Analytic perturbation of Sylvester and Lyapunov matrix equations

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Abstract

We consider an analytic perturbation of the Sylvester matrix equation. Mainly we are interested in the singular case, that is, when the null space of the unperturbed Sylvester operator is not trivial, but the perturbed equation has a unique solution. In this case, the solution of the perturbed equation can be given in terms of a Laurent series. Here we provide a necessary and sufficient condition for the existence of a Laurent series with a first order pole. An efficient recursive procedure for the calculation of the Laurent series' coefficients is given. Finally, we show that in the particular, but practically important case of semisimple eigenvalues, the recursive procedure can be written in a compact matrix form.

1 Introduction

The Sylvester matrix equation $AX + XB = C$ (or Lyapunov equation when $B = A^T$) have numerous applications in control and system theory (see for example [12, 17, 22]). It is well known [6, 19] that when A and B are nearly singular, then solving the Sylvester equation becomes difficult and most standard methods are likely to provide a wrong and meaningless solution. One way to overcome this problem is to use deflation techniques as proposed in [6]. Another possible approach that we investigate in this paper is to use analytic *singular* perturbation techniques. To the best of our knowledge, the analytic singular perturbation technique have not yet been applied to the perturbation analysis of Sylvester equation. In this latter approach, the matrices A and B are considered to be analytical perturbations $A(z)$ and $B(z)$ of some nominal matrices $A(0)$ and $B(0)$, respectively. One then has to solve the perturbed Sylvester (Lyapunov) equation

$$A(z)X + XB(z) = C(z), \tag{1}$$

that is, to obtain a solution $X(z)$ as a Laurent series. There are at least two advantages of this approach. First, all the computations needed to obtain the coefficients of the Laurent series for $X(z)$ are numerically stable, because we are using a *reduction process* which works on spaces of small dimension. Second, the Laurent series expansion of $X(z)$ permits one to compute a good approximation of the exact solution and provides interesting information on the behavior of the approximate solution for small values of z.

For other applications involving parametric Sylvester matrix equations (1) see for example [19, 25] and the references therein.

Therefore we will study (1) where $A(z) = \sum_{k=0}^{\infty} z^k A^{(k)} \in \mathbb{C}^{m\times m}$, $B(z) = \sum_{k=0}^{\infty} z^k B^{(k)} \in \mathbb{C}^{n\times n}$, $C(z) = \sum_{k=0}^{\infty} z^k C^{(k)} \in \mathbb{C}^{m\times n}$. These series are assumed to be convergent in some disc in the complex plane, around the origin $z = 0$. In particular, when $z = 0$, the equation

$$A(0)X + XB(0) = C(0) \tag{2}$$

is called the unperturbed Sylvester equation.

We distinguish between *regular* and *singular* perturbations. The perturbation is said to be regular if the unperturbed equation (2) has a unique solution, whereas the perturbation is said to be singular if the unperturbed equation (2) possesses none or multiple solutions, but the perturbed equation (1) has a unique solution for any $z \neq 0$ and sufficiently small. The difference between regular and singular perturbations can also be explained in terms of the eigenvalues of $A(z)$ and $B(z)$. Let

$$\delta(z) := \mathrm{dist}[\sigma(A(z)), \sigma(B(z))] = \min_{i,j} |\lambda_i(z) - \mu_j(z)|,$$

where $\lambda_i(z)$ and $\mu_i(z)$ are the respective eigenvalues of $A(z)$ and $B(z)$. Then, in the case of a regular perturbation, $\delta(0) \neq 0$ whereas in the case of a singular perturbation, $\delta(0) = 0$ but $\delta(z) > 0$ for $z \neq 0$ and sufficiently small. If $A(z)$ and $B(z)$ are normal matrices, then the following bounds are available [5]:

$$||X(z)|| \leq \frac{c}{\delta(z)} ||C(z)||. \tag{3}$$

Thus, we can see that the bounds grow to infinity in the case of singular perturbations. This is another justification for using the term "singular perturbations". Apart

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from the above bounds there are many other bounds available for additive perturbations (e.g. see [9, 10, 18] and references therein). Since all of them are given in terms of some norm, they only indicate that in the singular case the solution may be very large, but they do not provide any direction for the deviation of the perturbed solution. By contrast, using a more specific analytic approach, we are able to exhibit the asymptotic behaviour of the perturbed solution. Moreover, as will be demonstrated in the sequel, if the unperturbed system has multiple solutions, then the perturbed solution may converge to some particular solution of the unperturbed problem. In the latter case, the normwise estimation (3) does not reflect the real situation. Finally, we would like to mention the paper [15] that uses the special analytic perturbation $B(z) = B + zI$ to obtain feasibility conditions for the unperturbed Sylvester equation.

2 Preliminaries on Sylvester matrix equation

Let A, B, C be complex-valued matrices in $\mathbb{C}^{m\times m}$, $\mathbb{C}^{n\times n}$, $\mathbb{C}^{m\times n}$ respectively and consider the Sylvester equation in $\mathbb{C}^{m\times n}$, that is:

$$AX + XB = C. \tag{4}$$

In this section, we first discuss the structure of the null space as well as the feasibility conditions of equation (4) in terms of eigen-elements of matrices A and B. Our exposition follows closely the paper [14] (see also related work [21]).

For any two vectors (x, y) in $\mathbb{C}^m$, $< x, y >$ denotes the usual scalar product, that is $< x, y >= y^*x$, where y^* denotes the conjugate transpose of y. For any two matrices X, Y in $\mathbb{C}^{m\times n}$, $\langle X, Y\rangle$ denotes the usual scalar product $tr(Y^*X)$, where again Y^* denotes the conjugate transpose matrix of Y.

Let α_i be an eigenvalue of matrix A. Then $-\alpha_i$ is an eigenvalue of $-A$ and there exists a Jordan chain of generalized eigenvectors of $-A$ in $\mathbb{C}^m$ satisfying

$$(-A + \alpha_i I_m)x_{ir} = x_{i,r-1}, \quad r = 1, ..., \rho_i.$$

By convention, $x_{i0} = 0$. Let $y_{ir}, r = 1, ..., \rho_i$ be a system of generalized eigenvectors for the adjoint matrix $-A^*$. These vectors satisfy the following equations.

$$(-A^* + \alpha_i^* I_m)y_{ir} = y_{i,r-1}, \quad r = 1, ..., \rho_i.$$

In addition, the systems of vectors $\{x_{ir}\}$ and $\{y_{ir}\}$ are biorthogonal, that is:

$$< x_{ip}, y_{jr} >= 0, \quad \text{if} \quad i \neq j,$$

and if $i = j$,

$$< x_{ip}, y_{ir} >= \begin{cases} 1, & p + r = \rho_i + 1, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Similarly, let β_j be an eigenvalue of the matrix B. Then there exists a system of eigenvectors $u_{js}, s = 1, ..., \sigma_j$ in $\mathbb{C}^n$, satisfying

$$(B - \beta_j I_n)u_{js} = u_{j,s-1}, \quad s = 1, ..., \sigma_j.$$

By convention, $u_{j0} = 0$. Let $v_{js}, s = 1, ..., \sigma_j$ be a system of the generalized eigenvectors for the adjoint matrix B^* such that

$$(B^* - \beta_j^* I_n)v_{js} = v_{j,s-1}, \quad s = 1, ..., \sigma_j,$$

with $v_{j0} = 0$. The vectors $\{u_{js}\}$ and $\{v_{js}\}$ also form a biorthogonal system. As above,

$$< u_{is}, v_{jl} >= 0, \quad \text{if} \quad i \neq j,$$

and if $i = j$,

$$< u_{is}, v_{il} >= \begin{cases} 1, & s + l = \sigma_i + 1, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Now, consider the linear operator $L : \mathbb{C}^{m\times n} \to \mathbb{C}^{m\times n}$ defined by $X \mapsto LX := AX + XB$, $X \in \mathbb{C}^{m\times n}$. All eigenvalues of L are of the form $\alpha_i + \beta_j$, where α_i and β_j are eigenvalues of the matrices A and B, respectively. Suppose that $\alpha_i + \beta_j = 0$ for some indicies i and j. Then, the operator L has a nontrivial null space spanned by the basis elements

$$X_{ijk} = \sum_{s=1}^{k} x_{i,k+1-s}v_{js}^*, \quad k = 1, ..., \mu_{ij}, \tag{7}$$

where $\mu_{ij} = \min(\rho_i, \sigma_j)$.

Similarly, the null space of the adjoint operator $L^* : \mathbb{C}^{m\times n} \to \mathbb{C}^{m\times n}$ defined by $Y \mapsto L^*Y := A^*Y + YB^*$, $Y \in \mathbb{C}^{m\times n}$ is spanned by the basis elements

$$Y_{ijk} = \sum_{s=1}^{k} y_{i,k+1-s}u_{js}^*, \quad k = 1, ..., \mu_{ij}. \tag{8}$$

Now we can state the necessary and sufficient feasibility condition for the Sylvester equation (4) in terms of the left and right eigenvectors of the matrices A and B.

Proposition 1 *The Sylvester equation (4) is feasible if and only if*

$$\sum_{l=1}^{r} y_{p,r+1-l}^* C u_{ql} = 0, \quad r = 1, ..., \mu_{pq}, \tag{9}$$

with $\mu_{pq} = \min(\rho_p, \sigma_q)$.

PROOF: The feasibility condition for the Sylvester equation (4), which can also be rewritten as $LX = C$, reads

$$\langle Y_{pqr}, C\rangle = tr(Y_{pqr}^* C) = 0 \ \ \forall p, q \,\text{s.t.}\, \alpha_p + \beta_q = 0. \tag{10}$$

Next, by taking Y_{pqr} as in (8) and substituting it into (10), we get the feasibility condition (9).

□

3 Analytic perturbation under non-degeneracy assumption

In this section we briefly discuss the implementation of the algebraic reduction technique of [1] under the non-degeneracy assumption. For more details on the inversion of an analytically perturbed operator the interested reader is also referred to the following papers and books: [7, 8, 13, 20, 23, 24].

Let $(X, \langle .,. \rangle)$ be a finite dimensional Hilbert space and consider a family of linear operators $L^{(i)} : X \to X$, $i = 0, 1, \ldots$. Let $L(z) : X \to X$ be the linear operator

$$L(z) := L^{(0)} + zL^{(1)} + z^2 L^{(2)} + \ldots, \qquad (11)$$

that is, $L(z)$ is an analytic perturbation of $L^{(0)}$. Consider the linear equation $L^{(0)}x = c^{(0)}$, with $c^{(0)} \in X$, and its perturbed version

$$L(z)x(z) = c(z), \qquad (12)$$

with $c(z) = c^{(0)} + zc^{(1)} + z^2 c^{(2)} + \ldots$, and z a scalar parameter. Suppose that the unperturbed operator $L^{(0)}$ is not invertible, whereas $L(z)$ is invertible for sufficiently small and nonzero z, which is the case of *singular* perturbations. Note that the perturbed solution $x(z)$ can have a singularity at $z = 0$. Let the vectors $\{u_i\}_{i=1}^p$ form a basis for the null space of $L^{(0)}$ and let the vectors $\{v_i\}_{i=1}^p$ form a basis for the null space of the adjoint operator $L^{(0)*}$. Following [16], consider the *non-degeneracy* assumption

$$\det(\{< v_i, L^{(1)}u_j >\}_{i,j=1}^n) \neq 0. \qquad (13)$$

If (13) holds, then as one can conclude from the results of [1] the Laurent expansion for the perturbed solution $x(z)$ has at most a simple pole. That is,

$$x(z) = \frac{1}{z}x^{(-1)} + x^{(0)} + zx^{(1)} + \ldots. \qquad (14)$$

It turns out that the non-degenerate case, that is, when (13) holds, describes the most common case. Furthermore, the case of a higher order pole reduces to the case of a first order pole via a so-called "reduction technique" (see for example [1, 4] for a detailed development of this technique). Therefore, in the sequel, and for the sake of clarity of exposition, we voluntarily restrict ourselves to the non-degenerate case, that is, we will assume that (13) holds. Substituting series (11) and (14) into equation (12) and equating terms with same powers of z, we obtain the following system of fundamental equations:

$$\begin{array}{rcll} L^{(0)}x^{(-1)} & = & 0 & (f0) \\ L^{(0)}x^{(0)} + L^{(1)}x^{(-1)} & = & c^{(0)} & (f1) \\ & \vdots & & \vdots \end{array}$$

Then, from equation (f0) we conclude that

$$x^{(-1)} = \sum_{j=1}^{p} \gamma_j u_j. \qquad (15)$$

The coefficients γ_j can be found from the feasibility conditions of the second fundamental equation (f1), that is:

$$\langle v_i, c^{(0)} - L^{(1)}x^{(-1)} \rangle = 0, \quad i = 1, \ldots, p.$$

Substituting (15) into the above equations, the coefficients $\{\gamma_i\}$ must be solutions to the linear system of equations

$$\sum_{j=1}^{p} \langle v_i, L^{(1)}u_j \rangle \gamma_j = \langle v_i, c^{(0)} \rangle, \quad i = 1, \ldots, p. \qquad (16)$$

In the non-degenerate case, the system (16) has a unique solution. In particular, if $\langle v_i, c^{(0)} \rangle = 0$ for all $i = 1, \ldots, p$, that is, unperturbed equation $L^{(0)}x = c^{(0)}$ is feasible, the singular term in the Laurent series (14) vanishes and the first regular coefficient $x^{(0)}$ corresponds to the best approximation for the perturbed problem. Next, assuming that the coefficients $x^{(-1)}, \ldots, x^{(k-1)}$ are known, the next coefficient $x^{(k)}$ can be calculated by a recursive formula. Indeed, the (fk+1) fundamental equation

$$L^{(0)}x^{(k)} = c^{(k)} - \sum_{i=1}^{k+1} L^{(i)}x^{(k-i)}$$

has a general solution of the form

$$x^{(k)} = x_*^{(k)} + \sum_{j=1}^{p} \gamma_j^{(k)} u_j \qquad (17)$$

where $x_*^{(k)}$ is any particular solution. For instance, one may choose $x_*^{(k)} := L^{(0)\dagger}(c^{(k)} - \sum_{i=1}^{k+1} L^{(i)}x^{(k-i)})$, where $L^{(0)\dagger}$ is the Moore-Penrose generalized inverse of $L^{(0)}$. Then, substituting again the expression (17) into the feasibility conditions for the next fundamental equation (fk+2), we obtain

$$\langle v_i, c^{(k+1)} - \sum_{s=1}^{k+2} L^{(s)}x^{(k+1-s)} \rangle = 0, \quad i = 1, \ldots, p.$$

which yields the linear system of equations

$$\sum_{j=1}^{p} \langle v_i, L^{(1)}u_j \rangle \gamma_j^{(k)} = \qquad (18)$$

$$\langle v_i, c^{(k+1)} - L^{(1)}x_*^{(k)} - \sum_{s=2}^{k+2} L^{(s)}x^{(k+1-s)} \rangle, i = 1, \ldots, p.$$

The latter system uniquely determines the coefficients $\{\gamma_i^{(k)}\}$. It is worth noting that the linear systems of equations (16) and (18) have the same coefficient matrix in the left hand side. This fact leads to efficient computational procedures. For example, if a LU-decomposition is used for solving (16), it can be stored and re-used for solving (18).

4 Perturbation analysis of the Sylvester equation

First, note that that the perturbed Sylvester equation (1) can be viewed as the perturbed linear system $L(z)X(z) = C(z)$ with perturbed linear operator $L(z) : X \mapsto A(z)X + XB(z)$. The linear operator $L(z)$ acts on the Hilbert space of matrices $\mathbb{C}^{m\times n}$ endowed with the scalar product $\langle X, Y\rangle = tr(X^*Y)$. As already mentioned before, we only study the case of first-order singular perturbations, that is, when the Laurent series for the perturbed solution $X(z)$ has a simple pole at $z = 0$. In this case, the system of fundamental equations for the Sylvester operator takes the form

$$A^{(0)}X^{(-1)} + X^{(-1)}B^{(0)} = 0 \ (fs0)$$

$$(A^{(0)}X^{(0)} + X^{(0)}B^{(0)})+$$
$$+(A^{(1)}X^{(-1)} + X^{(-1)}B^{(1)}) = C^{(0)} \ (fs1)$$

and so on. To solve the above system, we will follow the general method briefly outlined in the previous section. From the fundamental equation (fs0), $X^{(-1)}$ can be written as the following linear combination

$$X^{(-1)} = \sum_{ijk} \gamma_{ijk}^{(-1)} X_{ijk},$$

where X_{ijk} is as in (7) and where the summation is taken over all pairs of indicies (i, j) such that $\alpha_i + \beta_j = 0$. The coefficients $\gamma_{ijk}^{(-1)}$ are solutions of the linear system of equations induced by the feasibility conditions of the second fundamental equation (fs1). This requires computing the quantities $\langle Y_{pqr}, L^{(1)}X_{ijk}\rangle$, with $L^{(1)} : X \mapsto A^{(1)}X + XB^{(1)}$, and with X_{ijk}, Y_{pqr} as in (7) and (8).

$$\langle Y_{pqr}, L^{(1)}X_{ijk}\rangle = \langle Y_{pqr}, A^{(1)}X_{ijk} + X_{ijk}B^{(1)}\rangle \quad (19)$$

The matrix of the linear system (16) in the unknown variables $\gamma_{ijk}^{(-1)}$ is filled-up with the above coefficients $\langle Y_{pqr}, L^{(1)}X_{ijk}\rangle$ defined in (19). Under the non-degeneracy assumption (13), which now reads

$$\det\left[\langle Y_{pqr}, L^{(1)}X_{ijk}\rangle\right] \neq 0, \quad (20)$$

the matrix is non-singular so that there is a unique solution $\gamma_{ijk}^{(-1)}$. We now develop $\langle Y_{pqr}, A^{(1)}X_{ijk}\rangle$ and $\langle Y_{pqr}, X_{ijk}B^{(1)}\rangle$ separately. Using the feasibility condition (9) given in terms of eigen-elements of A and B, and using the biorthogonality condition (6), we obtain

$$\langle Y_{pqr}, A^{(1)}X_{ijk}\rangle = \sum_{l=1}^{r} y_{p,r+1-l}^* A^{(1)} X_{ijk} u_{ql}$$

$$= \begin{cases} 0, & j \neq q, \\ \sum_{l=1}^{r} y_{p,r+1-l}^* A^{(1)} \sum_{1\le s\le k;\ s+l=\sigma_q+1} x_{i,k+1-s}, & j = q, \end{cases}$$

Similarly, using again (9) and biorthogonality condition (5), we get

$$\langle Y_{pqr}, X_{ijk}B^{(1)}\rangle = \sum_{l=1}^{r} y_{p,r+1-l}^* X_{ijk} B^{(1)} u_{ql}$$

$$= \begin{cases} 0, & i \neq p, \\ \sum_{l=1}^{r} \sum_{1\le s\le k;\ s+r+1-l=\rho_p+1} v_{j,k+1-s}^* B^{(1)} u_{ql}, & i = p, \end{cases}$$

We are now able to formulate our main result:

Theorem 1 *Under the non-degeneracy assumption (20), the linear system of equations in the unknown variables* $\gamma_{ijk}^{(-1)}$

$$\sum_{ik} \gamma_{iqk}^{(-1)} \left(\sum_{l=1}^{r} y_{p,r+1-l}^* A^{(1)} \sum_{1\le s\le k;\ s+l=\sigma_q+1} x_{i,k+1-s} \right) +$$

$$\sum_{jk} \gamma_{pjk}^{(-1)} \left(\sum_{l=1}^{r} \sum_{1\le s\le k;\ s+r+1-l=\rho_p+1} v_{j,k+1-s}^* B^{(1)} u_{ql} \right) =$$

$$\sum_{l=1}^{r} y_{p,r+1-l}^* C^{(0)} u_{ql}, \ r = 1, ..., \mu_{pq}. \quad (21)$$

has a unique solution and the perturbed solution of the Sylvester equation (1) has a Laurent series expansion with a simple pole

$$X(\varepsilon) = \frac{1}{\varepsilon}X^{(-1)} + X^{(0)} + \varepsilon X^{(1)} + ... \quad (22)$$

where $X^{(-1)} = \sum_{ijk} \gamma_{ijk}^{(-1)} X_{ijk}$ *and the other coefficients can be calculated recursively by*

$$X^{(k)} = X_*^{(k)} + \sum_{ijk} \gamma_{ijk}^{(k)} X_{ijk},$$

where $X_*^{(k)}$ *is any particular solution of (fs k+1) and the coefficients* $\gamma_{ijk}^{(k)}$ *are determined by*

$$\sum_{ik} \gamma_{iqk}^{(-1)} \left(\sum_{l=1}^{r} y_{p,r+1-l}^* A^{(1)} \sum_{1\le s\le k;\ s+l=\sigma_q+1} x_{i,k+1-s} \right) +$$

$$\sum_{jk} \gamma_{pjk}^{(-1)} \left(\sum_{l=1}^{r} \sum_{1\le s\le k;\ s+r+1-l=\rho_p+1} v_{j,k+1-s}^* B^{(1)} u_{ql} \right) =$$

$$\sum_{l=1}^{r} y_{p,r+1-l}^* (C^{(k+1)} - A^{(1)}X_*^{(k)} - X_*^{(k)}B^{(1)} -$$

$$\sum_{s=2}^{k+2} [A^{(s}X^{(k+1-s)} + X^{(k+1-s)}B^{(s)}])u_{ql}. \quad (23)$$

Again we would like to emphasize that the linear systems (21) and (23) have the same coefficient matrix on the left hand side, which permits an efficient calculation of the regular terms in the Laurent series (22). Note also that if the unperturbed system (2) is feasible and the non-degeneracy assumption (13) holds, then the Laurent series (22) has only positive powers of ε and $X^{(0)}$ represents the limit solution as $\varepsilon \to 0$, of the perturbed problem.

Consider now a particular but practically important case which yields a concise matrix representation. Suppose that there is only one pair of eigenvalues (α, β) such that $\alpha + \beta = 0$, and in addition, assume that α and β are semisimple eigenvalues with respective multiplicities p and q. This case of a unique pair of semisimple eigenvalues was also considered in [19] in the context of a Lyapunov equation. Let us form the following matrices $X = [x_1, ..., x_p]$ and $Y = [y_1, ..., y_p]$, where x_i, y_i are the respective right and left eigenvectors of $-A$ corresponding to the eigenvalue α. Similarly, let the matrices U and V be formed with the right and left eigenvectors of B corresponding the eigenvalue β. Then, the results of Theorem 1 can be rewritten in a more elegant matrix form, as indicated below.

Corollary 1 *Let A and B possess only one pair of respective eigenvalues (α, β) such that $\alpha + \beta = 0$. In addition, assume that both α and β are semisimple with respective associated bases of right eigenvectors (X and U) and bases of left eigenvectors (Y and V). Under the non-degeneracy assumption (20), the* reduced *Sylvester equation*

$$(Y^* A^{(1)} X)\Gamma^{(-1)} + \Gamma^{(-1)}(V^* B^{(1)} U) = Y^* C^{(0)} U, \quad (24)$$

has a unique solution and the perturbed solution of the Sylvester equation (1) can be expanded in the Laurent series (22) with

$$X^{(-1)} = X\Gamma^{(-1)}V^* \quad (25)$$

and

$$X^{(k)} = X_*^{(k)} + X\Gamma^{(k)}V^*, \quad (26)$$

where $X_*^{(k)}$ is any particular solution of (fs k+1) and the matrix $\Gamma^{(k)}$ is a solution of the following Sylvester equation

$$(Y^* A^{(1)} X)\Gamma^{(k)} + \Gamma^{(k)}(V^* B^{(1)} U) =$$

$$= Y^*(C^{(k+1)} - A^{(1)} X_*^{(k)} - X_*^{(k)} B^{(1)} -$$

$$\sum_{s=2}^{k+2} [A^{(s} X^{(k+1-s)} + X^{(k+1-s)} B^{(s)}])U \quad (27)$$

Again, we would like to emphasize that to construct the Laurent series (22), we need to solve a series of new *reduced* Sylvester equations but with coefficient matrices of much smaller dimensions. This fact is especially useful in the case of higher order singularities. Namely, after each reduction step (see [1, 3]), we obtain new fundamental equations with coefficient matrices of smaller dimensions but the structure of these fundamental equations is essentially the same, that is, in performing the reduction process, one has to solve a series of Sylvester equations of smaller and smaller dimensions.

Finally we would like to note that detail proofs and some examples can be found in the journal version of this paper [2].

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