Growing Networks Through Random Walks Without Restarts

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Abstract Network growth and evolution is a fundamental theme that has puzzled scientists for the past decades. A number of models have been proposed to capture important properties of real networks. In an attempt to better describe reality, more recent growth models embody local rules of attachment, however they still require a primitive to randomly select an existing network node and then some kind of global knowledge about the network (at least the set of nodes and how to reach them). We propose a purely local network growth model that makes no use of global sampling across the nodes. The model is based on a continuously moving random walk that after s steps connects a new node to its current location, but never restarts. Through extensive simulations and theoretical arguments, we analyze the behavior of the model finding a fundamental dependency on the parity of s, where networks with either exponential or a conditional power law degree distribution can emerge. As s increases parity dependency diminishes and the model recovers the degree distribution of Barabási-Albert preferential attachment model. The proposed purely local model indicates that networks can grow to exhibit interesting properties even in the absence of any global rule, such as global node sampling.

1 Introduction

The growth and evolution of networks is a fundamental problem in Network Science specially in the light that networks are constantly changing over time. Explaining how and why different real networks grow and evolve the way they do has kept researchers busy for the past decades. Not surprising, various mathematical models for network growth and evolution have been proposed in the literature, either ad-hoc models tailored to specific domains, or general models aiming to capture general principles. A celebrated general network growth model is the Barabási-Albert (BA) model [1] which embodies the principle of *preferential attachment* found in various real networks.

A recognized drawback of most proposed network growth and evolution models is the assumption of global information about the network [2, 4, 5, 8]. For example, the BA model has a primitive to randomly select a node from the existing network according to the degree distribution. To relax such assumption, models that attach

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new nodes and edges to the existing network using local attachment rules, such as the Random Walk Model [9, 10], have been proposed. Clearly, random walks require knowledge of the current node degree and its neighbors, a much more localized information. Moreover, it seems more reasonable that new nodes connect to nearby nodes (through some local process) rather than selecting new neighbors from the entire population (through some global process). However, the Random Walk Model studied in [9, 10] and others [5, 8] still require a primitive to randomly select a node from the network (for the purpose of restarting the walker, for example) and are thus not purely local, because they need to know the number and the identity of all network nodes as well as a way to reach them. Such models have local attachment rules, but global "entry point" selection. More recently, models that have no global primitives have started to be explored [6, 7]. A drawback of these other models is that they rely on an initial network already containing all nodes such as a lattice or a regular tree, that is then modified according to local rules, and thus are technically not growth models.

In this work we propose and explore a network growth model that is purely local, requiring no global selection over the nodes or any initial network. The model works as follows:

0. Start a network with a single node with a self-loop and place a random walk on this node.
1. Let the random walk take exactly s steps.
2. Connect a new node to the node where the walker resides.
3. Stop if the number of nodes in the network is n, otherwise go to Step 1.

Intuitively, the random walk moves around continuously and after every s steps a new node is added and connected to its current location. The new node immediately becomes part of the network and the walker sees no difference between it and any other node. Note that the model has two parameters s and n and grows an undirected tree (apart from the self-loop at the initial node) since every new node starts with degree one. Moreover, the random walk is uniform on the neighbors and is never restarted, thus its name NRRW (No Restart Random Walk) model. Figure 1 illustrates a sample path of network growth with $s = 1$ and $s = 2$. Can such purely local model give rise to interesting network structures such as networks that exhibit a power-law degree distribution?

Interestingly, we uncover various non-trivial features of this model such as the fundamental dependency on the parity and magnitude of s and its relationship to the degree distribution. If s is odd and small we find that networks generated by the model tend to have very short-tailed degree distribution and very long distances. On the contrary, if s is even and small, networks exhibit a special kind of power law degree distribution (to be formalized later) and very short distances! As s increases the effect of parity decreases and networks exhibit a heavy-tailed degree distribution. Interestingly, with s large enough, the observed degree distribution follows a power law with exponent identical to the network generated by the BA model, recovering the effect of preferential attachment. We also rigorously prove that for $s = 1$ the random walk is transient and the degree of every node is bounded from above by a geometric distribution. Other interesting features will be highlighted in what follows.

The model here proposed is very related to the Random Walk Model [9] which also allows a random walk to take s steps before connecting a new node. The key difference is that in [9], after a new node is attached to the network, the random walk is restarted uniformly at random across all existing nodes in the network. Our random walk never restarts, and is therefore a purely local model. Interestingly, the authors of [9] show (through simulations and approximations) that their model is closely related to the BA model and yields a power law degree distribution independently of s. However, recently this finding has been questioned and for $s = 1$ it was mathematically proven that this is not the case [3]. Our model and findings contributes to this debate and possibly sheds light on how both results could be reconciled (more on this on Section 6).

The remainder of this paper is organized as follows. Section 2 discusses the model and its intuitive behavior, as well as the connection with prior works. Section 3 presents the evolution of node degree induced by the model. Section 4 analyzes the depth of the tree generated by the model. Section 5 presents our theoretical findings for the case $s = 1$, showing the transient nature of the model in this case. Finally, we summarize our findings and present a brief discussion in Section 6.

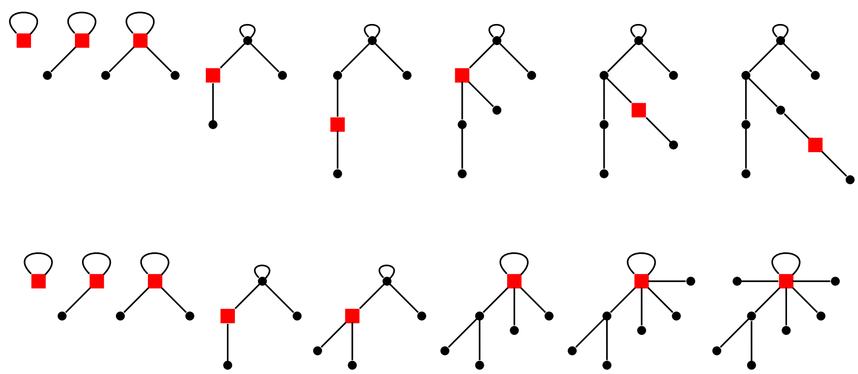

Fig. 1 Examples of sample path for network growth for $s = 1$ (top) and $s = 2$ (bottom). The red square denotes the walker position. The snapshots represent the growing network just after the new node is connected.

2 Network Growth Model

As presented in Section 1, NRRW (No Restart Random Walk) model can be interpreted as a simple random walk that attaches a new node to its current location every s steps. Similar proposed random walk models for growing networks assume that the random walk *restarts* either after connecting a new node or adding some number of edges to the new node [9, 10]. A restart consists of randomly selecting a node from the existing network (usually uniformly) and placing the random walk on that node. Despite the similarities, the lack of restarts makes NRRW fundamentally different from models with restart. In particular, the restart significantly reduces the correlation between consecutive node additions since it is very unlikely that the random walk will visit the previous new node when walking to add a new node. Intuitively, the random walk loses memory at every restart. Moreover, restarts have the drawback of assuming knowledge of all network nodes and random access to any such node, and is thus not a purely local growth model.

What is intuitively the behavior of NRRW? In a sense, when s is large the random walk will have little memory between node additions. However, this behavior is different from restarts since the random walk will not find itself on a node chosen uniformly at random but on a node chosen randomly proportional to its degree.[1] Thus, when s is large the NRRW seems similar to the BA model since new nodes connect to random nodes chosen proportionally to their degree. However, since s is fixed and the network grows, will NRRW indeed exhibit a behavior similar to BA model when $s << n$ and then s will finally become small in comparison to the network size?

What about small values for s? Intuitively, the random walk will frequently stumble over the newly created nodes. Interestingly, this local behavior depends fundamentally on the parity of s. If $s = 1$ then the random walk can always walk to the newly created node and add a new node to it. Such behavior is just not possible if $s = 2$ and the walker is not on the root. This qualitative difference is not limited to $s = 1$ and $s = 2$. When s is odd the walker can always land to the most recently added node after s steps and then add a new node. For s even, this is impossible unless the walker does not traverse the loop at the initial node.

The above observation justifies why in the NRRW model we consider a single node with a self-loop as a starting point. If this was not the case, for any s even the random walk would only add nodes to the original node, trivially constructing a star since it can never step on a newly created node. The loop allows a change of "parity" with respect to the levels of the tree where new nodes can be added. In fact if the random walk is at level k of the tree and s is even, the random walk can only add new edges at the levels $k+2h$ for $h = -\lfloor \frac{k}{2} \rfloor, \ldots, -1, 0, 1, 2, \ldots$ until it does not traverse the loop. For s odd this is not necessary as the walker can step on a newly created node to be able to add to nodes in any level of the tree without returning to the root. Thus, yet another fundamental difference between s even and odd.

Will these differences between small and large s and even and odd s manifest themselves in structural properties of the trees generated by the model? In particular, will the degree distribution fundamentally depend on s? In what follows we explore the degrees and other properties of the trees generated by the model showing in fact, that s plays a key role.

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[1] Recall that the steady state distribution of a random walk on a fixed network is given by $d_i / \sum_j d_j$, where d_i is the degree of node i.

2.1 Simulations

In order to study the model we designed and implemented an efficient simulator (in C++) for the NRRW model which has as parameters s, n and r, with r denoting the number of independent runs. For each run, we start with a single node with a self loop, move the random walk s steps, connect a new node to its current location, and repeat. We collect statistics for the various properties merging the results across the r simulation runs, such as degree distribution (fraction of nodes with degree k across all runs). The worst case time complexity of a simulation run is $O(sn\log n)$ but the amortized time complexity is $O(sn)$, as we use a growing vector to represent the neighbors of a node that doubles its capacity when needed. Thus, a walker step requires $\Theta(1)$ time and a node addition takes $O(1)$ amortized time.

3 Degree Behavior

In this section we study the degree distribution of NRRW through extensive simulations illustrating its behavior and dependencies. Figure 2 shows the Complementary Cumulative Distribution Function (CCDF) of nodes' for various values for s. Surprisingly, when s is small (between 1 and 8) the respective degree distributions are fundamentally different, exhibiting a kind of power law for s even and an exponential tail for s odd. Note that when $s = 1$ we do not observe nodes with degree larger than 40 while for $s = 2$ a non-negligible fraction of nodes have degree greater than 10^5. We also observe opposite trends in the degree distribution as s increases. For s odd, increasing s yields a distribution with heavier tails, while for s even increasing s yields a distribution with a lighter tails. As s increases into a medium range (between 15 and 64) the trends continue and the two distributions approach each other. For even larger s (between 127 and 256) the degree distributions become very close, being almost indistinguishable. Interestingly, with large s the degree follows a power law distribution, suggesting that the effect of even s value dominates the dynamics. Moreover, for large s the CCDF exhibits a power law with exponent approximately -2 as it is also the case for the BA model which is based on linear preferential attachment.[2] This supports our initial intuition that when s is large, the random walk samples nodes (adding a new node and connecting to it) with probability proportional to their degree, behaving similarly to the BA model.

Figure 3 shows the degree distribution for $s = 2$ but over different values for n. Interestingly, note that independent of n the degree distribution exhibits the same power law exponent. However, as n increases the fraction of nodes greater than k becomes smaller for any fixed $k > 0$ (with the exception of the cut-off regime which occurs when k is near n). This implies that the fraction of nodes with $k = 1$, the mini-

[2] Recall that if D follows a power law distribution, then $P(D = k) \sim k^{-\alpha}$ where $\alpha > 1$ is the power law exponent, and it follows that $P(D \geq k) \sim k^{-(\alpha-1)}$. Thus, the CCDF has an exponent that is one unit less than the PDF.

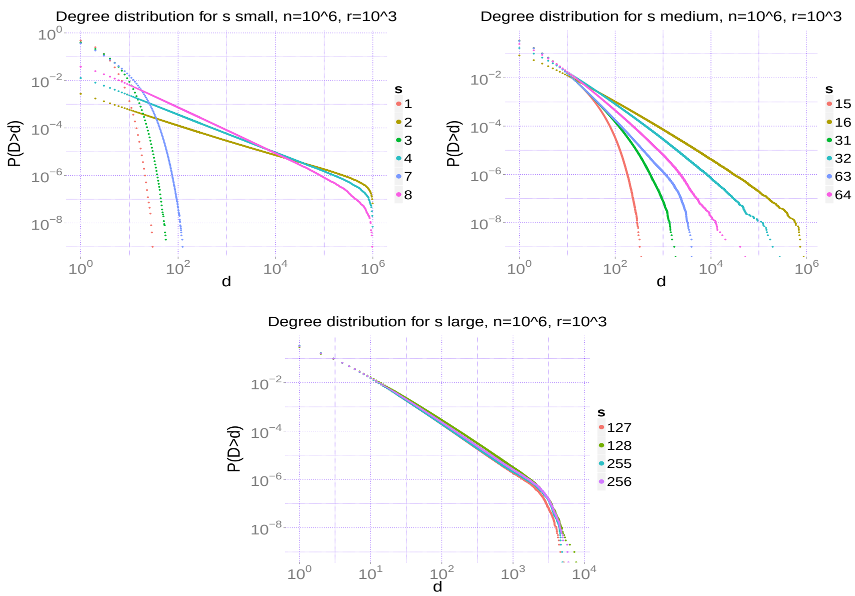

Fig. 2 Empirical degree Complementary Cumulative Distribution Function (CCDF) for various values of s in $\log - \log$ scale ($n = 10^6$, $r = 10^3$).

mum degree, is increasing with n. This is clear by observing $d = 1$ (leftmost point in x-axis) and noting that the fraction of nodes with degree greater than 1 is decreasing with n. Note that such behavior does not occur for $s = 1$ which maintains its degree distribution as n increases (the dots for different n values are barely distinguishable in plot).

If for $s = 2$ the fraction of nodes with degree 1 increases and converges to 1 as n goes to infinity, then we cannot claim that the degree distribution follows a power law. However, we can consider the degree distribution of the nodes that do not have degree 1. In particular, the conditional degree distribution, conditioned on $D > 1$, is shown in Figure 3. Note that the conditional degree distribution does not show dependance on n and moreover seems to follow a power law. This finding is quite interesting since the fraction of nodes with degree 1 can converge to 1 (as $n \to \infty$) while the remainder of nodes can still follow a power law. This may shed new light on the contrasting results in [9] and [3]. We return to this discussion in Section 6.

Figure 3 also shows the fraction of nodes with degree greater than 1, $P(D > 1)$, as a function of n for different even s values (for s odd, it does not depend significantly on n as shown in the top right part of Figure 3 for $s = 1$). Note that for small s the fraction goes to zero reasonably fast (and thus, the fraction of nodes with degree 1 goes to one). As s increases the rate at which $P(D > 1)$ decreases also decreases. Note that for large s (128 or 256) this decrease is barely noticeable, despite being present. Interestingly, as s odd increases, $P(D > 1)$ decreases but without showing any dependency on n. When $s = 257$, $P(D > 1)$ approaches the value shown in Figure 3 for $s = 256$ (result not shown due to space constraints).

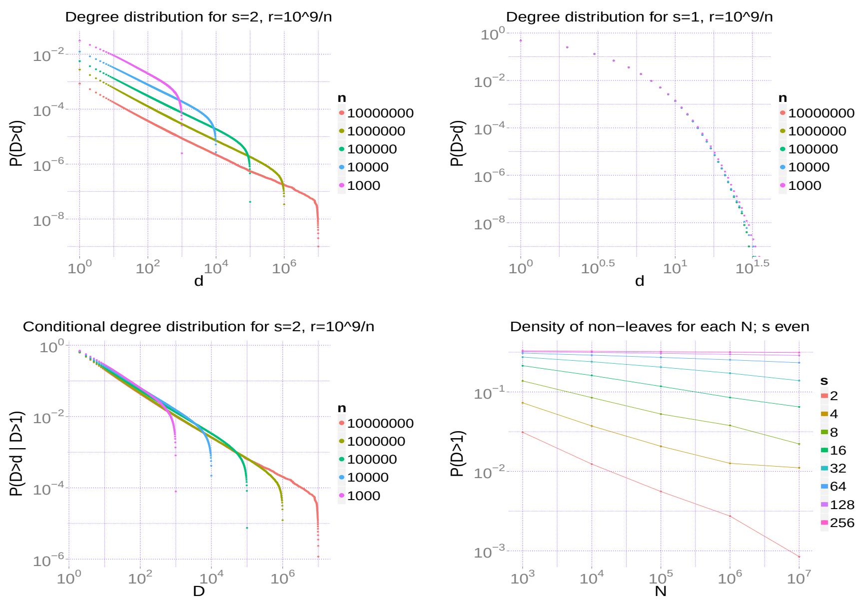

Fig. 3 Empirical degree CCDF for various values of n in $\log - \log$ scale (top plots). Empirical degree CCDF conditioned on the degree being greater than 1; fraction of nodes with degree greater than 1 (bottom plots).

As shown, the NRRW model has a very particular behavior with respect to the degree distribution. In Section 6 we provide a further discussion with a few conjectures for its asymptotic behavior with n.

4 Level Behavior

We now investigate the level of the nodes on the trees generated by the NRRW model.[3] As we have shown above, the model dynamics has a fundamental dependence on the parity of s, specially when s is small. Indeed, this dependence also manifests itself on the level of the nodes. Figure 4 shows the level distribution (fraction of nodes at level larger than ℓ) for a few small values of s separated into odd and even, respectively. The level distribution for s even decreases very fast. Note that although $n = 10^6$, when $s = 2$, 90% of nodes are at level 4 or less and no node

[3] Recall that the level of node on a tree is given by its distance to the root, and thus the root is at level zero.

is at level 7 or higher. As s even increases the level distribution decreases relatively slower, with 90% of nodes found at level greater than 4 when $s = 8$. Still, no node is found at level greater than 10. The behavior is completely different for s odd, and the level distribution seems to be uniform (straight line on a linear-linear plot). For $s = 1$ the distribution has the heaviest tail with about 4 nodes per level, giving rise to $2.5 \cdot 10^5$ different levels. For $s = 7$ there are about 40 nodes per level, giving rise to $2.5 \cdot 10^4$ different levels. Interestingly, as s even increases the level distribution becomes heavier while as s odd increases the level distribution becomes lighter. Figure 4 also shows the level distribution for large s. Indeed, as s increases the level distributions for s even and s odd become more similar and the dependency on the parity diminishes. This behavior is similar to what observed for the degree distribution, illustrated in Section 3.

Note that from the level distribution we can infer the kind of trees that NRRW generates. When s is small and even, the trees generated are "fat and short", with most nodes near the root and a few with very large degrees. When s is small and odd, the trees are "thin and long" with few nodes spread across many levels and no node with large degree. As s increases, the two kind of trees move in each other's direction, becoming more and more similar.

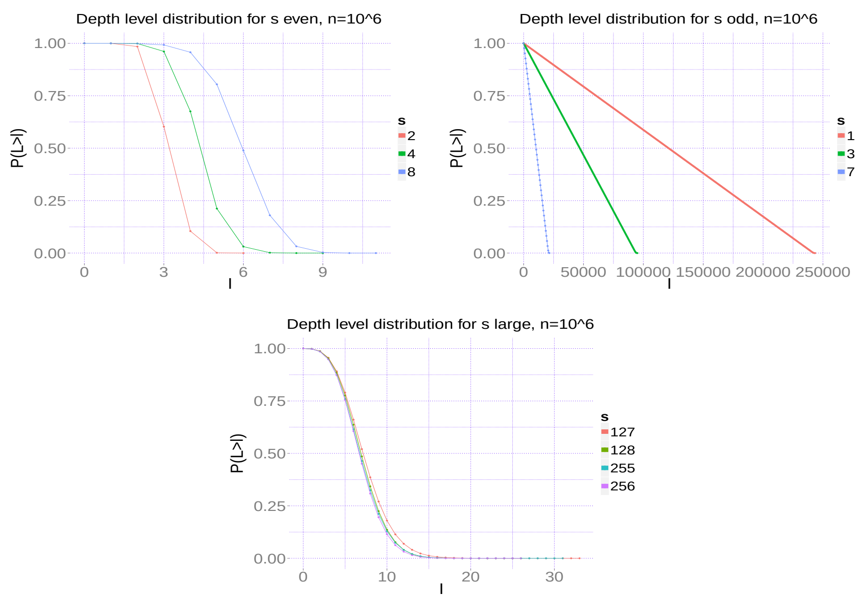

Fig. 4 Empirical CCDF of the node level for different s values ($n = 10^6$, $r = 10^3$).

5 Theoretical findings for $s = 1$

The numerical simulations with s odd and in particular with $s = 1$ suggest that trees grow in depth as the number of nodes increases. In particular, the growth in depth seems linear on the number of nodes. This is an indication that the random walk is continuously pushing the tree to lower depths just never to return to its origins. In a nutshell, the random walk is transient and visits each node in the tree only a relative small number of times, with high probability. The following Theorem rigorously formalizes this intuition.

Theorem 1. *In the NRRW model with* $s = 1$*, the number of visits to a node is stochastically dominated by* 1 *plus a geometric random variable with support on* $\mathbb{Z}_{>0}$.

Proof. We consider here that the initial network consists of a single node with no self-loop. This simplifies the notation and does not compromise the main argument.

Let r denote the initial node of the growing network hereafter referred to as the root of the tree (at any step the growing network is a tree) where the walker resides at time zero. Note that r is the only node at level zero. Let X_n be the level (i.e. the distance from the root) of the node visited by the NRRW at step n. We call the process $\{X_n, n \in \mathbb{Z}_{\geq 0}\}$ the *level process*. Note that the random walk visits r the same number of times that the level process visits level zero. At step $n > 0$ the NRRW is in a node v_n with at least two edges: the one the NRRW has arrived from and the new one added as a consequence of the NRRW's arrival. Let $d_n \geq 2$ denote the degree of node v_n. If $v_n \neq r$, the NRRW jumps from v_n to a node with larger level with probability $\frac{d_n - 1}{d_n} \geq \frac{1}{2}$ and with the complementary probability $\frac{1}{d_n} \leq \frac{1}{2}$ to a node with smaller level. If $v_n = r$, then the level can obviously only increase. Note that, due to the fact that degrees keep changing because of the arrival of new edges, the level process is non-homogeneous (both in time and in space).

We now study the evolution of the level process every two steps, i.e. we consider the process $Y_n \triangleq X_{2n}$. Given that the network is a tree and $X_0 = 0$, the two-step level process can be seen as a non-homogeneous reflecting 'lazy' random walk on $2\mathbb{Z}_{\geq 0} = \{0, 2, 4, \ldots\}$. We denote by $p_{k,h}(n)$ the probability that the level at step $n+1$ is h conditioned on the fact that it is k at step n. Although the notation hides it, we observe that the probabilities $p_{k,h}(n)$ depend on the whole history of the NRRW until step n. The reason to consider the two-step level process is that we can get bounds on the transition probabilities $p_{k,h}(n)$ that allow a simple comparison with a (biased) homogeneous random walk. The bounds derived above for X_n lead immediately to conclude that $p_{k,k+2}(n) \geq \frac{1}{2}\frac{1}{2} = \frac{1}{4}$ for any level $k \geq 0$ and $p_{k,k-2}(n) \leq \frac{1}{2}\frac{1}{2} = \frac{1}{4}$ for $k \geq 2$, but we can get a tighter bound for $p_{k,k-2}(n)$. If the NRRW is at level k, all the nodes on the path between its current position and the root r have degree at least 2. If it then moves to node v at level $k-1$, a new edge is attached to v, whose degree is now at least 3. The probability to move from v further closer to the root to a node with level $k-2$, is then at most $\frac{1}{3}$. It follows then that $p_{k,k-2}(n) \leq \frac{1}{2}\frac{1}{3} = \frac{1}{6}$ for $k \geq 2$.

We consider now a homogeneous biased lazy random walk $(Y_n^*)_{n \geq 0}$ on $2\mathbb{Z}_{\geq 0}$ starting from 0 with transition probabilities $p^*_{k,k+2} = \frac{1}{4}$ for all $k \in 2\mathbb{Z}_{\geq 0}$ and $p^*_{k,k-2} =$

$\frac{1}{6}$ for $k \in 2\mathbb{Z}_{\geq 0}$ and $k \neq 0$. We show that if $(Y_n^*)_{n\geq 0}$ also starts in 0 ($Y_0^* = 0$), it is stochastically dominated by $(Y_n)_{n\geq 0}$. We prove it by coupling the two processes as follows. Let $(\omega_n)_{n\geq 0}$ be a sequence of independent uniform random variables over $[0,1]$. We use them to generate sample paths for both processes $(Y_n)_{n\geq 0}$ and $(Y_n^*)_{n\geq 0}$ as follows:

$$Y_{n+1} = \begin{cases} Y_n - 2, & \text{if } \omega_n \in [0, p_{k,k-2}(n)) \\ Y_n + 2, & \text{if } \omega_n \in [1 - p_{k,k+2}(n), 1] \\ Y_n & \text{otherwise} \end{cases} \quad Y_{n+1}^* = \begin{cases} Y_n^* - 2, & \text{if } \omega_n \in [0, p_{k,k-2}^*) \\ Y_n^* + 2, & \text{if } \omega_n \in [1 - p_{k,k+2}^*, 1] \\ Y_n^* & \text{otherwise} \end{cases}$$

where $p_{k,k-2}(n)$ and $p_{k,k-2}^*$ are 0 if $k = 2$. We start observing that if Y_n and Y_n^* have the same value k, then every time Y_n increases also Y_n^* increases because $p_{k,k+2}^* = \frac{1}{4} \leq p_{k,k-2}(n)$. On the contrary if Y_n^* decreases (as it can happen only for $k \geq 2$), then Y_n may decrease or not because $p_{k,k-2}(n) \leq \frac{1}{6} = p_{k,k-2}^*$. It follows that if Y_n and Y_n^* are at the same level, then $Y_{n+1}^* \leq Y_{n+1}$.

We are going to prove by induction on n that $Y_{n+1}^* \leq Y_{n+1}$ for every n. With a slight abuse of terminology we say that Y_n increases (resp. decreases) if $Y_{n+1} > Y_n$ (resp. $Y_{n+1} < Y_n$). We start observing that indeed $Y_0^* \leq Y_0$, because both processes start in 0. Let us assume that $Y_n^* = h \leq k = Y_n$. For all values of h, every time Y_n^* increases also Y_n increases because $p_{k,k+2}^* = \frac{1}{4} \leq p_{k,k+2}(n)$ and then $Y_{n+1}^* = h + 1 \leq k + 1 = Y_{n+1}$. If $h \geq 2$, then $p_{h,h-2}^* = \frac{1}{6} \geq p_{k,k-2}(n)$ and if Y_n decreases then Y_n^* must also decrease ($Y_{n+1}^* = h - 1 \leq k - 1 = Y_{n+1}$). It follows that for $h \geq 2$ then $Y_{n+1}^* \leq Y_{n+1}$. The only case when Y_n may decrease without Y_n^* decreasing is when $h = 0$ and $k \neq 0$, but in this case $Y_{n+1}^* = 0$ and $Y_{n+1} \geq 0$. This proves that $Y_{n+1}^* \leq Y_{n+1}$ for every n.

Given that $Y_n^* \leq Y_n$ and both processes start at level zero, the number of visits of $(Y_n)_{n\geq 0}$ to level zero is bounded by the number of visits of $(Y_n^*)_{n\geq 0}$ to level zero. The homogeneous biased lazy random walk $(Y_n^*)_{n\geq 0}$ is transient since $p_{k,k+2}^* = 1/4 > p_{k,k-2}^* = 1/6$. Thus, the probability of the first return time to level 0 is $f_0 < 1$. By the strong Markov property, the number of visits to level 0 is geometrically distributed on the set $\mathbb{Z}_{>0}$ with parameter equal to $1 - f_0$. Since a visit to level zero in $(X_n)_{n\geq 0}$ (one level process) implies a visit to level zero in $(Y_n)_{n\geq 0}$ (two level process), then it follows that the number of visits of $(X_n)_{n\geq 0}$ to level zero is bounded by a geometric random variable and then even more so by 1 plus the same geometric random variable.

Now let us consider any node v in the growing network. If the NRWW never visits v, then the degree of b is 1 and the thesis follows immediately. Otherwise, let consider the first time the NRWW visits v to be time $t = 0$ and let consider v to be the root of the current tree. We can retrace the same reasoning and conclude that the number of visits to v for $t > 0$ is bounded by a geometric random variable on $\mathbb{Z}_{>0}$ with parameter equal to $1 - f_0$. Then the total number of visits to v is bounded by 1 plus such random variable. This concludes the proof. □

Corollary 1. *In the NRRW model with $s = 1$, the degree distribution of any node is bounded by a geometric distribution.*

This follows since the degree of every node equals the number of visits of the random walk to the node plus 1 (the plus 1 accounts for the fact that any node joining the network, although not yet visited by the walker, has degree 1).

6 Discussion and Conclusion

As we have shown, the NRRW model exhibits interesting features that fundamentally impact the networks it generates. For $s = 1$ the random walk is transient and node degree is bounded by a geometric distribution (Theorem 1). For $s = 2$, the fraction of nodes with degree 1 seems to converge to 1 as $n \to \infty$. However, the conditional degree distribution seems to follow a power law. Can such results be made mathematically rigorous? Other interesting questions also emerge from our analysis of the NRRW model. In particular, our numerical simulations seem to indicate that for any s even, the fraction of nodes with degree 1 will converge to 1 as $n \to \infty$. On the other hand, our simulations also indicate that this is not the case for any s odd. So will there be a fundamental difference between a fixed but arbitrarily large even and odd s? It is hard to imagine that $s = 2^{10}$ and $s = 2^{10} - 1$ would have fundamentally different behavior, since in both cases the random walk moves quite a lot before adding a new node. Of course, any fixed s will be small as $n \to \infty$. Thus, we make the following conjecture:

Conjecture 1. For any fixed s even, the fraction of nodes with degree one converges to 1, as $n \to \infty$. For any fixed s odd, the fraction of nodes with degree one converges to a number strictly less than 1, as $n \to \infty$.

If true, such conjecture would imply that the degree distributions are also never identical, for any fixed s even or odd. However, our numerical results indicate that the conditional degree distribution (conditioned on degree being greater than one) for s even, seems to converge to a power law as $n \to \infty$. On the other hand, for $s = 1$ we have proved that the random walk is transient and degree distribution is bounded by a geometric distribution (Theorem 1). Can fixed odd s values really generate power laws? If this is the case, then there would be a phase transition on s, from inducing a network with degree distribution with an exponential tail ($s = 1$) to a power law tail. Despite the numerical results indicating the heavy tail degrees for $s = \{127, 255\}$, we make the following conjecture:

Conjecture 2. For any fixed s even, the conditional degree distribution is bounded from below by a power law, as $n \to \infty$. For any fixed s odd, degree distribution is bounded from above by an exponential, as $n \to \infty$.

Such conjectures consider that n diverges. In practice n must be finite when generating a network with NRRW model. Thus, for a fixed n, the differences induced by an even or odd s may diminish as s increases. In particular, the degree distribution generated by even and odd s values may become arbitrarily close as s increases, as we have observed in numerical simulations for a fixed n (Figure 2).

Last, we return to the recent dispute if the Random Walk model with restarts generates a power law degree distribution, independently of s [3, 9]. It has been mathematically proved that when $s = 1$ the fraction of nodes with degree one converges to 1, as $n \to \infty$ [3]. At the same time, simulation results suggest that the degree distribution follows a kind of power law [9]. We can attempt to reconcile such findings by leveraging our own findings on NRRW model. When $s = 1$ the new node is connected to a given existing node u if i) a neighbor of u is selected at the restart and then ii) the random walk moves to u. A node whose neighbors are all leaves would be selected with a probability proportional to its degree. Now it has been shown that when $n \to \infty$ the fraction of nodes that are leaves converges to 1, then most of the neighbors of a non-leaf node are leaves and this node is essentially selected proportionally to its degree, similarly to the BA model embodying preferential attachment. Thus, the conditional degree distribution, leaving out degree 1 nodes, will follow a power law distribution with the same exponent as in the BA model. In some sense, this reconciles the findings of the two prior works [3, 9].

To conclude, as exemplified above, a fundamental understanding of NRRW model adds to our understanding of purely local network growth models. In particular, besides requiring a less strict assumption to operate, models that do not rely on any global primitive can also generate networks with rich and diverse structural properties.

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References

1. Barabási, A.L., Albert, R.: Emergence of scaling in random networks. science **286**(5439), 509–512 (1999)
2. Blanchard, P., Krueger, T., Ruschhaupt, A.: Small world graphs by iterated local edge formation. Phys. Rev. E **71**, 046,139 (2005)
3. Cannings, C., Jordan, J.: Random walk attachment graphs. Electronic Communications in Probability **18**, 1–5 (2013)
4. Colman, E.R., Rodgers, G.J.: Local rewiring rules for evolving complex networks. Physica A: Statistical Mechanics and its Applications **416**, 80–89 (2014)
5. Gabel, A., Redner, S.: Sublinear but never superlinear preferential attachment by local network growth. Journal of Statistical Mechanics: Theory and Experiment **2013**(02), P02,043 (2013)
6. Ikeda, N.: Network formation determined by the diffusion process of random walkers. Journal of Physics A: Mathematical and Theoretical **41**(23), 235,005 (2008)
7. Ikeda, N.: Network formed by movements of random walkers on a bethe lattice. In: Journal of Physics: Conference Series, vol. 490, p. 012189. IOP Publishing (2014)
8. Li, M., Gao, L., Fan, Y., Wu, J., Di, Z.: Emergence of global preferential attachment from local interaction. New Journal of Physics **12**(4), 043,029 (2010)
9. Saramäki, J., Kaski, K.: Scale-free networks generated by random walkers. Physica A: Statistical Mechanics and its Applications **341**, 80–86 (2004)
10. Vázquez, A.: Growing network with local rules: Preferential attachment, clustering hierarchy, and degree correlations. Physical Review E **67**(5), 056,104 (2003)