Competitive Routing in Networks with Polynomial Costs

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Abstract— **We study a class of noncooperative general topology networks shared by N users. Each user has a given flow which it has to ship from a source to a destination. We consider a class of polynomial link cost functions adopted originally in the context of road traffic modeling, and show that these costs have appealing properties that lead to predictable and efficient network flows. In particular, we show that the Nash equilibrium is unique, and is moreover efficient. These properties make the polynomial cost structure attractive for traffic regulation and link pricing in telecommunication networks. We finally discuss the computation of the equilibrium in the special case of the affine cost structure for a topology of parallel links.**

Keywords. Networks; routing; nonzero-sum games; noncooperative equilibria.

I. Introduction

We consider in this paper a routing problem in networks where noncooperating agents, to whom we refer to as *players* or *users*, wish to establish paths from agent-specific sources to agent-specific destinations so as to transport a fixed amount of total traffic (per agent). In the context of telecommunication networks, players correspond to different traffic sources which have to route their traffic over a shared network. A similar setting has also been studied in the context of road traffic networks [11], where a player can be viewed as a transportation company which is to ship a flow of vehicles.

A natural framework within which this class of problems can be analyzed is that of noncooperative game theory, and an appropriate solution concept is that of Nash equilibrium (NE) [4]: a composite routing policy for the users constitutes a NE if no user can gain by unilaterally deviating from his own policy.

There exists a rich literature [8], [14], [17], [19] on the analysis of equilibria in networks, particularly in the context of the Wardrop equilibrium [23], which pertains to the case of infinitesimal users (e.g. a single car in the context of road traffic). In such a framework, the impact of a single user on the congestion experienced by other users is negligible. Our focus here, however, is *competitive routing*, where the network is shared by several users, with each one having a non-negligible amount of flow. This concept has attracted considerable attention in recent years, as reflected in the works [1], [11], [12], [15], [18], which will be our starting point here. These papers have presented conditions for the existence and uniqueness of an equilibrium. This has allowed, in particular, the design of network management policies that induce efficient equilibria [15]. This framework has also been extended to the context of repeated games in [16], in which cooperation can be enforced by using policies that penalize users who deviate from the equilibrium.

A desired property for an equilibrium is efficiency, i.e. social optimality. It is well known that Nash equilibria in routing games are generally non-efficient. In particular, this non-efficiency can lead to paradoxical phenomena (e.g. the Braess's paradox [4]) where adding a link to the network could result in an increase of cost to all users. For specific examples of non-efficient behavior and paradoxes, we cite [8] for road-traffic, [7], [15] for telecommunications, and [12], [13] for distributed computing.

In this paper we study the equilibria that arise in networks of general-topology under some polynomial cost functions introduced originally in the context of road traffic by the US Bureau of Public Roads, and we obtain conditions for the uniqueness of the equilibrium. The uniqueness result is important, since to date there is not much known on uniqueness in the case of general topologies; only for a few special cases the uniqueness has been established, see [1], [12], [18]. We further present in the paper conditions under which the NE is efficient.

The fact that under a class of nonlinear costs the equilibrium is unique and efficient for general-topology networks may be especially useful for pricing purposes; indeed, if the network operator chooses prices that correspond to a unique and efficient equilibrium, it can enforce a globally optimal use of the network.

In the context of communications, an example of an architecture where our routing framework (and the spe-

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cific results obtained) would be of particular use is MPLS (MultiProtocol Label Switching) [2], [3]. One of the most important traffic engineering problems in MPLS, as identified in [3], is how to partition traffic into traffic trunks, or equivalently, how to distribute the arriving traffic over a given set of links. This problem has recently been studied in the framework of a single criterion that is optimized globally [10], but our framework here is more suitable for MPLS, since in practice routing decisions are made (and implemented) locally by each source, and typically in a noncooperative manner.

In the next section, we introduce the general model considered in this paper. In Section III, we establish conditions for the uniqueness of the NE. In Section IV, we establish the existence of an *efficient* NE under appropriate conditions. We then focus, in Section V, on the computation of the equilibrium for the special case of affine costs and parallel links. The paper ends with the concluding remarks of Section VI.

II. The model

The topology is given by a directed graph $\{V, L, f\}$ where V is the set of nodes, L is the set of directed arcs, and $f = (f_l, l \in L)$ where $f_l : \mathbb{R} \to \mathbb{R}$ is the cost function of link l, which gives the cost per unit traffic as a function of the total load λ_{lt} on that link. We take it as

$$f_l(\lambda_{lt}) = a_l \cdot \left(\lambda_{lt}\right)^{p(l)} + b_l\,, \quad l \in L\,, \tag{1}$$

where a_l, b_l, $p(l)$ are link-specific positive parameters. The special structure where $p(l)$ does not depend on l has been widely used in road traffic studies, such as [6], [20]. It is the cost adopted by the US Bureau of Public Roads [22], which we thus call the BPR cost. The additive term b_l here could be interpreted as an additional fixed toll per packet (or per unit traffic) for the use of link l.

In the model we adopt here, there are N users, where user $i \in \mathcal{N} := \{1, \ldots, N\}$ has a fixed amount Λ^i of flow to ship from a source $s(i)$ to a destination $d(i)$. Each user i has to determine how to distribute its flow over the available links (connecting $s(i)$ to $d(i)$). Introduce:

λ_l^i:= the rate of traffic user i sends on link l,

λ_{lt}^{-i}:= the total rate of traffic sent on link l, excluding the traffic from source i.

Thus, $\lambda_{lt} = \sum_{j=1}^N \lambda_l^j$, $\lambda_{lt}^{-i} := \lambda_{lt} - \lambda_l^i$.

We visualize this problem as a noncooperative network, i.e., a network in which the rate at which traffic is sent is determined by selfish players, each having its own utility (or cost). The cost for player i is given by

$$J^i(\lambda_l^i, \lambda_{lt}, l \in L) = \sum_{l \in L} J_l^i(\lambda_l^i, \lambda_{lt})\,, \tag{2}$$

where $J_l^i(\lambda_l^i, \lambda_{lt}) = \lambda_l^i f_l(\lambda_{lt})$. It should be noted that, with f_l as given by (1), J^i is a convex function of the flow vector of user i, which is a property we shall henceforth make frequent use of. Also, by a slight abuse of notation, we will occasionally write $J^i(\lambda_l^i, \lambda_{lt}, l \in L)$ as $J^i(\lambda)$ or $J^i(\lambda^i, \lambda^{-i})$, where $\lambda^i := \{\lambda_l^i; l \in L\}$, $\lambda^{-i} := \{\lambda^j; j \neq i,\ j \in \mathcal{N}\}$, and $\lambda := \{\lambda^j; j \in \mathcal{N}\}$.

The flows of each user i have to satisfy the feasibility conditions: $\lambda_l^i \geq 0$, and $\forall\, v \in V$,

$$r^i(v) + \sum_{l \in In(v)} \lambda_l^i = \sum_{l \in Out(v)} \lambda_l^i, \tag{3}$$

where

$$r^i(v) = \begin{cases} \Lambda^i & \text{if } v = s(i)\,, \\ -\Lambda^i & \text{if } v = d(i)\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and $In(v)$ (respectively, $Out(v)$) is the set of links which are input (respectively, output) to node v.

The solution concept adopted is NE, i.e., we seek a feasible multi-policy $\lambda = \{\lambda^i, i \in \mathcal{N}\}$ such that

$$J^i(\lambda) = J^i(\lambda^i, \lambda^{-i}) = \min_{\tilde{\lambda}^i} J^i(\tilde{\lambda}^i, \lambda^{-i})\,,\ i \in \mathcal{N},$$

where the minimum is taken over all policies $\tilde{\lambda}^i$ which lead to a feasible multi-policy together with λ^{-i}.

For the game model just introduced, we know from [18] that a NE indeed exists.

III. Uniqueness of the NE

Having settled the question of existence of a NE, we now establish its uniqueness under appropriate conditions. Let us first introduce the quantity

$$p^* := (3N-1)/(N-1)\,,$$

and note that $p^* > 3$ for all $N \geq 2$.

Theorem 1: For the general topology network with cost (2), where $0 < p(l) < p^*$, the NE is unique.

Proof: Introduce the gradient column vector

$$g_l(\lambda_l) = \left[\nabla_1 J_l^1(\lambda_l^1, \lambda_{lt}), \ldots, \nabla_N J_l^N(\lambda_l^N, \lambda_{lt})\right]^T \tag{4}$$

where ∇_i denotes the gradient operator with respect to λ_l^i, and $\lambda_l := \{\lambda_l^i; i = 1, \ldots, N\}$, which we will also view as an N-dimensional row vector. Then, it follows from

Theorem 2 of [21], in view of Corollary 3.1 of [18] (which translates Rosen's *diagonal strict concavity* condition to a similar condition applied to individual links), that the NE is unique if for any set of vectors λ_l and $\tilde{\lambda}_l$, $l \in L$ ($\lambda_l \neq \tilde{\lambda}_l$, $\forall\, l \in L$, in the vector sense), satisfying the flow constraints, we have

$$\sum_{l\in L} \left(\lambda_l - \tilde{\lambda}_l\right)^T \left(g_l(\lambda_l) - g_l(\tilde{\lambda}_l)\right) > 0\,.$$

We will show that in fact, for any l for which $\lambda_l \neq \tilde{\lambda}_l$, every term in the summation above is positive, i.e.,

$$\left(\lambda_l - \tilde{\lambda}_l\right)^T \left(g_l(\lambda_l) - g_l(\tilde{\lambda}_l)\right) > 0\,. \qquad (5)$$

We first consider the case $p(l) = 1$, for which the expression in (5) is quadratic in $(\lambda_l - \tilde{\lambda}_l)$. Then, all we need to show is that the Jacobian, $G_l(\lambda_l)$, of $g_l(\lambda_l)$ with respect to λ_l is positive definite. Simple calculation yields:

$$G_l(\lambda_l) := \left\{ \frac{\partial^2 J_l^i(\lambda_l^i, \lambda_{lt})}{\partial \lambda_l^i \partial \lambda_l^j} \right\}_{i,j} = (\,\underline{1}\,\underline{1}^T + I\,)\, a_l\,,$$

where $\underline{1}$ denotes the N-dimensional vector with entries all 1, and I is the identity matrix. G_l is clearly positive definite, and hence the positivity condition (5) holds. Therefore, the NE is unique (for this special case of $p(l) = 1$).

Next, we consider the case of a general $p(l) \in (0, p^*)$, $p(l) \neq 1$. The Jacobian of $g_l(\lambda_l)$ in this case is:[1]

$$G_l(\lambda_l) = a_l p(l) \left(\lambda_{lt}\right)^{p(l)-2} (\mathcal{G}_l + \lambda_{lt} I),$$

where $\mathcal{G}_l = q_l \underline{1}^T$, $q_l^i := \lambda_{lt} + (p(l) - 1)\lambda_l^i$, and q_l is an N-dimensional vector whose ith entry is q_l^i. It follows from Theorem 6 of [21], brought to the link level in view of Corollary 3.1 of [18], that a sufficient condition for (5) is for the symmetric matrix $G_l(\lambda_l) + G_l(\lambda_l)^T$ to be positive definite for all nonnegative λ_l. A careful look at the proof of Theorem 6 of [21] will actually reveal that it will be sufficient to require positive definiteness for all but isolated values of λ_l, and in particular for all λ_l such that the total flow over that link is positive (that is, at least one of the components of λ_l is positive). In accordance with this, it will be sufficient (for uniqueness) to show that for $\lambda_{lt} > 0$, the eigenvalues of $(\mathcal{G}_l + \lambda_{lt} I) + (\mathcal{G}_l + \lambda_{lt} I)^T$ are all positive, or equivalently $\Xi_l := q_l \underline{1}^T + \underline{1} q_l^T > -\lambda_{lt} I$, which is what we do next. Note that being the sum of the outer product of two vectors and its transpose, the symmetric matrix Ξ_l can have rank at most *two*, and its rank is precisely *two* if q_l is not a multiple of $\underline{1}$ (that is, λ_l^i's are not all equal). For the case when the rank is *one*, the only nonzero eigenvalue of Ξ_l is its trace, which is clearly positive (and hence larger than $-\lambda_{lt}$). In view of this, we henceforth assume (for the remainder of the proof) that q_l and $\underline{1}$ are linearly independent.

[1] Note that for $\lambda_{lt} > 0$ and $p(l) = 1$, the expression for $G_l(\lambda_l)$ below is consistent with the one obtained earlier for this special case.

We now first show that the two nonzero eigenvalues of Ξ_l are $z + r - 2\lambda_{lt}$ and $z - r - 2\lambda_{lt}$, where

$$z := \underline{1}^T q_l + 2\lambda_{lt} = (N + 1 + p(l))\lambda_{lt}, \quad r := \sqrt{N q_l^T q_l}\,.$$

Since Ξ_l is symmetric, the eigenvectors corresponding to its two nonzero eigenvalues have to be linear combinations of its columns, and therefore have the form $\alpha q_l + \beta \underline{1}$, where α and β are some constants. Hence, if x is an eigenvalue, we have the equation

$$(q_l \underline{1}^T + \underline{1} q_l^T)(\alpha q_l + \beta \underline{1}) = x(\alpha q_l + \beta \underline{1})\,,$$

for some α and β. Since, q_l and $\underline{1}$ are linearly independent, their coefficients in the equation above should separately vanish, yielding the pair of equations:

$$\alpha[(\underline{1}^T q_l) - x] + N\beta = 0\,,\ \alpha(q_l^T q_l) + \beta[(q_l^T \underline{1}) - x] = 0\,.$$

In order for a nontrivial solution to exist for α and β, the determinant of their coefficient matrix in the above set of equations has to be zero, i.e., $(\underline{1}^T q_l - x)^2 - N(q_l^T q_l) = 0$. This is a quadratic equation in terms of the unknown eigenvalues, whose solutions are $z+r-2\lambda_{lt}$ and $z-r-2\lambda_{lt}$.

To complete the proof of the theorem, we now finally have to show that x_1 and x_2 are both greater than $-2\lambda_{lt}$. Since $x_1 > x_2$, it suffices to show that $x_2 > -2\lambda_{lt}$, or equivalently, $z - r > 0$. Toward this end, note that

$$\begin{aligned}
\frac{r^2}{N} &= \sum_{i=1}^{N} (\lambda_{lt} + (p(l) - 1)\lambda_l^i)^2 \\
&= (N-1)(\lambda_{lt})^2 + (\lambda_{lt})^2 \\
&\quad + (p(l) - 1)^2 \sum_{j=1}^{N} (\lambda_l^j)^2 + 2(\lambda_{lt})^2 (p(l) - 1) \\
&\leq (N-1)(\lambda_{lt})^2 + (\lambda_{lt})^2 \\
&\quad + (p(l) - 1)^2 (\lambda_{lt})^2 + 2(\lambda_{lt})^2 (p(l) - 1) \\
&= (N - 1 + p(l)^2)(\lambda_{lt})^2. \qquad (6)
\end{aligned}$$

It then follows from (6) that $z - r > 0$ if $z^2 > N(N - 1 + p(l)^2)(\lambda_{lt})^2$, or equivalently if $(N + 1 + p(l))^2 > N(N - 1 + p(l)^2)$, which can be rewritten as

$$\begin{aligned}
0 &< 1 + 3N + 2(N+1)p(l) + (1 - N)p(l)^2 \\
&= -(p(l) + 1)((N - 1)p(l) - (3N + 1)).
\end{aligned}$$

For any $p(l) \in (0, p^*)$, the above indeed holds, thus proving that $z - r > 0$.

Hence the sufficient condition (5) holds for $p(l) \in (0, p^*)$, and thereby the NE is indeed unique. $\diamond$

IV. Efficiency of the NE

We consider in this section the special case of a *homogeneous* link cost for a general topology network. That is,

f_l is now given by $f_l(\lambda_{lt}) = a_l \cdot (\lambda_{lt})^p$, $p > 0$, $a_l > 0$, where the power p is a constant. As mentioned earlier, this is the BPR cost which was studied earlier in [9] (see also [5]). It was shown in these references that under this cost the Wardrop equilibrium and the globally optimal solution coincide. Our result here complements this earlier one. We show that when all users have the same source and the same destination, the NE is globally optimal (with respect to a single cost function) and that the corresponding link flows of different users are proportional to their total input traffic.

The global optimization problem is the minimization, over all total link flows $\{\lambda_{lt},\ l \in L\} =: \lambda_t$ and subject to the feasibility constraint (3), the total additive cost:

$$J(\lambda_t) = \sum_{l \in L} J_l(\lambda_{lt}), \tag{7}$$

where $J_l(\lambda_{lt}) = \lambda_{lt} f_l(\lambda_{lt})$. Note that the total cost (7) depends only on the total flows over each link and not on the flows of the individual users, and hence to make the feasibility constraints (3) compatible with this, we can take instead of (3) its sum over all users i, that is

$$r(v) + \sum_{l \in In(v)} \lambda_{lt} = \sum_{l \in Out(v)} \lambda_{lt} \tag{8}$$

where we have also made use of the assumption that $r^i(v)$ is independent of i (and hence is denoted by $r(v)$). Therefore the global optimization problem is

$$\min_{\lambda_t} J(\lambda_t) \quad \text{such that} \quad (8) \quad \text{holds.} \tag{9}$$

Since $J(\lambda_t)$ is strictly convex over a linear constraint set, this minimization problem admits a unique solution, which we denote by $\overline{\lambda}_t \equiv \{\overline{\lambda}_{lt},\ l \in L\}$. Now, to bring this solution down to the level of an individual user, we distribute the total flow, λ_{lt}, on a link l among the users in proportion with their total flows, that is $\overline{\lambda}_l^i / \overline{\lambda}_l^j = \Lambda^i / \Lambda^j$. This leads to

$$\overline{\lambda}_l^i = \alpha_i \, \overline{\lambda}_{lt}\ ;\ \ i \in \mathcal{N};\ l \in L, \quad \alpha_i := \Lambda^i / \sum_{j=1}^N \Lambda^j . \tag{10}$$

It should be noted that the flows (10) satisfy the flow feasibility constraints (3) whenever $\overline{\lambda}_t$ satisfies (8). We now show that (10) is in fact a NE (and the unique one, by Theorem 1), and hence the NE is efficient.

Theorem 2: Let $\overline{\lambda}_t$ be the globally optimal solution, and $\overline{\lambda}_t^i = \alpha_i \overline{\lambda}_{lt}$, as in (10). This also constitutes the unique NE, with the corresponding cost for user i being $J^i(\overline{\lambda}^i, \overline{\lambda}_t) = \alpha_i J(\overline{\lambda}_t)$.

Proof: Let us first note that a flow N-tuple $\hat{\lambda}$ is in NE if and only if it is feasible and the following holds for all i, for any feasible λ^i (see [11]):

$$\sum_{l \in L} \frac{\partial J_l^i(\hat{\lambda}_l^i, \hat{\lambda}_{lt})}{\partial \lambda_l^i} \cdot (\hat{\lambda}_l^i - \lambda_l^i) \leq 0 \, . \tag{11}$$

On the other hand, $\overline{\lambda}_t$ is globally optimal if and only if it is feasible (i.e. it satisfies the traffic constraints (3), or equivalently (8) in this case) and the following holds for any feasible λ:

$$\sum_{l \in L} \frac{\partial J_l(\overline{\lambda}_{lt})}{\partial \lambda_{lt}} \cdot (\overline{\lambda}_{lt} - \lambda_{lt}) \leq 0 \, . \tag{12}$$

For the cost adopted, the partial derivative in (11) is

$$\frac{\partial J_l^i(\hat{\lambda}_l^i, \hat{\lambda}_{lt})}{\partial \lambda_l^i} = a_l (p (\hat{\lambda}_{lt})^{p-1} \hat{\lambda}_l^i + (\hat{\lambda}_{lt})^p) \, . \tag{13}$$

Letting $\hat{\lambda}_l^i = \alpha_i \overline{\lambda}_{lt}$ in (13) above, we have

$$\begin{aligned} &\sum_{l \in L} \frac{\partial J_l^i(\hat{\lambda}_l^i, \hat{\lambda}_{lt})}{\partial \lambda_l^i} \cdot (\hat{\lambda}_l^i - \lambda_l^i) \qquad\qquad (14) \\ &\quad = \quad \alpha_i (p \alpha_i + 1) \sum_{l \in L} a_l (\overline{\lambda}_{lt})^p (\overline{\lambda}_{lt} - \tilde{\lambda}_{l,i}) \, , \end{aligned}$$

where $\tilde{\lambda}_{l,i} := \lambda_l^i / \alpha_i$. Note that in the global optimization problem,

$$\begin{aligned} &\sum_{l \in L} \frac{\partial J_l(\overline{\lambda}_{lt})}{\partial \lambda_{lt}} \cdot (\overline{\lambda}_{lt} - \lambda_{lt}) \\ &\quad = \quad (p+1) \sum_{l \in L} a_l (\overline{\lambda}_{lt})^p (\overline{\lambda}_{lt} - \lambda_{lt}) \leq 0 \, , \end{aligned}$$

where the inequality follows from (12). By combining this with (14), we conclude that (11) holds with $\hat{\lambda} = \overline{\lambda}$, and this establishes the proof. ◇

V. Computation of the NE: affine costs

We consider in this section the case of a network consisting of M parallel links and with affine cost function for all links: $f_l(\lambda_{lt}) = a_l \lambda_{lt} + b_l$. This is a special case of the BPR cost for $p(l) = 1$. A possible interpretation for this type of cost in the context of telecommunication systems is that it is the expected delay of a packet in a light traffic regime. For example, the expected value of the delay D_l in link l for an M/G/1 queue is given by

$$E[D_l] = \frac{(\beta_l)^2}{2(1 - \lambda_{lt} \beta_l)} \lambda_{lt} + \beta_l .$$

Under a light traffic regime, i.e., with $\lambda_{lt} \beta_l << 1$, this can be approximated by $a_l \lambda_{lt} + \beta_l$, where $a_l = (\beta_l)^2 / 2$.

It follows from standard nonlinear (convex) programming theory that the following Kuhn-Tucker conditions are both necessary and sufficient for a solution $\lambda = \{\lambda_l^i\}_{l,i}$ to constitute a NE:

$$K_l^i(\lambda) = a_l\lambda_{lt} + b_l + a_l\lambda_l^i = \mu^i \quad \text{if } \lambda_l^i > 0, \forall i, l, \tag{15}$$

$$K_l^i(\lambda) = a_l\lambda_{lt} + b_l \geq \mu^i \quad \text{if } \lambda_l^i = 0, \forall i, l \tag{16}$$

$$\sum_{l=1}^{M} \lambda_l^i = \Lambda^i, \ \forall i, \quad \lambda_l^i \geq 0 \quad \forall i, l. \tag{17}$$

Since (15)-(17) constitute a system of linear equalities and inequalities, they can be solved by standard LP methods. We present below an alternative approach that yields explicitly the equilibrium when the bias terms or their variability are relatively small.

Lemma 1: (i) If at equilibrium each user sends a positive amount of flow on each link, the unique NE is

$$\lambda_l^i = \frac{1/a_l}{\sum_{j\in L}(1/a_j)} \left(\Lambda^i + \frac{1}{N+1}\sum_{j\in L}\frac{b_j - b_l}{a_j}\right), \ \forall l, i. \tag{18}$$

(ii) At equilibrium, each user sends a positive amount of flow on each link if and only if:

$$\min_i R(i) > \max_l b_l/(N+1) \tag{19}$$

where $$R(i) := \frac{1}{\sum_{j\in L}(1/a_j)}\left(\Lambda^i + \frac{1}{N+1}\sum_{j\in L}\frac{b_j}{a_j}\right).$$

Proof: We will initially assume that at equilibrium all users send positive flows on all links. This will allow us to obtain an equilibrium which is consistent with this assumption, which in turn will be unique by the result of Section III. Now, by summing (15) over all users:

$$N(a_l\lambda_{lt} + b_l) + a_l\lambda_{lt} = (N+1)(a_l\lambda_{lt}) + Nb_l = \sum_{i=1}^{N}\mu^i .$$

Thus, $a_l\lambda_{lt} + b_l = \left(\sum_i \mu^i + b_l\right)/(N+1)$. Substituting in the Kuhn-Tucker condition (15), we obtain:

$$a_l\lambda_l^i + \frac{b_l}{N+1} = \mu^i - \frac{1}{N+1}\sum_{j\in\mathcal{N}}\mu^j \tag{20}$$

Dividing by a_l and summing over all the links, we get

$$\Lambda^i + \frac{1}{N+1}\sum_{l\in L}\frac{b_l}{a_l} = \left(\mu^i - \frac{1}{N+1}\sum_{j\in\mathcal{N}}\mu^j\right)\sum_{l\in L}\frac{1}{a_l}.$$

Hence

$$\mu^i - \frac{1}{N+1}\sum_{j\in\mathcal{N}}\mu^j \tag{21}$$

$$= \frac{1}{\sum_{l\in L}(1/a_l)}\left(\Lambda^i + \frac{1}{N+1}\sum_{l\in L}\frac{b_l}{a_l}\right) = R(i).$$

Then, from (20), $a_l\lambda_l^i = R(i) - b_l/(N+1)$. Thus, if $\min_i R(i) > \max_l b_l/(N+1)$, the equilibrium is given by $\lambda_l^i = \left([R(i) - b_l/(N+1)]/a_l\right) > 0$, which is compatible with the initial hypothesis on positivity. ◇

Corollary 1: If all b_l's are equal, the unique NE is

$$\lambda_l^i = \left(\frac{1}{a_l} / \sum_{k\in L}\frac{1}{a_k}\right)\Lambda^i, \quad i\in\mathcal{N}.$$

We next consider some further properties of the NE.

Lemma 2: (i) Consider a NE which dictates a particular user to send all its traffic over a single link. If there is some other user who sends all its traffic also over a single link, then these two links are the same.
(ii) All users send their traffic over a particular link l if and only if $a_l\Lambda + b_l + \max_i a_l\Lambda^i \leq \min_{n\neq l} b_n$, where $\Lambda := \sum_{i\in\mathcal{N}}\Lambda^i$.

Proof: (i) Consider a NE where player i uses only link l and player j uses only link m, with $l \neq m$. From (15), we get: $\mu^i = a_l\lambda_{lt} + b_l + a_l\Lambda^i$, and $\mu^j = a_m\lambda_{mt} + b_m + a_m\Lambda^j$. From (16), we get: $\mu^j \leq a - l\lambda_{lt} + b_l$, and $\mu^i \leq a_m\lambda_{mt} + b_m$. Combining the last two relationships, we obtain the following contradiction:

$$\begin{aligned} a_l\lambda_{lt} + b_l + a_l\Lambda^i = \mu^i &\leq a_m\lambda_{mt} + b_m \\ = \quad \mu^j - a_m\Lambda^j &\leq a_l\lambda_{lt} + b_l - a_m\Lambda^j . \end{aligned}$$

(ii) By (15)-(16), all users send their traffic to link l if and only if $\mu^i = a_l\Lambda + b_l + a_l\Lambda^i$, and $\mu^i \leq b_m$, $\forall l \neq m$, which then leads to the desired result. ◇

Lemma 3: Consider the case of two links.
Let $\mathcal{I}$ be a nonempty set of players who at equilibrium send all their traffic on link 2. Then the equilibrium flows for the remaining players $i \notin \mathcal{I}$ are given by, for $l, k = 1, 2$; $k \neq l$, and with $\Lambda(\mathcal{I}) := \sum_{i\in\mathcal{I}}\Lambda^i$,

$$\lambda_l^i = \frac{a_k}{a_1 + a_2}\left(\Lambda^i + \frac{1}{N+1}\frac{b_k - b_l + a_2\Lambda(\mathcal{I})}{a_2}\right). \tag{22}$$

Proof: Let $\lambda_l(\mathcal{I}^c) := \sum_{i\notin\mathcal{I}}\lambda_l^i$, $\quad b_2^{\mathcal{I}} := b_2 + a_2\Lambda(\mathcal{I})$. Each player $i \notin \mathcal{I}$ faces the problem of minimizing the cost

$$\begin{aligned} J_i(\lambda) &= \lambda_1^i(a_1\lambda_{1t} + b_1) + \lambda_2^i(a_2\lambda_{2t} + b_2) \\ &= \lambda_1^i(a_1\lambda_{1t}(\mathcal{I}^C) + b_1) + \lambda_2^i(a_2\lambda_{2t}(\mathcal{I}^C) + b_2^{\mathcal{I}}). \end{aligned}$$

This is equivalent to the original game where users from $\mathcal{I}$ do not participate, and the corresponding value of b_2 is changed to $b_2^{\mathcal{I}}$. It follows from Lemma 2 that each user in this new game sends positive flow over both links at equilibrium. Using Lemma 1 then completes the proof. $\diamond$

By comparing (18) and (22), we finally arrive at:

Corollary 2: Consider a two-link game with $b_1 > b_2$. Let $\hat{\mathcal{I}}$ be the set of users for which λ_1^i in (18) is negative. Then $\mathcal{I} \subset \hat{\mathcal{I}}$, where $\mathcal{I}$ is as defined in Lemma 3.

VI. Conclusions

We have studied the problem of static competitive routing to parallel queues with polynomial link holding costs. We have established the uniqueness of the Nash equilibrium (NE) for a general topology with the BPR cost [22] and have obtained a simple relationship with the globally optimal solution. We have further obtained some explicit results for the special case of affine link costs.

The results of this paper should prove useful for the analysis of networks with source-determined routing, when the link cost functions can be approximated by polynomial (and in particular affine) costs of the type considered here. The fact that the NE was shown to be efficient under a class of nonlinear costs can be used *as a starting point* for designing pricing mechanisms so as to obtain a socially optimal use of the network.

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