Spatial SINR Games Combining Base Station Placement and Mobile Association

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Abstract—We study in this paper the question of determining locations of base stations (BSs) that may belong to the same or to competing service providers, taking into account the impact of these decisions on the behavior of intelligent mobile terminals who can connect to the base station that offers the best utility. We first study the SINR association-game: we determine the cells corresponding to each base stations, i.e. the locations at which mobile terminals prefer to connect to a given base station than to others. The Signal to Interference and Noise Ratio (SINR) is used as the quantity that determines the association. We make some surprising observations: (i) displacing a base station a little in one direction may result in a displacement of the boundary of the corresponding cell to the opposite direction; (ii) A cell corresponding to a BS may be the union of disconnected sub-cells. We then study the Stackelberg equilibrium in the combined BS location and mobile association problem: we determine where to locate the BSs so as to maximize the revenues obtained at the induced SINR mobile association game. We consider the cases of single frequency band and two frequency bands of operation. Finally, we also consider Stackelberg equilibria in two frequency systems with successive interference cancellation.

I. INTRODUCTION

As mobile communication technologies rapidly develop, intelligent mobile terminals capable of accessing multiple radio access technologies will decide for themselves the wireless access technology to use and the access point with which to connect. These capabilities should be taken into account while designing and deploying wireless networks.

In this paper we study some hierarchical decision making problems arising in the uplinks of cellular networks. We first address the problem of association: given multiple base stations (BS) capable of providing services to a mobile, to which BS should the mobile connect? This is studied in a non-cooperative context where each mobile connects to the BS that provides it with the best signal to interference and noise ratio (SINR). The associations determine the *cells* corresponding to each BS. We characterize the nature of cells as a function of BS locations.

We then consider the problem of determining the location of base stations, taking into account the behavior of the mobiles that will be induced by the location decisions. We study cases where the BSs cooperate (for e.g., they belong to the same service provider) and those where they compete with each other for throughput. The later scenario results in a location game between the BSs.

Related work: Plastria [4] presented an overview of the research on locating one or more new facilities in an environment where competing facilities already exist. Gabszewicz & Thisse [1] provided another general overview on location games.

Mazalov & Sakaguchi [3] and references therein studied competition over prices of goods between facilities that have fixed positions. They then derived the equilibrium allocation of customers. Such games, as well as hierarchical games in which firms compete for the location or over prices which then determine the customer-allocation equilibrium, were introduced by Hotelling [2] in 1929. When considering such games over a finite line segment with two firms, the models under appropriate conditions give rise to a partition of the segment into two convex subsegments or "cells" as introduced in our context.

An interesting difference between the settings above and our setting, which is also defined on a finite line segment, is that in our case more complex cells are obtained at equilibrium. Another difference is the cost structure that arises in the cellular context. Hotelling [2] considered a general cost related to the distance between the customer and the firm it chooses; this cost however depended only on the distance and not on the actual location of the firm. This does not hold in our case: the throughput of the mobile depends on the interference at the base station which in turn depends on the location of the base station. We finally note that in our model, the throughput of a mobile, which can be considered as the "demand", is not fixed and depends on the location of the base stations.

A description of the model studied in this paper and the notation used can be found in Section II and Appendix A.

Our contribution: We derive analytical expressions for the cell boundaries (Section III-A). This allows us to study the geometric properties of cells as a function of the locations of the BSs. We then study the Stackelberg equilibrium in the combined BS location and mobile association problem: we determine where to locate the BSs so as to maximize the revenues obtained at the induced SINR-based mobile association game. We consider cases where BSs are on the same frequency band (Section III-B) or on different frequency bands (Section IV). Subsequently, we consider BSs capable of successive interference cancellation (SIC) decoding. After a discussion on the association problem and the single frequency band case, we analyse the case of different frequency bands and give a complete characterization of the equilibria that arise (Section V).

II. THE MODEL AND NOTATION

Our focus is on communication in the uplink direction, i.e., from the mobiles to base stations (BS). Mobiles and BSs lie in the two-dimensional plane $\mathbb{R}^2$. A large number of mobiles are placed uniformly over the segment $[-L, L]$ on the first of the coordinate axes. As the number of mobiles becomes large, we obtain the fluid

approximation with uniform density of mobiles in the segment $[-L, L]$. For details see Appendix A. There are two BSs, BS 1 and BS 2, located at $(x_1, 1)$ and $(x_2, 1)$, respectively (say on the top of a flat building whose height is one unit). BSs cooperate if they belong to the same operator, and compete if they belong to different operators. We permit placements of BSs outside the area where mobiles exist, i.e., $x_j < -L$ and $x_j > L$ are allowed for $j = 1, 2$. In the following, we use only the first coordinates to specify locations (with the understanding that second coordinates are 0 in case of mobiles, and 1 in case of BSs).

Transmitters are point sources radiating in two-dimensional space with circular wavefronts (respectively, three-dimensional with spherical wavefronts). We consider a power law path loss model with exponent α; i.e., the power from a radio transmitter attenuates as distance raised to the power α (see Appendix A). A mobile located at y has a channel "gain" of $[(y - x_j)^2 + 1]^{-\alpha/2}$ to BS j. All mobiles are assumed to transmit at a power such that the power density along the line is unit power per unit length. Thermal noise at the BSs is assumed to be Gaussian with noise variance σ^2 per sample.

At any time, each mobile is *associated* with exactly one BS. Let $A_j \subset [-L, L]$ be such that the mobiles in A_j are associated with BS j. A_j will be called *cell j*. The utility of a mobile at y is assumed to be a non-decreasing function of the *SINR density* at y, as seen at the BS to which the mobile is associated. The SINR density depends on the interference model under consideration and will soon be specified.

A. Interference models

Mobiles that connect to a particular BS may or may not cause interference to the other BS depending on whether the BSs operate on the same or different radio frequency (RF) bands. We consider both the cases in this paper. The case in which the same frequency band (channel) is used at both the BSs occurs if the wireless network operates in an unlicensed band (for example, WiFi networks); in such a case, base stations belonging to different networks (or providers) may use the same RF band. We call this *the single-frequency case*. If the wireless network operates in licensed RF bands, two neighboring BSs would operate in disjoint RF bands. We call this *the two-frequencies case*. We now discuss the useful energy collected and interference seen at a BS in the single- and two-frequencies cases. For this purpose, it is useful to define the following functions. Define

$$g(y) := [1 + y^2]^{-\alpha/2}. \tag{1}$$

For a set $S \subseteq [-L, L]$ and candidate BS location x, define

$$E(x, S) := \int_S g(y - x) \, dy. \tag{2}$$

and $E^o(x) := E(x, [-L, L])$. The dependence of g, E, and E^o on α is understood.

1) The single-frequency case: In this case, power from all the mobiles is received at both the BSs. The total received power at BS j located at x_j is therefore given by $E(x_j, [-L, L]) = E^o(x_j)$. Assuming that the receiver treats other users' received signals as Gaussian noise, all of this received signal will clearly be interference to a mobile at y because the mobile's own contribution to this is infinitesimal.

With this interference interpretation for $E^o(x_j)$, we now highlight some of its properties. It is straightforward to see via change of variables that

$$E^o(x) = \int_{-L-x}^{L-x} g(y) \, dy = \int_{\arctan(-L-x)}^{\arctan(L-x)} (\cos \theta)^{\alpha-2} \, d\theta \tag{3}$$

Closed form expressions are available for E^o when α takes integer values. In particular, for $\alpha = 2$ we get

$$E^o(x) = \arctan(L - x) + \arctan(L + x), \tag{4}$$

and for $\alpha = 1$ we get

$$E^o(x) = \operatorname{arcsinh}(L - x) + \operatorname{arcsinh}(L + x). \tag{5}$$

The above expressions motivate the following definition of the α-parametric function

$$\arctan_\alpha (x) := \int_0^x g(y) \, dy, \ x \in \mathbb{R}.$$

Then clearly $\arctan_\alpha (\cdot)$ is an odd function[1] that is increasing, differentiable with derivative g, and sigmoidal[2]. We may therefore write the received energy at location x_j (and therefore the interference in the single-frequency case) as

$$E^o(x_j) = \arctan_\alpha (L - x_j) + \arctan_\alpha (L + x_j). \tag{6}$$

The following is a useful property of E^o.

Proposition 2.1: E^o is an even function with a unique maximum at 0. Moreover, $E^o(|x|)$ monotonically decreases with $|x|$.

Proof: See Appendix B. ∎

2) The two-frequencies case: Unlike the previous setting, in which the two BSs operate on the same RF band, in the two-frequencies case the total interference at each BS depends on the association decisions of mobiles. Indeed, the interference power at BS j is the total power received at that BS from all mobiles that actually associate with it. The total received power at BS j is thus given by $E(x_j, A_j)$.

For example, suppose $A_1 := [-L, \theta]$ and $A_2 := (\theta, L]$ denote the two *cells* for some $\theta \in [-L, L]$. Then the interference power at BS 1 is $E(x_1, A_1) = \arctan_\alpha (\theta - x_1) - \arctan_\alpha (-L - x_1)$. The formula for $E(x_2, A_2)$ is obtained analogously.

B. The SINR-equilibrium association

We shall first consider the case in which the BSs locations are fixed, and each mobile has option of associating with one of the BSs. The continuum of mobiles constitute the players in this association game.

Consider a mobile at y. Its utility is a nondecreasing function of the throughput density at y. From the fluid model (see Appendix A), the throughput density at location y increases linearly with SINR density. Thus, this mobile chooses a BS with the higher SINR density. Let I_j be the set of interferers as seen at BS j. If the mobiles at point y are associated with BS j, the SINR density there is

$$\operatorname{SINR}(y, x_j, I_j) := \frac{g(y - x_j)}{E(x_j, I_j) + \sigma^2} \tag{7}$$

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[1]It is an odd function because $\arctan_\alpha (-x) = -\arctan_\alpha (x)$.

[2]A function is sigmoidal, if it is concave to the right of a particular point and convex to its left. The second derivative of $\arctan_\alpha (x) = g'(x) = -\alpha x [1 + x^2]^{-(1+\alpha/2)}$. Inflection point for $\arctan_\alpha (\cdot)$ is therefore 0.

A mobile at $y \in [-L, L]$ will therefore prefer to associate with BS 1 if $\mathrm{SINR}(y, x_1, I_1) \geq \mathrm{SINR}(y, x_2, I_2)$.

We observe that in the single-frequency case, $I_j = [-L, L]$. Thus, the SINR density at a location, as seen at BS j is fixed. However, in the two-frequencies case, $I_j = A_j, j = 1, 2$. Hence, the SINR density at a location, as seen at BS j is a function of the cell A_j.

Definition 2.1: The cell partition (A_1, A_2) is said to be an SINR-equilibrium if the following holds: $y \in A_1$ if and only if $\mathrm{SINR}(y, x_1, I_1) \geq \mathrm{SINR}(y, x_2, I_2)$.

Remark 2.1: This definition of equilibrium is similar to the Wardrop equilibrium in road traffic [6]. Note, however, that usually, in Wardrop equilibrium the utility of choosing a resource (a link) depends on the set of users that make the same choice through their total "number" (their fraction or their mass). Extensions exist to the case where there is a finite number of user classes and the utility of using a link for a user in a given class depends on the amount of users of each one of the classes that use the link [5]. Our model leads to such a multiclass Wardrop equilibrium with a continuum of classes.

C. Hierarchical equilibrium problem

We shall also consider placement of BSs taking into account the SINR-equilibrium that follows when mobiles associate to maximize their SINR density. The BSs play a location game: BS j decides to place itself at $(x_j, 1)$ where $x_j \in \mathbb{R}, j = 1, 2$. The utility of a BS is a monotone function of the aggregate throughput of all the mobiles associated with it. Since the throughput density at location y increases linearly with SINR density, we may simply set the integral of SINR density over the cell of a BS as its utility. Thus for BS j with cell A_j and interferers I_j, the utility is

$$\frac{1}{2} \int_{A_j} \mathrm{SINR}(y, x_j, I_j) \, dy = \frac{1}{2} \int_{A_j} \frac{g(y - x_j) \, dy}{E(x_j, I_j) + \sigma^2}$$

Once the BSs choose their locations, A_j, I_j, and thus the utility of BS j are determined by the association game played by the mobiles. We thus have a Stackelberg-like game with the lead players being the two BSs (who may either cooperate or compete) and the followers the continuum of users (who compete to maximize their respective SINR densities). We refer to this as the hierarchical equilibrium problem.

III. CDMA: The Single-Frequency Case

A. SINR-equilibrium association

We begin by describing some surprising features of the SINR-equilibrium (see Definition 2.1) that distinguish this from other association games. We set $L = 10$ (so that mobiles are concentrated over the interval $[-10, 10]$) and the noise parameter $\sigma = 0.3$. We place BS 1 at one of the fixed locations x_1 where $x_1 = -10, -5, -2, 0$. For each of these, we vary the location of BS 2 from $x_2 = 0$ to $x_2 = 20$ (see Figure 1). The left column of plots corresponds to a path loss exponent of $\alpha = 2$ and the right one to $\alpha = 1$. The equilibrium sets A_j turn out to have the form $A_1 = [\theta_1, \theta_2]$, $A_2 = [-L, \theta_1) \cup (\theta_2, L]$ for $x_1 = 0, -2$, and $A_1 = [-L, \theta_2]$, $A_2 = (\theta_2, L]$ for $x_1 = -5, -10$. The top (respectively bottom) row of plots depict the threshold θ_2 (respectively θ_1) as a function of BS 2 location x_2. See the following for more details.
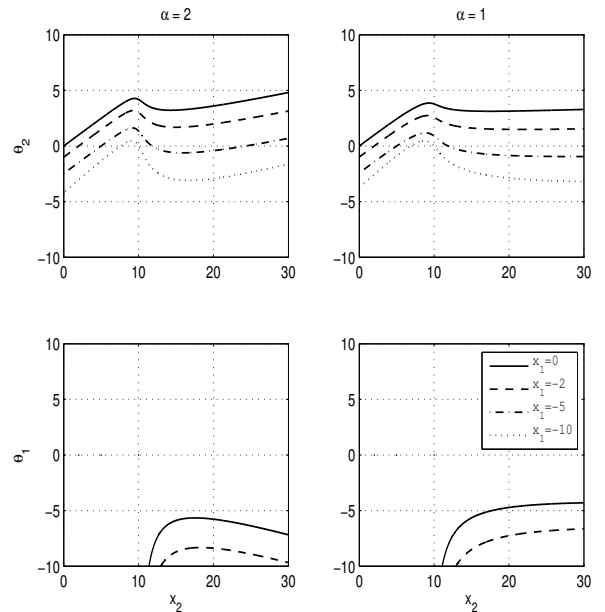
1) Observations:

Fig. 1. Thresholds determining the cell boundaries (vertical axis) as a function of the location of BS 2 (horizontal axis) for various locations of BS 1. The path loss exponent α is 2 in the left Figure and 1 in the right one.
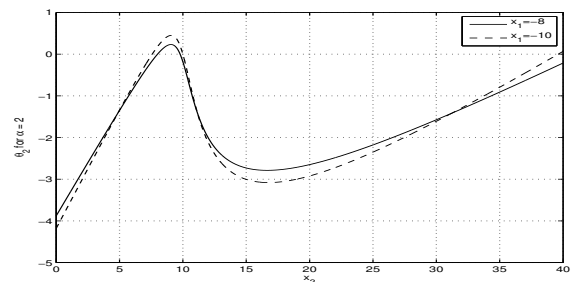

Fig. 2. Upper cell boundary of cell A_1 (vertical axis) as a function of the location of BS 2 (horizontal axis) for various locations of BS 1.

a) Non-convex cells: For all the locations of BS 1, $x_1 = -10, -5, -2, 0$, mobiles in $(\theta_2, L]$ have a better SINR density at BS 2. Let us concentrate on the curves corresponding to $x_1 = -2$ in Figure 1. When BS 2 is located sufficiently far to the right of the origin, the interference at BS 1 is large compared to that at BS 2 (see Proposition 2.1). Thus, mobiles sufficiently far away and to the left of BS 1 (those in $[-L, \theta_1)$) also have a better SINR density at BS 2 despite BS 2 being the farther base station. Thus, in this case, $A_2 = [-L, \theta_1) \cup (\theta_2, L]$, a non-convex set. A_2 is similarly non-convex when $x_1 = 0$ and x_2 is sufficiently far to the right (or left).

b) Non-monotonicity of the cell boundaries: We observe a surprising non-monotonicity of the threshold θ_2 (θ_1 also, in the the curves corresponding to $x_1 = -2, 0$ and $\alpha = 2$) as a function of the location x_2 of BS 2. θ_2 first increases with x_2 until about $x_2 = 8$, then it decreases with x_2 until around $x_2 = 14$; finally, for larger x_2, θ_2 again increases. Analogous observations can be made for θ_1.

The dashed line in Figure 2 shows a zoomed-in view of the $x_1 = -10$ case of the top-left plot of Figure 1. The threshold,

θ_2, increases beyond 0 until x_2 is about 8 units to the right of the origin, and then returns to 0 when $x_2 = 10$. This can be understood as follows. Clearly, for $x_2 = 10$ the interferences at both the BSs are the same, hence, $\theta_2 = 0$, the midpoint. Now imagine moving BS 2 a little to the left (i.e., decreasing x_2). Now $|x_2| < |x_1|$. Thus, from Proposition 2.1, the interference $E^o(x_2)$ at BS 2 is larger than $E^o(x_1)$, the interference at BS 1. This makes it advantageous for mobiles a little to the right of the origin also to associate with BS 1; hence θ_2 increases as x_2 decreases from $x_2 = 10$. Further decrease in x_2 makes BS 2 more proximate to mobiles on the negative x-axis, thus ultimately causing θ_2 to return to 0, and even cross below 0, as x_2 decreases further. As x_2 increases beyond 10, the interference perceived by it decreases, thus making it advantageous for mobiles a little to the left of the origin also to associate with BS 2; hence θ_2 decreases as x_2 increases beyond $x_2 = 10$. Once BS 2 is moved far from the region where the mobiles exist, the signal power to x_2 becomes smaller and smaller, and association with BS 1 becomes increasingly better for mobiles to the right of the origin, causing θ_2 to increase.

The top row of plots in Figure 1 suggests that θ_2 is perhaps monotone in the position of BS 1. But this is not true because a closer look at the θ_2 curves in Figure 2 for $x_1 = -10, -8$ shows that they cross each other several times.

2) Discussion: The form of equilibria displayed in the SINR-association examples is unusual in the class of location games. The reason for the unusual features lies in the SINR criterion:

- If a mobile is very close to a BS, path gain from the mobile to the BS will be very high. Thus, the mobile connects to this BS, even if the interference suffered by this BS is relatively higher.
- If a mobile is located sufficiently far from both BSs, then the relative difference in the powers received at the BSs will be small. Thus the mobile will prefer to connect to BS that suffers from less interference.
- If a mobile is at moderate distance from both the BSs, it takes into account both the factors (i) path gains to the BSs and (ii) interferences suffered the BSs, while making association decision.

3) Closed form expressions for cell boundaries: In this section, we provide closed form expressions for cell boundaries. Without loss of generality we assume that BS 2 is located closer to the origin than BS 1, i.e., $|x_1| \geq |x_2| \geq 0$. Define the αth root of the ratio of the net interferences (including thermal noise) at the two base stations to be

$$B_\alpha(x_1, x_2) = \left(\frac{E^o(x_1) + \sigma^2}{E^o(x_2) + \sigma^2} \right)^{1/\alpha}.$$

On account of $|x_1| \geq |x_2| \geq 0$ and Proposition 2.1, we have $B_\alpha(x_1, x_2) \leq 1$.

Proposition 3.1: Let BS 1 be located at x_1 and BS 2 at x_2 where $|x_1| \geq |x_2| \geq 0$. The set of mobile locations that connect to BS 2 is nonempty if and only if

$$\tau := |x_2 - x_1| \cdot \frac{B_\alpha(x_1, x_2)}{1 - B_\alpha^2(x_1, x_2)} \geq 1. \qquad (8)$$

If the condition holds then the set of locations that connects to BS 2 is given by the interval[3]

$$\frac{x_2 - x_1 B_\alpha^2(x_1, x_2)}{1 - B_\alpha^2(x_1, x_2)} + \left(-\sqrt{\tau^2 - 1}, \sqrt{\tau^2 - 1} \right).$$

Proof: Mobiles that connect to BS 2 will have a higher SINR density to BS 2, i.e.,

$$\frac{[(y - x_2)^2 + 1]^{-\alpha/2}}{E^o(x_2) + \sigma^2} > \frac{[(y - x_1)^2 + 1]^{-\alpha/2}}{E^o(x_1) + \sigma^2}$$

which implies that

$$(y - x_2)^2 + 1 < \left((y - x_1)^2 + 1 \right) B_\alpha^2(x_1, x_2).$$

As $B_\alpha^2(x_1, x_2) \leq 1$, the above inequality holds when a convex quadratic function of y is strictly negative. The positive discriminant condition straightforwardly yields that the set connecting to BS1 is nonempty if and only if (8) holds. The roots of the convex quadratic equation are given by the ends of the specified interval. Since the convex quadratic function is strictly negative in the interval between the roots, BS 2 has the higher SINR in this interval. ∎

When $|x_2| \geq |x_1| \geq 0$, the roles of BS 1 and BS 2 are switched: BS 1 sees more interference, its cell A_1 may be empty, and when nonempty, A_1 is an interval.

B. Hierarchical equilibrium

1) Single base station: Suppose there is only one BS. Given that the interference is maximum at the origin and decreases monotonically with distance from the origin, where should it be placed to maximize utility? The utility of the BS, when placed at x, is given by

$$\frac{1}{2} \int_{-L}^{L} \frac{g(y - x)}{E^o(x) + \sigma^2} \, dy = \frac{1}{2} \frac{E^o(x)}{E^o(x) + \sigma^2}$$

which is maximized when $E^o(x)$ is maximized, i.e., at $x = 0$. Despite the high interference, the origin is the best location to maximize the utility given the nature of the utility function.

2) Two cooperating base stations: We now consider optimal joint placement of two BSs to maximize the sum utility. This would be of interest when both BSs belong to the same operator. Recall that BSs make decisions keeping in mind the SINR-equilibrium associations of mobiles. Simulations indicate that sum utility is maximized when $-x_1 = x_2$, i.e., the BSs are equidistant from the vertical axis. We call such a placement as symmetric. The SINR-equilibrium cells turn out to be $[-L, 0]$ and $(0, L]$ in this symmetric case.

Figure 3 depicts the utility obtained by each BS for symmetric placement $-x_1 = x_2 = x$, as a function of x. We see that the origin and the extreme points (at distance 10 from the origin) are suboptimal locations. We also observe that the performance close to the optimal location is quite robust to perturbations of BS locations.

Further experimentations reveal that, as σ is increased, optimal distance of the BSs from the origin as well as the optimal utility, both decrease (see Figure 4). As $\sigma \to \infty$, the optimal symmetric locations of the BSs converge to -5 and 5. This is expected because at very large σ, interference does not play any role, and the BSs should be placed to maximize the total energy

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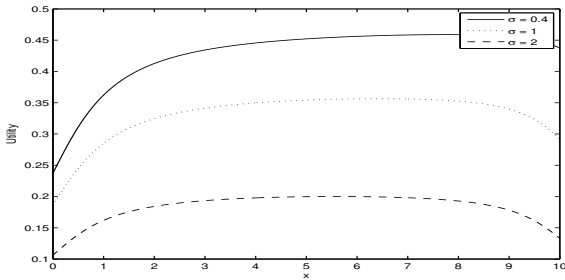[3]The notation $a + (b, c)$ is short for the interval $(a + b, a + c)$.

Fig. 3.   Optimal symmetric placement of base stations: cooperative case; here $L = 10, \alpha = 2$ and σ takes the values 0.4, 1, and 2.
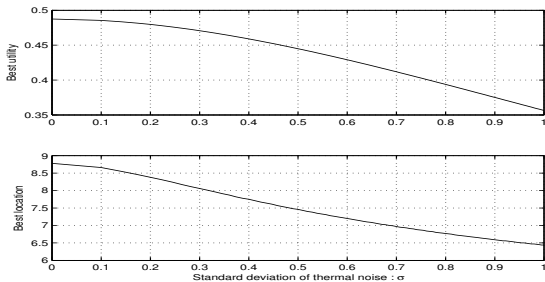

Fig. 4.   The top plot shows optimal utility and the bottom plot the optimal symmetric distance from origin for two BSs, as a function of thermal noise standard deviation.

collected from the respective cells, $E(x, (0, L]) + E(-x, [-L, 0])$. Proposition 2.1 says that this is maximized by choosing x and $-x$ to be the mid-points of the respective intervals, i.e., $x = L/2$, which is 5 in our example.
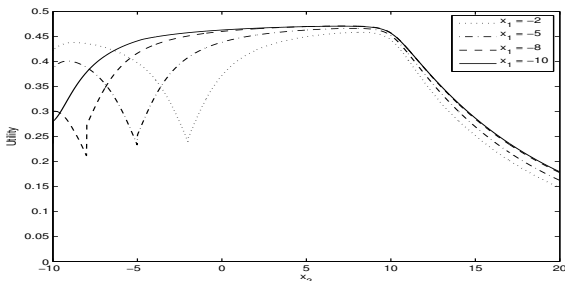

Fig. 5.   Utility of BS 2 as a function of its location when we position BS 1 at x_1 where $x_1 = -2, -5, -8, -10$.

3) Two non-cooperating base stations: We now consider a non-cooperative game between the two BSs. This may arise when these BSs belong to two different operators, but operate on the same RF band. The BSs move simultaneously and pick their respective locations, again keeping in mind the SINR-equilibrium associations of mobiles.

Figure 5 has in the horizontal axis the location x_2 of BS 2 and on the vertical axis the utility it achieves. The figures are obtained for $L = 10, \sigma = 0.3, \alpha = 2$. There are 4 curves that correspond to four locations of BS 1: $x_1 = -2, -5, -8, -10$. From these curves, one can conclude that the utility of BS 2 is quite robust to placement errors around the best response location, for the indicated values of BS 1 locations.

Figure 6 shows the best response of BS 2 to a BS 1 location. BS 1 is moved along the segment to left of the origin. In the figure the horizontal axis is $-x_1$, distance of BS 1 from the origin. A positive best response value indicates a location on the other side of the origin away from BS 1. The maximum utility itself does not change that much with location of BS 1. Numerical computations indicate the existence of a unique pure strategy equilibrium at $-x_1 = x_2 = 7.36$.
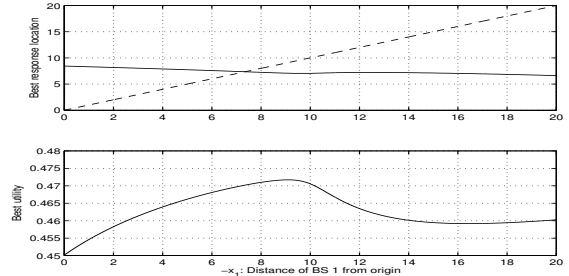

Fig. 6.   The top plot shows the best response of BS 2 when BS 1 at a distance indicated by abscissa to the left of the origin. The bottom plot shows the resulting throughput for BS 2.

In Table I we compare the optimal location of the cooperative case and the equilibrium location of the non-cooperative case, as a function of σ. We observe that at the non-cooperative equilibrium, the BSs are closer than at the cooperative optimum. In both cases the distances decrease in σ and tend to a limit which is $-x_1 = x_2 = 5$ for the cooperative case and $-x_1 = x_2 = 4.106$ for the non-cooperative case. (These answers are related to those of Section V).

TABLE I
COOPERATIVE AND NON-COOPERATIVE PLACEMENTS OF BSs AS A
FUNCTION OF σ

σ	0.1	0.4	1	2	40
Optimum dist. of BSs to 0 (cooperative)	8.658	7.745	6.435	5.591	5.002
Equilibrium dist. of BSs to 0 (non-cooperative)	8.10	6.95	5.50	4.667	4.09

IV. CDMA: THE TWO-FREQUENCIES CASE

We now consider the case when the BSs operate on disjoint RF bands.

A. SINR-equilibrium association

We study the properties of the SINR-equilibrium partition and arrive at a numerical method to compute it.

Recall that the interference at BS j in the two-frequencies case is $E(x_j, A_j)$. Without loss of generality, relabel indices so that

$$B := \left[\frac{E(x_1, A_1) + \sigma^2}{E(x_2, A_2) + \sigma^2} \right]^{1/\alpha} \leq 1.$$

At equilibrium, $y \in A_2$ if and only if

$$\frac{g(y - x_2)}{E(x_2, A_2) + \sigma^2} > \frac{g(y - x_1)}{E(x_1, A_1) + \sigma^2},$$
$$\iff \quad (y - x_2)^2 + 1 < \left((y - x_1)^2 + 1 \right) B^2,$$

where $B^2 \leq 1$. Proceeding exactly as in the proof of Proposition 3.1, we get A_2 to be the set of y such that a convex quadratic function of y is negative. Thus A_2 is an interval and its complement A_1 a union of at most two intervals. More precisely, the boundaries are given as follows. We first find

$$\tau(B) := |x_1 - x_2| \cdot \frac{B}{1 - B^2}. \tag{9}$$

If $\tau(B) \leq 1$, A_2 is empty, $B^2 = (E(x_1, A_1) + \sigma^2)/\sigma^2 > 1$, a contradiction. Thus $\tau(B) > 1$ and A_2 is given by the interval (as in the proof of Proposition 3.1)

$$
\begin{aligned}
&(g_1(B), g_2(B)) \\
&:= \frac{x_2 - x_1 B^2}{1 - B^2} + \left(-\sqrt{\tau(B)^2 - 1}, \sqrt{\tau(B)^2 - 1}\right).
\end{aligned}
$$

This gives expressions for the end points of intervals that make up A_1 and A_2 in terms of B. To emphasize that A_1 and A_2 depend only on B, we write $A_1(B)$ and $A_2(B)$. At SINR-equilibrium, we must therefore have

$$B^\alpha = \frac{E(x_1, A_1(B)) + \sigma^2}{E(x_2, A_2(B)) + \sigma^2},$$

for a $B \leq 1$. This can be written as an implicit equation in B when $\alpha = 2$ as

$$
B^2 = \\
\frac{\begin{array}{l}[\arctan(L - x_1) - \arctan(\lceil g_2(B)\rceil_L - x_1) \\ + \arctan(\lfloor g_1(B)\rfloor_{-L} - x_1) - \arctan(-L - x_1) + \sigma^2]\end{array}}{\arctan(\lceil g_2(B)\rceil_L - x_2) - \arctan(\lfloor g_1(B)\rfloor_{-L} - x_2) + \sigma^2},
$$

where $\lceil m \rceil_L = \min\{m, L\}$, $\lfloor m \rfloor_{-L} = \max\{m, -L\}$, and $\tau(B) > 1$. A similar equation holds for all α with a closed form expression for integral $\alpha \geq 1$. (For $\alpha = 1$, replace arctan by arcsinh, and B^2 on the left side by B). Since $\tau(B) > 1$ implies $B \in (\sqrt{d^2 + 1} - d, 1]$, where $d = |x_1 - x_2|/2$, we may numerically search for a B that solves the above equation through a suitably fine quantization of the specified interval.

Finally, if a $B \leq 1$ satisfying the above equation does not exist, we relabel the indices and repeat the procedure. Accordingly, we obtain the higher interference cell, and the corresponding cell boundaries, for a given pair of BS locations.

As a simple example, consider the symmetric case when $-x_1 = x_2$. It is easy to verify that the cell partition $(A_1, A_2) = (\,[-L, 0], (0, L]\,)$ is an equilibrium partition with $B = 1$.

B. Hierarchical equilibrium

1) Two cooperating base stations: The goal here is to place the two BSs so that the sum utility is maximized taking into account the SINR-equilibrium mobile associations.

Proposition 4.1: The locations $-x_1 = x_2 = L/2$ with SINR-equilibrium cell partition $(A_1, A_2) = (\,[-L, 0], (0, L]\,)$ maximizes the sum utility.

Proof: For a given pair of locations x_1 and x_2, let (A_1, A_2) be the SINR-equilibrium cell partition. For convenience let $u_j := E(x_j, A_j)$, $j = 1, 2$, be the received energy at BS j. Then the

sum utility satisfies the following:

$$\sum_{j=1}^2 \frac{1}{2} \frac{u_j}{u_j + \sigma^2} \quad \leq \quad \frac{(u_1 + u_2)/2}{(u_1 + u_2)/2 + \sigma^2} \tag{10}$$

$$\leq \quad \frac{u_{\max}/2}{u_{\max}/2 + \sigma^2} \tag{11}$$

$$= \quad \frac{E(L/2, [0, L])}{E(L/2, [0, L]) + \sigma^2} \tag{12}$$

where (10) follows from Jensen's inequality because the function $u/(u + \sigma^2)$ is concave in u; inequality (11) follows because the function $\frac{u/2}{u/2 + \sigma^2}$ is monotone increasing in u with $u_{\max}$ the maximum sum of received energies across any partition (not just SINR-equilibrium partitions). The last equality (12) follows from Proposition C.1 in Appendix C. The upper bound is independent of x_1 and x_2, and is achieved when $-x_1 = x_2 = L/2$. The corresponding intervals indeed constitute an SINR-equilibrium cell partition. ∎

2) Two non-cooperating base stations: We now consider the hierarchical game where the BSs compete with each other as in Section III-B3, but for the two-frequencies case.

Figure 7 yields the best response for BS 2 given BS 1's placement. Given a BS 1 location, the higher interference cell and the equilibrium ratio B are first found as discussed in Section IV-A, for each possible location of BS 2. Then the BS 2 location yielding the maximum utility is identified as the best response location and is plotted in the figure.
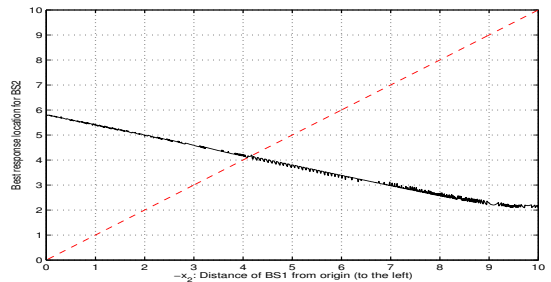

Fig. 7. Two frequency bands: The best response of BS 2 when BS 1 at a distance indicated by abscissa to the left of the origin. A positive best response indicates a location on the other side of BS 1.

Numerical computations indicate that there is a unique equilibrium for the chosen parameters at $-x_1 = x_2 = 4.1$. The corresponding SINR-equilibrium cell partition is $A_1 = [-L, 0]$ and $A_2 = (0, L]$. Note that any unilateral deviation will change the cell boundaries and yield lesser utility to the deviating BS. If the BSs were cooperative, the best locations are $-x_1 = x_2 = 5$. However, this latter set of locations is not an equilibrium under competition.

V. SUCCESSIVE INTERFERENCE CANCELLATION

A. Mobile Association

We now consider the effect of employing successive interference cancellation decoding by the BSs. For model description, see Appendix A. The SINR density seen at BS j, for a mobile at $y \in A_j$ can range from $\frac{g(y - x_j)}{\sigma^2}$ to $\frac{g(y - x_j)}{E(x_j, A_j) + \sigma^2}$, depending on the BS's decoding order. We assume that each mobile first associates with a BS. The BSs then choose an arbitrary decoding

order. In the absence of a clear policy for choosing the decoding order at the BSs, we assume that the mobiles simply associate to the nearest BS. We may interpret this as an association where the mobile optimistically believes that it will be decoded last and therefore expects to see an SINR density of $g(y - x_j)/\sigma^2$ with BS j. If the BSs are co-located, then either association is chosen with equal probability. Define $v = (x_1 + x_2)/2$. Then, the equilibrium cell partition is $A_1 = [-L, v]$, $A_2 = (v, L]$ if $x_1 < x_2$, $A_1 = [v, L]$, $A_2 = [-L, v)$, if $x_1 > x_2$, and an equiprobable choice at every y if $x_1 = x_2$.

B. Single-frequency case

Consider the single-frequency case first. Users connected to BS 1 cause interference at BS 2 and vice-versa. BS 1 and BS 2 have to choose their respective locations. They may do this cooperatively, or may compete in a simultaneous move game. Their actions will immediately fix the cell partition. Each BS then employs successive interference cancellation decoding for all mobiles in its cell. From the discussion in Appendix A, the utility of BS 1 is $\frac{1}{2}\log\left(1 + \frac{E(x_1, A_1)}{E(x_1, A_2) + \sigma^2}\right)$, independent of the decoding order. A similar expression is obtained for the utility of BS 2.

Characterization of jointly optimal locations and competitive equilibria are topics of current study. We do not pursue them in this paper, except for making the following interesting observation. In the cooperative case, if $\sigma^2 \approx 0$, it is (nearly) best if all mobiles can associate to one BS. This is because if there is a nonzero population of mobiles connected to one BS, it generates a nonzero interference to the other BS. On the other hand, with all the mobiles associated to one of the BSs, say BS 1, the sum utility is $\frac{1}{2}\log\left(1 + \frac{E^o(x_1)}{\sigma^2}\right) \to \infty$ when $\sigma^2 \to 0$. So BS 2 should be placed very far away so that its cell is nearly empty. Symmetric points cannot therefore be optimal.

C. Two-frequencies case

We now proceed to the two-frequencies case and give a complete characterization of both cooperative and competitive equilibria. Recall from Appendix A that for BS j, the utility with successive interference cancellation is $\frac{1}{2}\log(1 + \frac{E(x_j, A_j)}{\sigma^2})$, $j = 1, 2$.

1) The cooperative case: In this case, the two BSs cooperate to maximize sum utility.

Theorem 5.1: Consider the two-frequencies case with two cooperating BSs that employ successive interference cancellation decoding. The BS locations that maximize sum throughput are $-x_1 = x_2 = L/2$.

Proof: Recall the notation used in the proof of Proposition 4.1 where $u_j = E(x_j, A_j)$. The sum throughput may be upper bounded as

$$\sum_{j=1}^{2} \frac{1}{2}\log\left(1 + \frac{u_j}{\sigma^2}\right) \leq \log\left(1 + \frac{u_1 + u_2}{2\sigma^2}\right)$$
$$\leq \log\left(1 + \frac{u_{\max}}{2\sigma^2}\right)$$
$$= \log(1 + \frac{E(L/2, (0, L])}{\sigma^2}).$$

where the first inequality follows from Jensen's inequality, while the second follows as in the proof of Proposition 4.1, and third

follows from Proposition C.1. Finally, the upper bound is attained at $-x_1 = x_2 = L/2$ with cell partition ($[-L, 0], (0, L]$). This completes the proof. ∎

Consider now a case where the BSs are constrained to be co-located at x. Recall that mobiles pick one or the other BS with equal probability, so that the energy collected at each BS is $E^o(x)/2$ yielding a sum utility $\log\left(1 + \frac{E^o(x)}{2\sigma^2}\right)$. This attains its maximum when E^o does, which is at $x = 0$ (see Proposition 2.1).

2) The Non-cooperative Case: Define $a := 2^{2/\alpha}$. For $\alpha \in [1, \infty)$, we have $a \in (1, 4]$. Recall that if the two BSs are not co-located, the cell boundary is $(x_1 + x_2)/2$. It is not hard to see that best responses can lie only within $[-L, L]$. Thus, for equilibrium analysis, we only need to focus on $x_1, x_2 \in [-L, L]$. For $x_1, x_2 \in [-L, L]$, the energy collected by BS 2 is $r_2(x_1, x_2)$ as given in Table II, with a similar table for $r_1(x_1, x_2)$ of BS 1. The utility of each BS is a monotone function of the energy collected, and we may therefore assume that BS j's goal is to maximize $r_j(x_1, x_2)$.

TABLE II
ENERGY RECEIVED AT BS 2

$r_2(x_1, x_2)$	$x_2 \in$
$\arctan_\alpha\left(\frac{x_1 - x_2}{2}\right) + \arctan_\alpha\left(L + x_2\right)$	$[-L, x_1)$
$E^o(x_1)/2$	$\{x_1\}$
$\arctan_\alpha\left(L - x_2\right) + \arctan_\alpha\left(\frac{x_2 - x_1}{2}\right)$	$(x_1, L]$

Interestingly, the function $r_2(x_1, \cdot)$ as a function of x_2 is discontinuous at $x_2 = x_1$ unless $x_1 = 0$. A similar observation holds for $r_1(\cdot, x_2)$. We now characterize all pure strategy equilibria.

Theorem 5.2: (i) For $L \leq \sqrt{a - 1}$, there exists a unique equilibrium in pure strategies at $-x_1 = x_2 = 0$.
(ii) For $L > \sqrt{a - 1}$, there exists a unique equilibrium in pure strategies (up to a permutation) at

$$-x_1 = x_2 = \frac{1}{a - 1}\left(-L + \sqrt{aL^2 - (a - 1)^2}\right).$$

Proof: We only give an outline. Consider $L > \sqrt{a - 1}$. Let $x_2 > x_1 \leq 0$. Differentiating $r_2(x_1, x_2)$ with respect to x_2 and by equating it to 0, we get

$$\frac{\partial r_2(x_1, x_2)}{\partial x_2} = \frac{1}{2}g\left(\frac{x_2 - x_1}{2}\right) - g(L - x_2) = 0.$$

The best response x_2 to BS 1's location x_1 should satisfy

$$(L - x_2)^2 + 1 = a\left[1 + \left(\frac{x_2 - x_1}{2}\right)^2\right].$$

Similarly, the best response x_1 to BS 2's location x_2 should satisfy

$$(L + x_1)^2 + 1 = a\left[1 + \left(\frac{x_2 - x_1}{2}\right)^2\right].$$

Combining these two conditions and after ruling out an infeasible solution, we get $-x_1 = x_2 = x \geq 0$, where x satisfies $(L - x)^2 + 1 = a(1 + x^2)$, a quadratic equation in x since $a > 1$. The positive root of this equation yields the solution. The condition $L \geq \sqrt{a - 1}$ ensures that a solution always exists and lies in $[0, L]$. The proof of uniqueness, and the verification that this calculus-based procedure indeed yields a best response are omitted.

For the case $L \leq \sqrt{a-1}$, it is easy to verify that $-x_2 = x_1 = 0$ is an equilibrium. To prove uniqueness, one argues that if there were another equilibrium, then necessarily the derivative-based conditions above must hold and x_1 and x_2 must have the same sign. These and $L \leq \sqrt{a-1}$ yield a contradiction. ∎

Remark 5.1: (i) The equilibria locations do not depend on σ. (ii) As we saw already in Section III-B3 for the case of single user decoding, the competitive equilibrium locations of base stations are closer to each other than the optimum locations under cooperation, a statement that is easy to verify.

3) Convergence to equilibrium: We consider the best response dynamics in which the location of each of the two base stations is sequentially adjusted.

Theorem 5.3: Let $L \geq \sqrt{a-1}$. Assume that BSs follow the best response dynamics to adjust their positions. Then, starting from arbitrary initial positions x_1 and x_2, the best response sequence converges to the unique equilibrium.

Proof: Again, we only give an outline. It is sufficient to show the result for the case when, $x_1, x_2 \in [-L, L]$. The best response for BS 2 to $x_1 \leq 0$ is

$$\mathrm{BR}_2(x_1) = \begin{cases} \frac{4L - ax_1 - 2\sqrt{a(L-x_1)^2 + (4-a)(a-1)}}{4-a}, & a \in (1,4) \\ \frac{L+x_1}{2} - \frac{3}{2(L-x_1)}, & a = 4 \end{cases}$$

It can be seen that $0 < \frac{\partial \mathrm{BR}_2(x_1)}{\partial x_1} < 1 - \epsilon$ for some $\epsilon > 0$. The claim holds for $x_1 > 0$ also. Similar results can be shown for $\frac{\partial \mathrm{BR}_1(x_2)}{\partial x_2}$. Thus a slight change in the position of a BS causes an even smaller change (in the same direction) in the position of the other BS. This game is thus contracting and the best response dynamics converges to the equilibrium. ∎

As an example, consider $L = 10$ and $\alpha = 2$, i.e., $a = 2$. Figure 8 illustrates the fast convergence of dynamics from the starting locations ± 5 to the equilibrium locations ± 4.107.
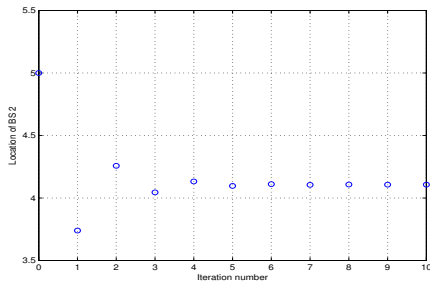

Fig. 8. Convergence to equilibrium for $L = 10$.

VI. Conclusions

We studied combined BS placements and mobile associations in a game-setting where the utilities were determined by SINR criteria. We saw that the SINR-equilibrium cells exhibited non-monotonicity and non-convexity properties that are not seen in the classical location game problems. These unusual properties arise because the SINR density that determines association is a function of distance between mobile and BS and also the BS location. We studied hierarchical equilibria in the CDMA single-frequency and two-frequencies cases. We saw evidence (via simulations in the CDMA single-frequency case and via analysis in the other cases) of a unique optimal pair of locations in the cooperative scenario.

We also saw evidence of a unique equilibrium pair of locations (up to permutation) in the competitive scenario. For the SIC case, we completely characterized the optimal cooperative locations and all pure-strategy competitive equilibria. Interestingly, in all scenarios considered, the BS locations are closer to each other in the competitive case than in the cooperative case.

Appendix A
Propagation, path loss, and fluid models

Propagation model: A mobile transmitter is modeled as a point source that radiates in two-dimensional space or three-dimensional space. The wavefronts emanating from the point source are circular (respectively, spherical in three-dimensional space). We assume that the far field model holds and that antenna couplings between neighboring transmitters and between transmitters and receiver are negligible, even in the limit as mobiles get closer to each other.

Path loss model: Under the far-field model for propagation in two dimensions with circular wavefronts, a receiver at a distance r from the point source and having aperture arc width $s \ll r$ will capture only $s/(2\pi r)$ of the total transmitted power, so that propagation loss is proportional to $1/r$. If there is further dissipation in the medium (analogous to shadowing of electromagnetic waves in three dimensions) we model the propagation loss as proportional to $1/r^\alpha$, where $\alpha \geq 1$. The path loss model $1/r^\alpha$ for three dimensional propagation with $\alpha \geq 2$ is of course the standard one.

Fluid model: Consider n mobiles located on a line at positions $-L + j\Delta y + \frac{\Delta y}{2}$, $j = 0, 1, \cdots, n-1$ with separation spacing $\Delta y = \frac{2L}{n}$. We use the letter y to represent the discrete location for finite n, and the continuum location $y \in (-L, L)$ when $n \to \infty$. Each mobile has power $\Delta p(y) = \Delta y$, so that we may think of transmitted power density per unit distance dp/dy as 1 power unit per unit distance, and the total transmitted power as $2L$ power units. Consider the BS located at x at a height of 1 unit from the line. The path loss for a mobile at y is $g(y - x) = \left[1 + (y - x)^2\right]^{-\alpha/2}$ (see (1)). The total received energy at the BS, if all of these are in the same frequency band, is

$$\begin{aligned} E_n(x) &:= \sum_{y=-L+\Delta y/2}^{L-\Delta y/2} g(y - x)\Delta y \\ &\to \int_{-L}^{L} g(y - x)dy = E^o(x), \end{aligned}$$

where the limit is taken as $n \to \infty$. Similarly, the total received energy from mobiles in a set $A \subseteq [-L, L]$ is

$$\begin{aligned} E_n(x, A) &= \sum_{y \in A} g(y - x)\Delta y \\ &\to \int_A g(y - x)dy = E(x, A). \end{aligned}$$

$E(x, A)$ was defined in (2) and $E^o(x)$ was defined as $E(x, [-L, L])$ immediately after.

SINR Density and throughput: Let $I \subseteq [-L, L]$ denote the set of locations that may be considered as interferer locations. The signal to interference and noise ratio is then

$$\mathrm{SINR}_n(y, x, I) = \frac{g(y - x)\Delta y}{\sigma^2 + \sum_{y \in I} g(y - x)\Delta y}$$

As $n \to \infty$, the denominator tends to $E(x,I) + \sigma^2$, the numerator goes to 0, and the ratio $\frac{\text{SINR}_n(y,x,I)}{\Delta y} \to \frac{g(y-x)}{E(x,I)+\sigma^2}$, so that the latter may be thought of as SINR density (SINR per unit distance). Using Shannon's capacity formula for Gaussian channels, the data rate for a mobile at location y is

$$\frac{1}{2} \log\left(1 + \text{SINR}_n(y,x,I)\right) \approx \frac{1}{2}\text{SINR}_n(y,x,I)$$

where the natural logarithm is employed and the unit of information is nats. (1 nat = $1/(\log 2)$ bits ≈ 1.44 bits). The aggregate throughput of users in a set $A \subseteq [-L, L]$ is

$$\sum_{y \in A} \frac{1}{2}\text{SINR}_n(y,x,I)$$
$$\to \ \frac{1}{2} \int_A \frac{g(y-x)}{E(x,I)+\sigma^2} \ dy = \frac{1}{2}\frac{E(x,A)}{E(x,I)+\sigma^2},$$

which is taken as the utility of a BS in the continuum case.

Successive Interference Cancellation (SIC): Let the interval $A \subseteq [-L, L]$ denote a set of locations associated with the BS at x. Suppose that the BS employs SIC. An arbitrary decoding order is chosen and communicated with the transmitters. For concreteness, let us assume that users are decoded in the decreasing order of y in A. Then all users in A that are to the left of a given user at y will become interferers to y. The throughput for user at $y \in A$ is therefore

$$\frac{1}{2} \log\left(1 + \frac{g(y-x)\Delta y}{\sigma^2 + \sum_{y' < y, y' \in A} g(y'-x)\Delta y}\right)$$
$$= \frac{1}{2} \log\left(\sigma^2 + \sum_{y' \leq y, y' \in A} g(y'-x)\Delta y\right)$$
$$- \frac{1}{2} \log\left(\sigma^2 + \sum_{y' < y, y' \in A} g(y'-x)\Delta y\right).$$

Summing these up over discrete $y \in A$, and passing to the limit, we get the aggregate throughput of all the users in set A to be

$$\frac{1}{2} \log\left(1 + \frac{\sum_{y \in A} g(y-x)\Delta y}{\sigma^2}\right)$$
$$\to \ \frac{1}{2} \log\left(1 + \frac{E(x,A)}{\sigma^2}\right),$$

a formula that is used in Section V for the utility of BS. Note that this remains the sum utility regardless of the decoding order chosen at the BS. Of course, the data rates for each mobile will depend on its position in the decoding order. The sender and the receiver should agree on this data rate and employ an appropriate code.

Discussion: It should be noted that the two-dimensional propagation (when $\alpha \in [1,2)$) and our treatment of mobiles as fluid particles on a line constitute a toy model. The purpose of their study is to get a qualitative feel for what one might expect in the three-dimensional propagation model with mobiles distributed in a plane and receiver antennas placed at a height from the plane.

APPENDIX B
PROOF OF PROPOSITION 2.1

That E^o is an even function, is obvious from (6). To see the monotonicity, for $x \geq 0$, write

$$E^o(0) - E^o(-x) = \int_{-L}^{L} g(y) \ dy - \int_{-L+x}^{L+x} g(y) \ dy \quad (13)$$
$$= \int_{-L}^{-L+x} g(y) \ dy - \int_{L}^{L+x} g(y) \ dy$$
$$= \int_{0}^{x} [g(y-L) - g(y+L)] \ dy \quad (14)$$

where (13) follows from (3), and (14) via a change of variable $y - L \leftarrow y$ in the first integral and $y + L \leftarrow y$ in the second. The integrand in (14) is positive for $y \in [0, x]$. This proves the monotonicity.

APPENDIX C
TWO-FREQUENCY CASE AND SUM RECEIVED POWER

Proposition C.1: Let $x_1 \leq x_2$ and let $v = (x_1 + x_2)/2$ denote the mid-point. Then following results hold.
1. Let (A_1, A_2) denote a partition of $[-L, L]$. Then

$$\max_{(A_1, A_2)} [E(x_1, A_1) + E(x_2, A_2)]$$
$$\leq E(x_1, [-L, v]) + E(x_2, (v, L]).$$

2. Furthermore, $E(x_1, [-L, v]) + E(x_2, (v, L])$ is maximized at $-x_1 = x_2 = L/2$. The cell partition in this case is $[-L, 0]$ and $(0, L]$.

Proof: The first statement is obvious once we write out the integrals and recognize that the integrand is nonnegative, symmetric, and $g(y)$ is decreasing in $|y|$. For the same reason, we may upper bound the sum in the second statement, by $E(x_1, I_1) + E(x_2, I_2)$ where I_1 is an interval of length $L + v$ centered at x_1 and I_2 is an interval of length $L - v$ centered at x_2. $((I_1, I_2)$ may not be a partition of $[-L, L]$). Constraining the sum of interval lengths to be $2L$, the upper bound is further maximized when the intervals are of equal length L. But this upper bound is achieved when $-x_1 = x_2 = L/2$. The corresponding intervals $[-L, 0]$ and $(0, L]$ constitute a cell partition, as required. This concludes the proof. ∎

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