From stochastic resonance to stationary measures of McKean-Vlasov type equations

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joint work with
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Outline

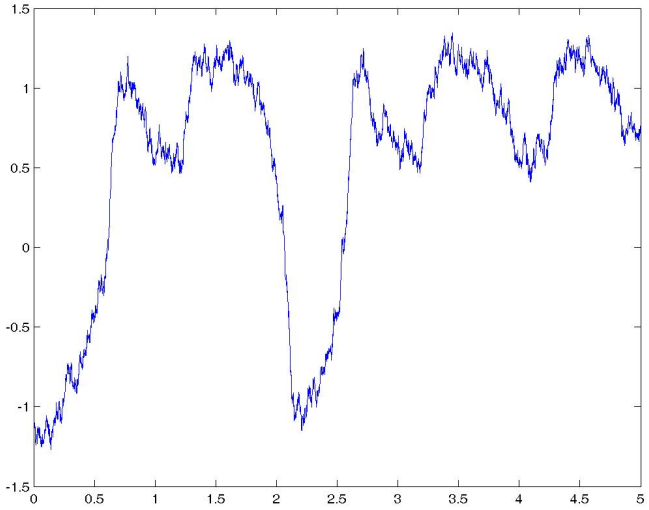
1. Introduction : link between exit problem and stochastic resonance
2. Exit time for a self-stabilizing process (McKean-Vlasov) living in a convex landscape: the inertia of a particular process attracted by its own law...
3. Exit time for a self-stabilizing diffusion in a double-well landscape

The stochastic resonance framework ➟ to understand how a weak deterministic and periodic input of a given dynamical system can be amplified by noisy perturbations.

In particular, there exists an optimal relation between the period $\log(T)$ (deterministic input) and the noise intensity ϵ so that the stochastic paths look like *the most periodic as possible* (quality measures are needed)

$$dX_t = \sqrt{\epsilon} dW_t - V'(X_t)dt + A_0 \cos(2\pi t/T)dt$$

where $(W_t,\ t \geq 0)$ is a Brownien motion, the diffusion coefficient is constant and V is a double-well potential.



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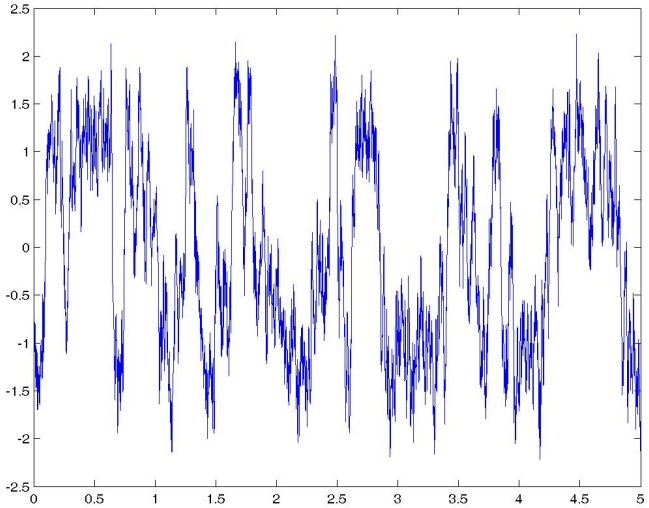

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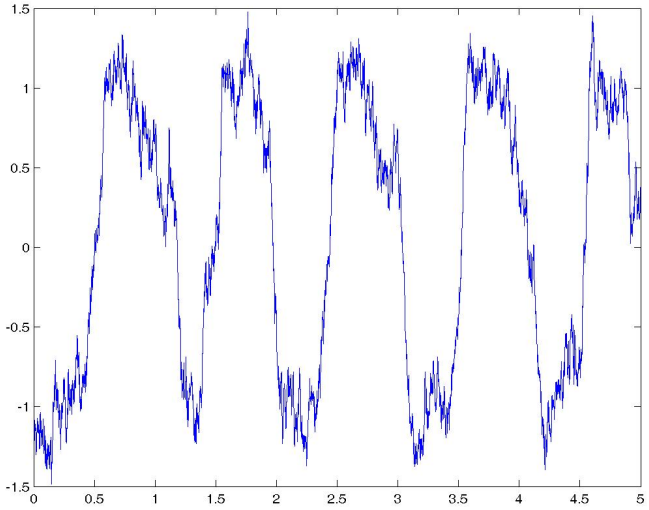

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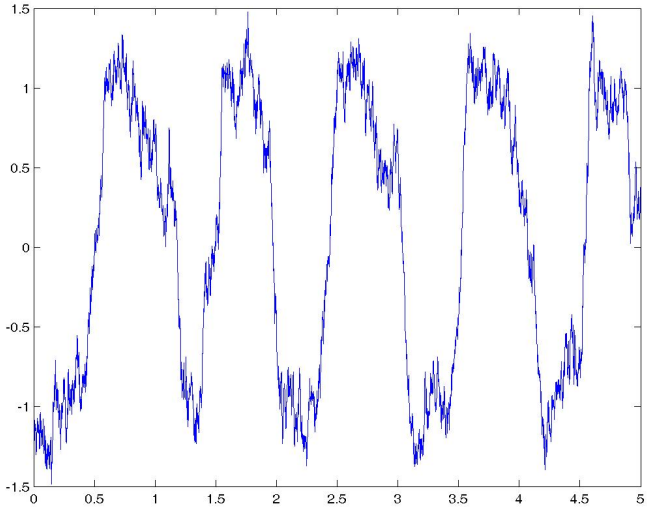


Study of the exit problem (from one well): joint work with P. Imkeller and D. Peithmann.

Several physicists (Jung, Behn, Pantazelou, Moss '92) introduced the *stochastic resonance* phenomenon for globally coupled systems (N individuals in interaction, for instance, neural networks):

$$dX_t^i = \sqrt{\epsilon} dW_t^i - V'(X_t^i)dt - \frac{1}{N}\sum_{j=1}^{N} F'(X_t^i - X_t^j)dt + A_0 \cos(2\pi t/T)dt,$$

where W^i are indep. BM, V is a double-well potential, F' is linear. The quality measures used in practice depends on invariant measures of simplified systems. ➟ Difficulties if ϵ is small and N large (mean field).

Propagation of chaos

The empirical measure $\frac{1}{N}\sum_{j=1}^{N} \delta_{X_t^j}$ converges towards u_t which corresponds to the distribution of the McKean-Vlasov solution:
$dX_t^\epsilon = \sqrt{\epsilon} dW_t - V'(X_t^\epsilon)dt - \int_{\mathbb{R}} F'(X_t^\epsilon - x)du_t^\epsilon(x)\, dt + A_0 \cos(2\pi t/T)dt.$

References: Sznitman '91, Dawson & Gärtner '87, McKean '66 '67, Stroock & Varadhan '79, Oelschläger '85, Funaki '84, Tamura '84 '87, Benachour, Roynette, Talay & Vallois '98, Benachour, Roynette & Vallois '98

In order to analyse *stochastic resonance* for the limit process, it suffices to describe the exit problem (transitions between the metastables states of the dynamical system) for the nonlinear stochastic process:

$$dX_t^\epsilon = \sqrt{\epsilon} dW_t - V'(X_t^\epsilon)dt - \int_{\mathbb{R}} F'(X_t^\epsilon - x) du_t^\epsilon(x)\, dt + A_0 \cos(2\pi t/T) dt.$$

First step: asymptotic behavior as $\epsilon << 1$ without the periodic perturbation.

Difficulties:

- the process is nonlinear
- the drift term depends on the time (non homogeneous)
- the drift term depends on the small parameter ϵ.

The study concerns a fonction V which represents:

- either a convex landscape
- or a double-well landscape

Assumption: V & F loc. Lipschitz, even with polyn. growth.

2. Exit problem: self-stabilizing process living in a convex landscape

$$dX_t^\epsilon = \sqrt{\epsilon} dW_t - V'(X_t^\epsilon)dt - b^\epsilon(t, X_t^\epsilon)\, dt,$$

$$b^\epsilon(t,x) = \int_{\mathbb{R}} F'(x-y) du_t^\epsilon(y)\, dt = \mathbb{E}[F'(x - X_t^\epsilon)].$$

Asymptotic behaviour of the exit time ($\epsilon \to 0$): Find the Large Deviations Principle associated to X^ϵ on the time interval $[0, T]$, then study on $\mathbb{R}_+$.

Convergence & Large Deviations

The self-stabilising process starting from x_0 converges in distribution towards the solution: $\dot{\psi}_t = -V'(\psi_t), \quad \psi_0 = x_0$.
X^ϵ satisfies LDP in $(\mathcal{C}([O,T]), \|\cdot\|_\infty)$ with the associated rate function:

$$I_T^{x_0}(\varphi) = \frac{1}{2}\int_0^T \|\dot{\varphi}_t + V'(\varphi_t) + F'(\varphi_t - \psi_t(x_0))\|^2 dt, \quad \text{si } \varphi \in H^1_{x_0}.$$

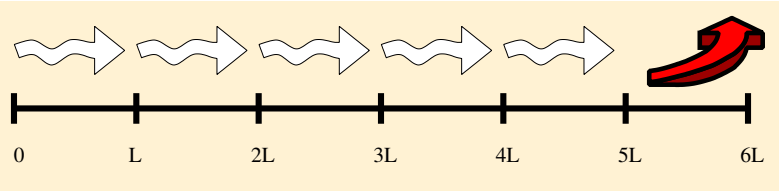
If the SDE is observed on $[s, s+T]$ then the LDP is associated with

$$I_T(\varphi) = \frac{1}{2}\int_0^T \|\dot{\varphi}_t + V'(\varphi_t) + F'(\varphi_t - \psi_{s+t}(x))\|^2\, dt.$$

✔ Exit time for diffusions in convex landscapes.

Intuitive idea: splitting the real axis in large time intervals of length L.

τ denotes the exit time of D which contains the stable point of V : x_{stable} .



1. On each interval time dependent LDP describing the probability to exit the domain D in this interval.
2. After a large number of intervals: the exit probability is close to a particular p associated with

$$I_L^\infty(\varphi) = \frac{1}{2}\int_0^L \|\dot{\varphi}_t + V'(\varphi_t) + F'(\varphi_t - x_{stable})\|^2\, dt$$

Minimal cost $\mathcal{E}^\infty = \inf\{I_t^\infty(\varphi) : \varphi \text{ cont.}, \varphi_0 = x_{\text{stable}}, \varphi_t \in D^c, t > 0\}$

- The convexity is essential : what happens in a double-well landscape ?

Self-stabilizing process in a double-well landscape V

$$dX_t^\epsilon = -V'(X_t^\epsilon)dt - \int_{\mathbb{R}} F'(X_t^\epsilon - x)du_t^\epsilon(x)\,dt + \sqrt{\epsilon}dW_t.$$

The minima of V are reached at $x = -a$ and a.

- Aim: to describe the transitions in the small noise limit.

- Simplification: stationary regime.
We replace u_t^ε by an invariant u^ε.

Invariant measures and asymptotic behavior as $\varepsilon \to 0$

In order to find an invariant measure we solve

$$\frac{\epsilon}{2}\,u_\epsilon''(x) + \big(u_\epsilon(x)(V'(x) + F' \star u_\epsilon(x))\big)' = 0.$$

Each invariant measure takes the exponential form

$$u_\epsilon(x) = \frac{1}{\lambda(u_\epsilon)} \exp\left[-\frac{2}{\epsilon}\left(\int_0^x F' \star u_\epsilon(y)dy + V(x)\right)\right] = \mathcal{A}(u_\epsilon).$$

We have to solve a fixed point problem associated with the application $\mathcal{A}$
➥ simplification in the particular linear $F'(x) = \alpha x$.

Linear case

If $F'(x) = \alpha x$ and V'' convex, there exist exactly:

- one symmetric invariant measure
- two asymmetric measures close to δ_a et δ_{-a}

If m is the mean of the invariant measure then $m = \Psi_\epsilon(m)$ with

$$\Psi_\epsilon(m) = \frac{\int_{\mathbb{R}} x \exp\left[-\frac{2}{\epsilon}\left(V(x)+\alpha\frac{x^2}{2}-\alpha m x\right)\right] dx}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon}\left(V(x)+\alpha\frac{x^2}{2}-\alpha m x\right)\right] dx}$$

There exist exactly 3 solutions corresponding to 3 invariant measures.

0 is an obvious solution (symmetric measure). The asymptotic behavior of the corresponding measure is emphasized by the Laplace method:

$$u^\epsilon(x) = \frac{1}{\mathcal{Z}^\epsilon} \exp -\frac{2}{\epsilon}\left\{ V(x) + \frac{\alpha x^2}{2} \right\}$$

➥ u^ϵ converges towards a limit measure ($\epsilon \to 0$) whose support belongs to the set of solutions of $\alpha x_0 + V'(x_0) = 0$.

- if $\alpha \geq -V''(0)$ then $u^\epsilon \to \delta_0$
- if $\alpha < -V''(0)$ then $u^\epsilon \to \frac{1}{2}\delta_{x_0} + \frac{1}{2}\delta_{-x_0}$.

Generalization to polynomial interaction functions

If V'' and F'' are both convex functions (+ suitable weak conditions), then there exist exactly **three invariant measures**: one is symmetric and two are asymmetric close to δ_a and δ_a. The symmetric one converges ($\varepsilon \to 0$) towards $\frac{1}{2}\delta_{-x_0} + \frac{1}{2}\delta_{x_0}$ with

$$V'(x_0) + \frac{1}{2}F'(2x_0) = 0 \text{ and } V(x_0) + \frac{F''(0)+F''(2x_0)}{2} \geq 0$$

If $F(0) \geq -V(0)$ then $x_0 = 0$ otherwise the support contains 2 points. We use the Schauder's fixed point theorem in order to solve $u^\epsilon = \mathcal{A}(u^\epsilon)$.

$$dX_t^\epsilon = -V'(X_t^\epsilon)dt - \int_{\mathbb{R}} F'(X_t^\epsilon - x)du_t^\epsilon(x)\,dt + \sqrt{\epsilon}dW_t.$$

Exit problem

For any $\delta > 0$, $\lim_{\epsilon\to 0} \mathbb{P}_x(e^{(\mathcal{E}^\infty+\delta)/\epsilon} > \tau > e^{(\mathcal{E}^\infty-\delta)/\epsilon}) = 1$
and $\lim_{\epsilon\to 0} \epsilon \ln \mathbb{E}_x[\tau] = \mathcal{E}^\infty$.

Here $\mathcal{E}^\infty = \inf\{I_T^\infty(\varphi) : \varphi \text{ cont.}, \varphi_0 = x_{\text{stable}}, \varphi_T \in D^c, T > 0\}$ with

$$I_T^\infty(\varphi) = \frac{1}{2}\int_0^T \|\dot{\varphi}_t + V'(\varphi_t) + \Phi(\varphi_t)\|^2\,dt$$

where Φ represents one of the following functions:

1. $\Phi(x) = F'(x)$ if u_ϵ is symmetric and $F''(0) \geq -V''(0)$
2. $\frac{1}{2}F'(x - x_0) + \frac{1}{2}F'(x + x_0)$ if u_ϵ is symmetric and $F''(0) < -V''(0)$
3. $F'(x - a)$ (resp. $F'(x + a)$) if u_ϵ converges towards δ_a (resp. δ_{-a}).

➥ Open questions: What's about the domain of attraction for these invariant measures ? What happens if we add a deterministic periodic perturbation ? Can we describe stochastic resonance ?