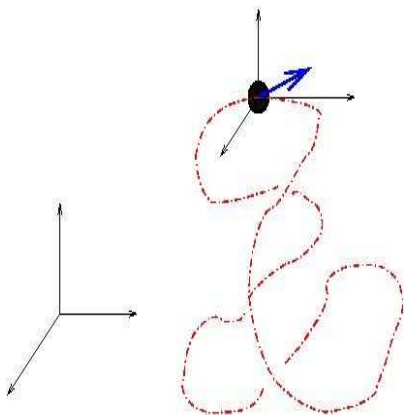
$$\langle \kappa_\sigma^s \mathbf{X} | \mathbf{\Phi} \rangle = \int \rho(d\mathbf{u}) \mathrm{D}\mathbf{G}_\mathbf{u}^\sigma \mathbf{X}^\mathbf{s}(\mathbf{u}).\nabla_{\mathbf{u}(\sigma)} \mathbf{\Phi}$$

$$\langle \kappa_\sigma^u \mathbf{X} | \mathbf{\Phi} \rangle = \rho\left[(\mathbf{\Phi} \circ \mathbf{G}^\sigma)(-\mathbf{div}^\mathbf{u}\mathbf{X}^\mathbf{u})\right] - \rho\left[\mathbf{\Phi}\right] \rho\left[-div^u \mathbf{X}^\mathbf{u}\right]$$

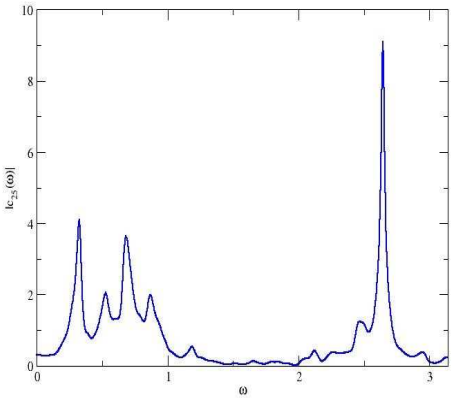
Linear response and resonances.



Linear response and resonances.



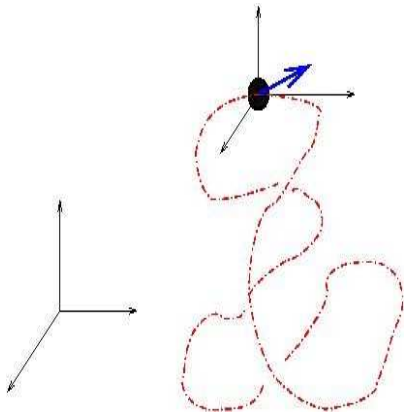
Local projection on the unstable manifold.
Average / SRB measure.
Ruelle-Pollicott resonances.

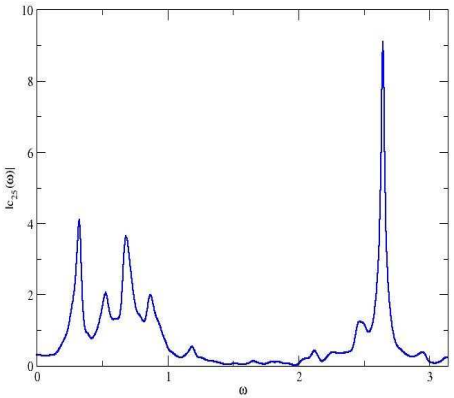


Correlation function

Linear response and resonances.

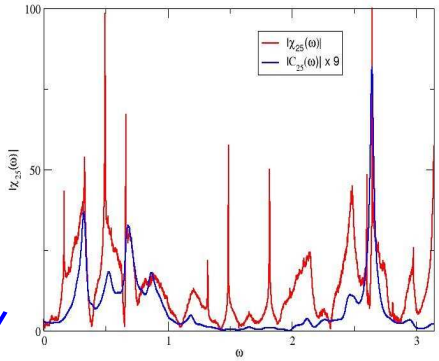
Local projection on the unstable manifold.
Average / SRB measure.
Ruelle-Pollicott resonances.





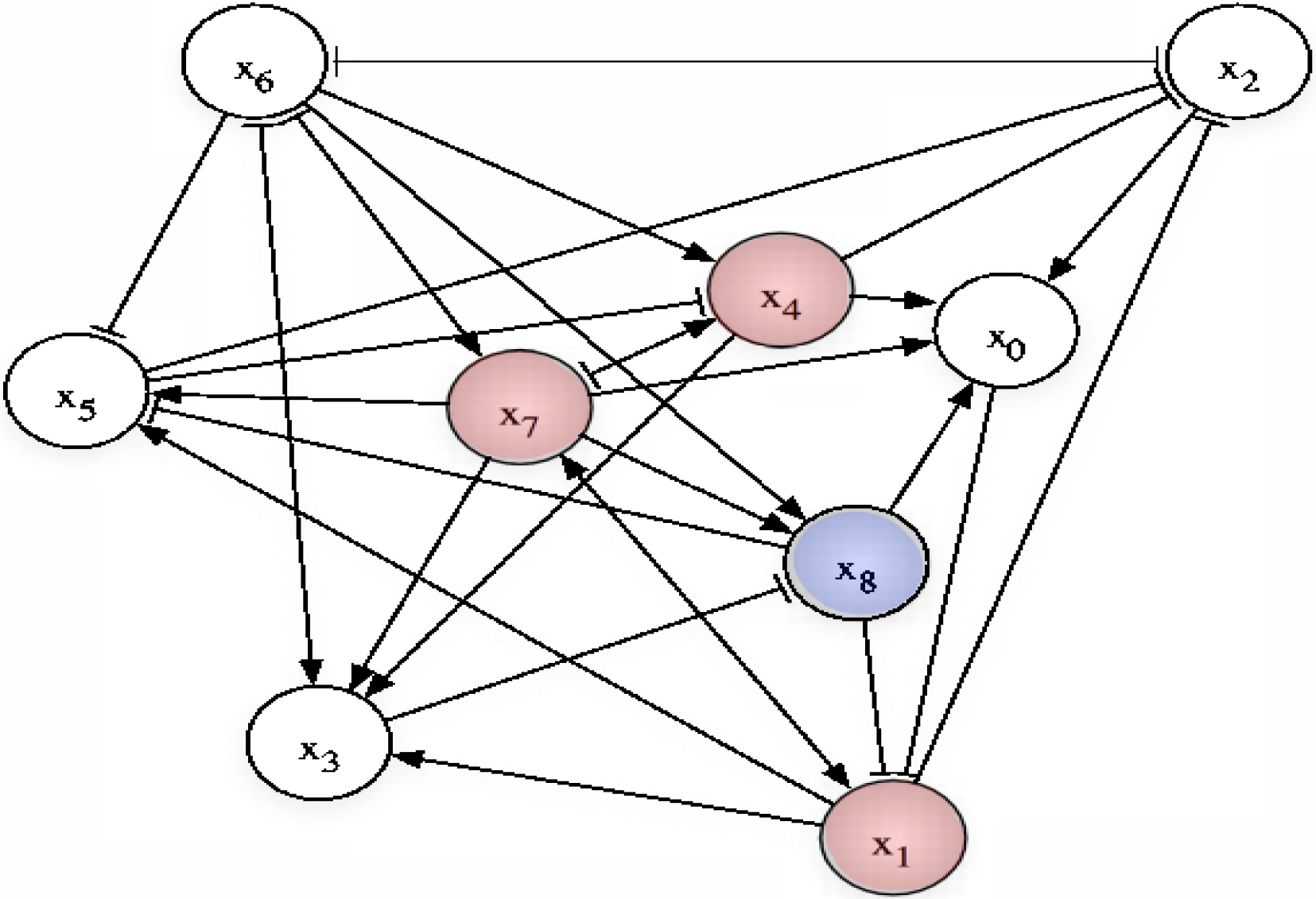
Correlation function

Sum of the local projections on the stable and unstable manifold.
Average / SRB measure.
Ruelle conjecture
(stable resonances).

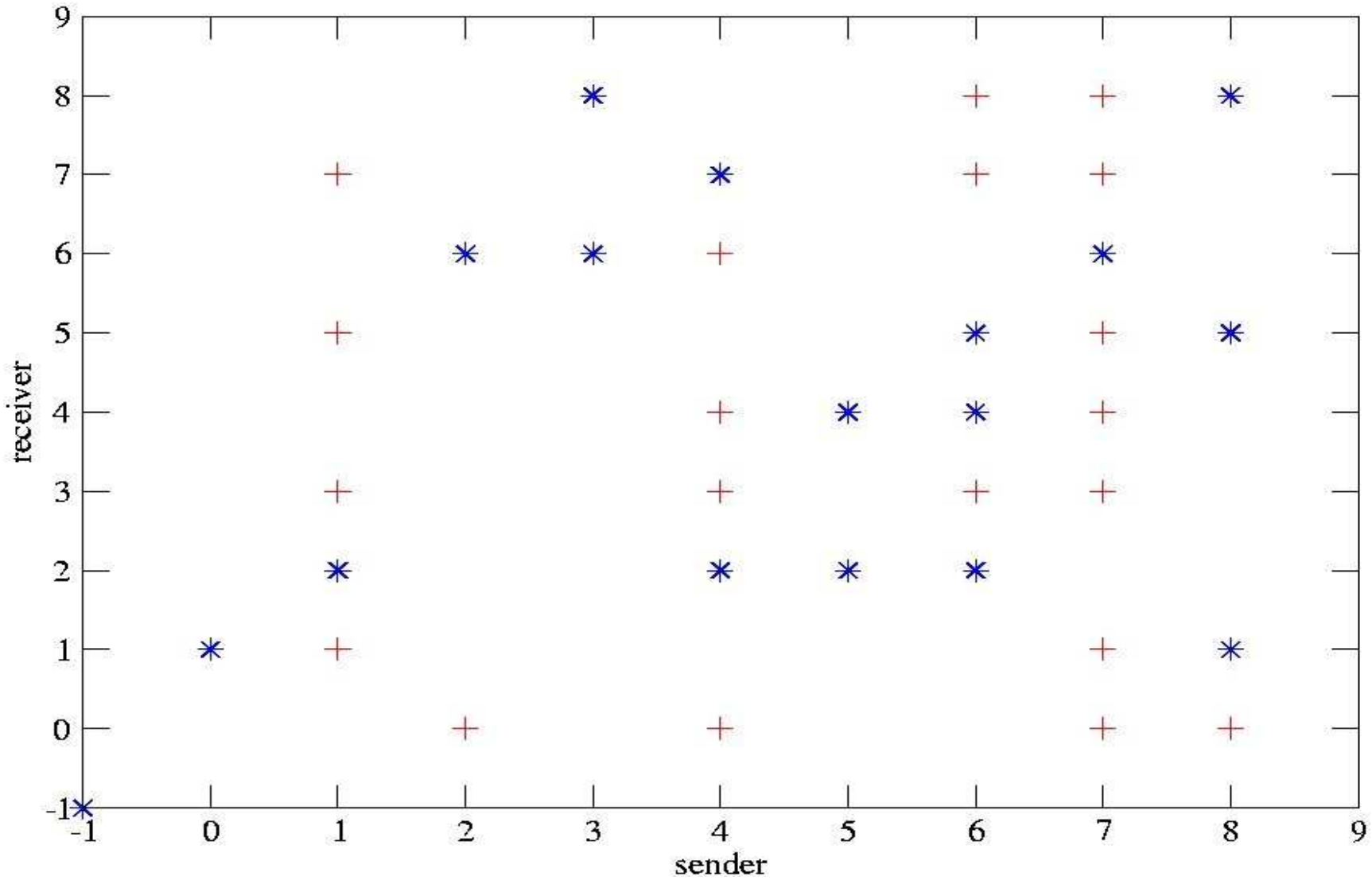


Linear response.

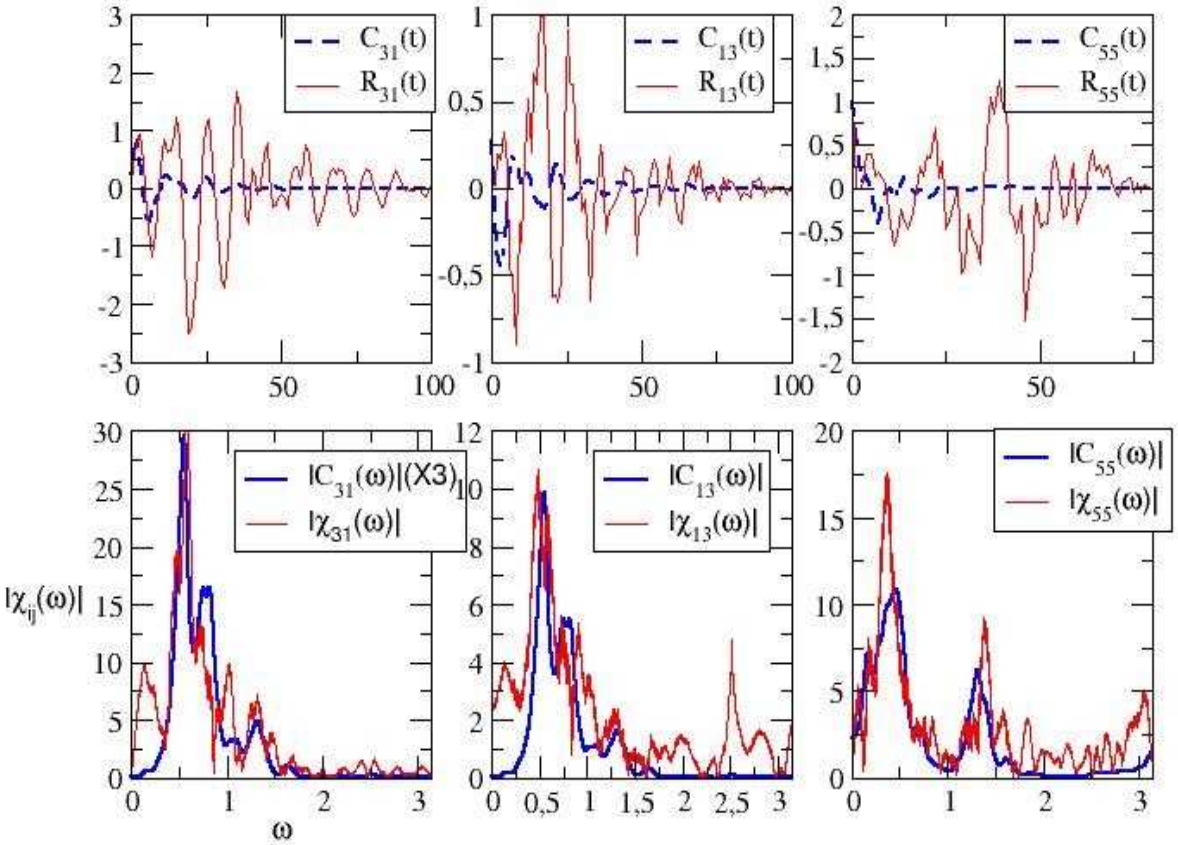
Synaptic network (Cessac-Sepulchre, 2006).

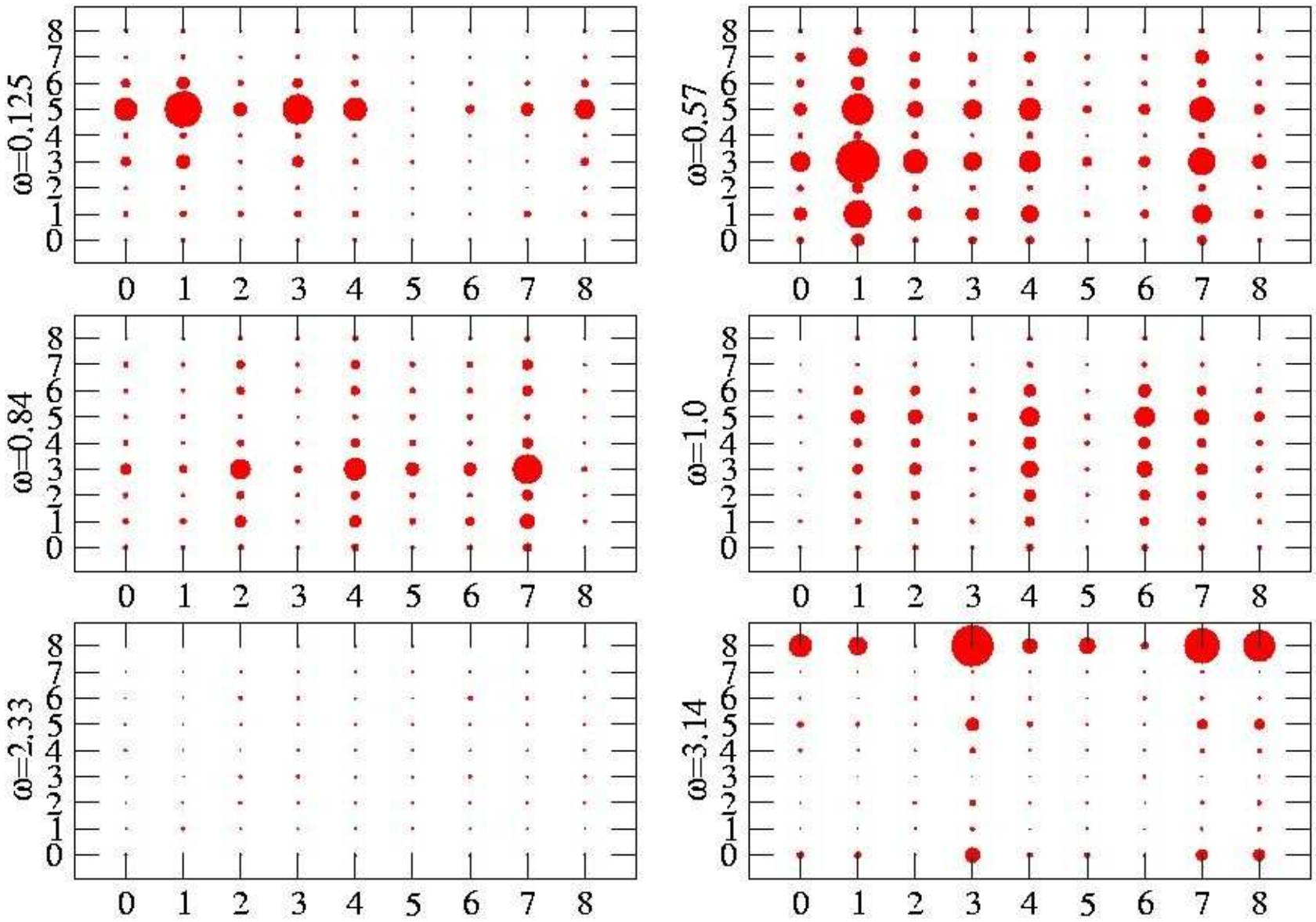


Connectivity matrix

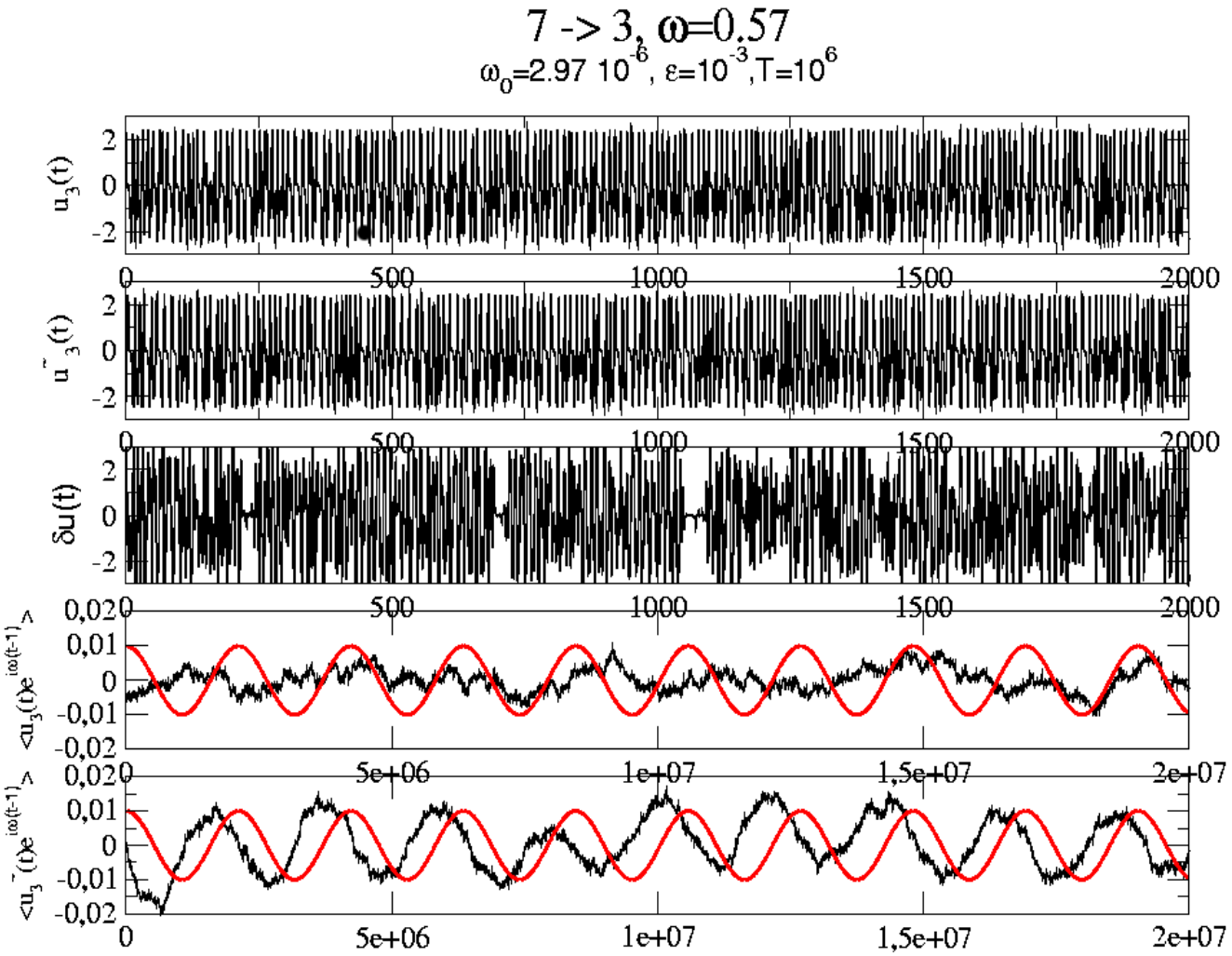


Correlations versus responses.

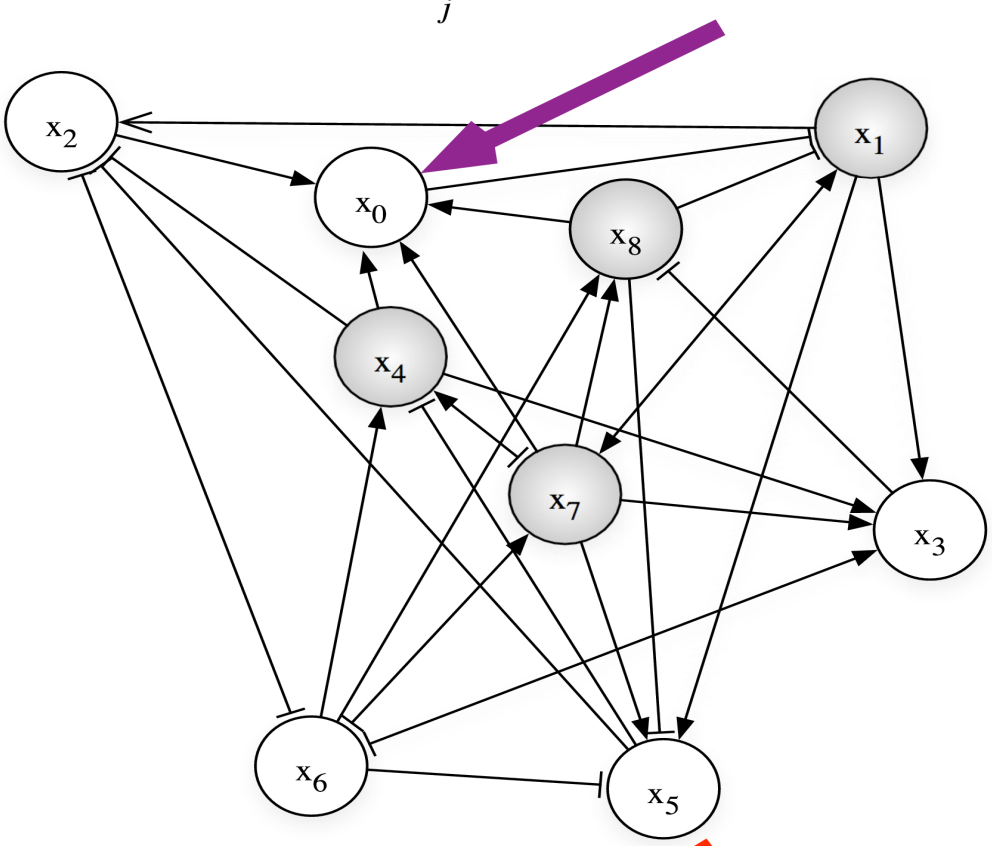



ω=0,125
ω=0,57
ω=0,84
ω=1,0
ω=2,33
ω=3,14
0 1 2 3 4 5 6 7 8

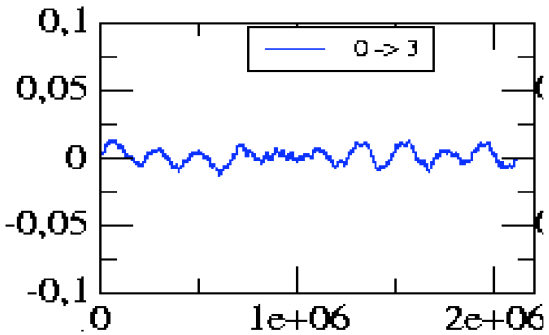
Transmitting a signal hidden in the chaotic background.



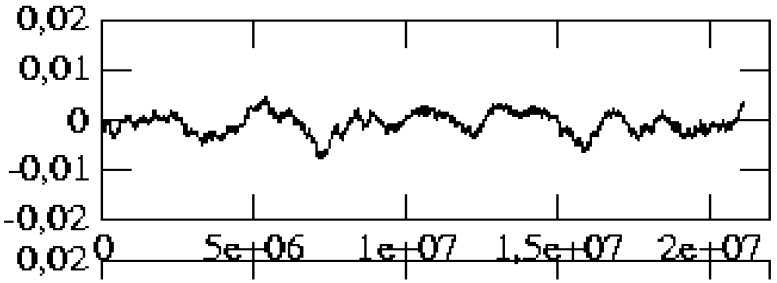
$$u_{0,t+1} = \sum_{j}^{N} J_{0j} f(u_{j,t}) + \varepsilon \cos(\omega_M t)\sin(\omega t) \qquad (\varepsilon \sim 10^{-3})$$



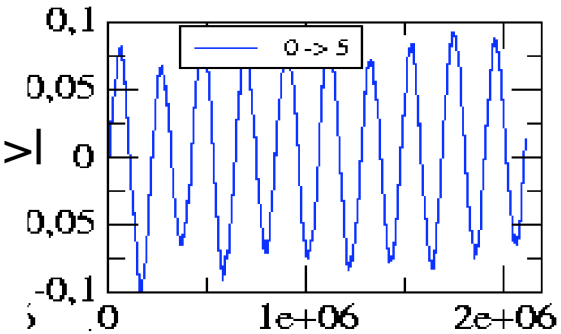
$|< u_{3,t+1} e^{i\omega t} >|$



$|< u_{j,t+1} e^{i\omega t} >| \qquad (\varepsilon = 0)$



$|< u_{5,t+1} e^{i\omega t} >|$



$$< u_{5,t+1} >= \varepsilon \chi_{50}(\omega)\cos(\omega_M t)\sin(\omega t + \phi_{50}(\omega)) + O(\varepsilon^2)$$