Simplex convexity , with application to open loop stochastic control in networks

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Abstract

In this paper, we present the notion of multimodular triangulation under a new geometrical point of view. We also show the link with multimodular functions by a new proof of the convexity theorem. This is used to define a partial ordering compatible with multimodularity called the *cone ordering*. An application in admission control in queues is then presented.

Keywords Multimodular functions, convexity, tiling, triangulation, admission control.

1 Introduction

The notion of multimodular triangulation was introduced in [3] as a generalization of a construction first presented in [5]. Here, we will give a completely new vision of multimodular triangulation using a geometric point of view. This approach has several advantages. First, we generalize the multimodular triangulations to meshes generated by n arbitrary independent vectors (instead of $\mathbb{Z}^n$). Also, the proofs are elementary and do not use set indexing techniques as in [3]. Finally, the link with multimodular functions is more natural and does not use lower envelopes as in [5]. This new approach of multimodularity allows us to construct a cone partition of the initial set. Within each cone, one can define an appropriate "norm" compatible with a multimodular function f, in the sense that when the norm of point x is larger than the norm of point y, then $f(x) \geqslant f(y)$. Finally an application to admission policies into G/G/1 queues in tandem is given.

2 Multimodular Triangulations

Let us start with a matrix D of size $(n+1)\times n$ of rank n such that the rows of matrix D define $n+1$ vectors $(d_0, \cdots, d_n)$ verifying $d_0 + \cdots + d_n = 0$. Such a matrix will be called a multimodular (m.m.) matrix in the following. For example, a multimodular matrix D can be constructed starting with any $n \times n$ matrix M with full rank and appending the opposite of sum of all the rows of M as the last row of D.

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Definition 2.1 *The mesh M_D associated with the m.m. matrix D is the set of all the points $\{a_0 d_0 + a_1 d_1 + \cdots + a_n d_n, \quad a_i \in \mathbb{Z}, \quad i = 0, \cdots, n\}$.*

Lemma 2.2 *The following properties are true.*
i) A point in $\mathbb{R}^n$ has a unique non-negative decomposition in $(d_0, \cdots, d_n)$ (up to the addition of $(a_0, \cdots, a_n)$ with $a_0 = \cdots = a_n$).
ii)A point in M_D has a unique integer non-negative decomposition in $(d_0, \cdots, d_n)$ (up to the addition of $(a_0, \cdots, a_n)$ with $a_0 = \cdots = a_n$).

Proof: i) Since $d_1, \cdots, d_n$ is a base of $\mathbb{R}^n$, then for any point x in $\mathbb{R}^n$ $x = \alpha_1 d_1 + \cdots + \alpha_n d_n$. Let α_i be the minimal coordinate. If $\alpha_i < 0$, then

$$\begin{aligned} x = &\ \alpha_1 d_1 + \cdot + \alpha_n d_n - \alpha_i (d_0 + \cdot + d_n) && (1)\\ = &\ -\alpha_i d_0 + (\alpha_1 - \alpha_i) d_1 + \cdot + (\alpha_n - \alpha_i) d_n && (2) \end{aligned}$$

where all the coordinates are non-negative. As for uniqueness, let $x = \alpha_0 d_0 + \cdots + \alpha_n d_n = \beta_0 d_0 + \cdots + \beta_n d_n$ where we may assume that all coordinates are non-negative and $\alpha_i = \beta_j = 0$. If $i = j$, then $\alpha = \beta$ because $(d_0, \cdots, d_n)\backslash d_i$ is a base of $\mathbb{R}^n$. If no coordinates are jointly null, then we can write $x = (\beta_0 - \beta_i) d_0 + \cdots + (\beta_n - \beta_i) d_n$ which means for the jth coordinate in the base $(d_0, \cdots, d_n)\backslash d_i$, is $\alpha_j = -\beta_i$, which is impossible by non-negativity.

ii) A point in M_D has a unique decomposition in $(d_1, \cdots, d_n)$, this decomposition being in $\mathbb{Z}$. By using the same method as in (2), then we transform this decomposition into a non-negative integer decomposition in $(d_0, d_1, \cdots, d_n)$. ■

Definition 2.3 *A D-atom is a simplex in $\mathbb{R}^n$, made of* *the* $n+1$ *points*

$$\begin{aligned} p_0 &= a, \\ p_1 &= a + d_{\sigma(0)}, \\ p_2 &= a + d_{\sigma(0)} + d_{\sigma(1)} \\ &\vdots \\ p_n &= a + \cdots + d_{\sigma(0)} + d_{\sigma(1)} + d_{\sigma(n-1)} \end{aligned} \tag{3}$$

where $a \in M_D$ *(the root) and* σ *is a permutation of* $\{0, \cdots, n\}$. *This atom will be denoted* $\mathcal{S}(p_0, \cdots, p_n)$.

Note that an atom is indeed a simplex since D is of rank n and that $\mathcal{S}(p_0, \cdots, p_n)$ and $\mathcal{S}(p_1, \cdots, p_n, p_0)$ are two notations for the same atom, when starting with p_0 (resp. p_1) as a root.

Definition 2.4 *A collection of simplexes is a triangulation of* E *(an arbitrary subset of* $\mathbb{R}^n$*) if*
E is the union of all the simplexes and
the intersection of two simplexes is either empty or a common face.

Theorem 2.5 *The set of all the* D*-atoms forms a triangulation of* $\mathbb{R}^n$*, called a multimodular triangulation.*

Proof: Let x be a point in $\mathbb{R}^n$. By Lemma 2.2, $x = \alpha_0 d_0 + \cdots + \alpha_n d_n$, with non-negative coordinates, one of which is 0. We construct $a = \lfloor \alpha_1 \rfloor d_1 + \cdots + \lfloor \alpha_n \rfloor d_n$ and σ such that $\alpha_{\sigma(i)} - \lfloor \alpha_{\sigma(i)} \rfloor \geqslant \alpha_{\sigma(i+1)} - \lfloor \alpha_{\sigma(i+1)} \rfloor$. We define $\beta_n = 0$, $\beta_{n-1} = \alpha_{\sigma(n-1)} - \lfloor \alpha_{\sigma(n-1)} \rfloor$, $\beta_i = \alpha_{\sigma(i)} - \lfloor \alpha_{\sigma(i)} \rfloor - \alpha_{\sigma(i+1)} - \lfloor \alpha_{\sigma(i+1)} \rfloor$, all of them verify $0 \leqslant \beta_i \leqslant 1$ and $\sum_{i=0}^{n-1} \beta_i \leqslant 1$. We have $x = a + \beta_0 d_{\sigma(0)} + \cdots + \beta_{n-1}(d_{\sigma(0)} + \cdots + d_{\sigma(n-1)})$. Therefore, x belongs to the atom with root a and permutation σ. Now, assume that a point x belongs to the interior of two different atoms with respective roots a and b and permutations σ and τ Since everything is shift invariant, we may assume that $b = 0$ and τ is the identity.

$$\begin{aligned} x &= a + \alpha_0 d_{\sigma(0)} + \cdots + \alpha_{n-1}(d_{\sigma(0)} + \cdots + d_{\sigma(n-1)}) \\ &= \beta_0 d_0 + \cdots + \beta_{n-1}(d_0 \cdots + d_{n-1}), \end{aligned}$$

with $\sum_{i=0}^{n-1} \beta_i \leqslant 1$ and $\sum_{i=0}^{n-1} \alpha_i \leqslant 1$. Since x is in the interior of both atoms, we also have $\beta_i > 0$ and $\alpha_i > 0$ for all $i = 0, \cdots, n-1$. Therefore, by uniqueness of the decomposition of x, and writing $a = a_0 d_0 + \cdots + a_n d_n$,

$$\begin{aligned} a_0 + \sum_{j=\sigma^{-1}(0)}^{n-1} \alpha_{\sigma(j)} &= \beta_0 + \cdots + \beta_{n-1} \\ &\vdots \\ a_{n-1} + \sum_{j=\sigma^{-1}(n-1)}^{n-1} \alpha_{\sigma(j)} &= \beta_{n-1} \\ a_n + \sum_{j=\sigma^{-1}(n)}^{n-1} \alpha_{\sigma(j)} &= 0. \end{aligned}$$

Since all the partial sums of the α_i or of the β_i are all smaller than one and since a_i are integer numbers, then, $a_i = 0$ for all $i = 0, \cdots, n$. Both atoms have the same root.

Now, the equality of the partial sums taken one by one imply first that $\sum_{j=\sigma^{-1}(n)}^{n-1} \alpha_{\sigma(j)} = 0$. Since $\alpha_i > 0$ for all i, then the only possibility is $\sigma^{-1}(n) = n$. Considering vectors d_k and d_{k+1}, we have:

$$\begin{aligned} \sum_{j=\sigma^{-1}(k)}^{n-1} \alpha_{\sigma(j)} - \sum_{j=\sigma^{-1}(k+1)}^{n-1} \alpha_{\sigma(j)} &= \beta_k \\ &> 0. \end{aligned}$$

This implies that $\sigma^{-1}(k) > \sigma^{-1}(k+1)$. This means that σ is the identical permutation. Therefore, both atoms are equal. ■

In the restricted case when the mesh is $\mathbb{Z}^n$ and D the incidence matrix of a graph, this theorem was proved in [3] using an intricate set indexing argument.

Lemma 2.6 *Combinatorial properties of m.m. triangulations.*

i) A point in M_D *belongs to* $(n+1)!$ D*-atoms.*

ii) The unit-cube in M_D *is partitioned into* $n!$ D*-atoms.*

Proof: i) This is a straightforward consequence of the definition of atoms. ii) The unit-cube U in M_D is the set of points of the form $a_1 d_1 + \cdots + a_n d_n$, with $a_i \in \{0, 1\}$ for all $1 \leqslant i \leqslant n$. Given a permutation σ, there exists a point $b \in U$ such that the atom $\mathcal{S}(b, b + d_{\sigma(0)}, \cdots, b + d_{\sigma(n-1)})$ is included in U. The point $b = b_1 d_1 + \cdots + b_n d_n$ is chosen in the following way:

$$b_i = \begin{cases} 0 & \text{if } \sigma^{-1}(i) < \sigma^{-1}(0), \\ 1 & \text{otherwise.} \end{cases} \tag{4}$$

Each atom in U has $n+1$ vertices, hence $n+1$ representation of the form $\mathcal{S}(b, b + d_{\sigma(0)}, \cdots, b + d_{\sigma(n-1)})$ Finally, $(n+1)!/(n+1)$ atoms are contained in U. Since all the atoms triangulate $\mathbb{R}^n$, those in U triangulate U. ■

Some multimodular triangulations have a special interest. The most used ones are the triangulations with a m.m. matrix D being the incidence matrix of a graph. An oriented tree $G = (V, E)$ with $n+1$ nodes (and n arcs) has an incidence matrix D of size $(n+1) \times n$ is defined by

$$D_{i,j} = \begin{cases} +1 & \text{if vertex } i \text{ is the start point of edge } j \\ -1 & \text{if vertex } i \text{ is the end point of edge } j \\ 0 & \text{otherwise} \end{cases}$$

First note that if D has rank n, then the graph has to be a tree. Also, since D is totally unimodular, then its mesh

M_D is $\mathbb{Z}^n$ (see for example [4] for a detailed presentation on totally unimodular matrices). The L-triangulation is the triangulation associated with a linear graph The associated m.m. matrix is

$$D = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}. \quad (5)$$

On the other hand, some triangulations of the space into simplexes are not multimodular triangulations. Such an example in dimension 3 is given in Figure 1. The triangulation in Figure 1 decompose the unit cube in 5 simplexes instead of 3!=6 for any multimodular triangulation.

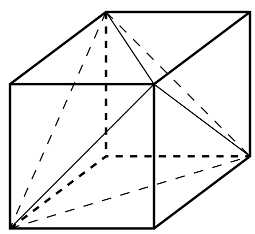

Figure 1: The triangulation of the unit cube of $\mathbb{R}^3$ with a minimal number of simplices is not multimodular

3 Multimodular Functions

Let D be a m.m. matrix with row vectors, $(d_0, \cdots, d_n)$. Sometimes in the following, the mention to D may be forgotten. Everything implicitly refers to D such as multimodularity and atoms.

Definition 3.1 *A function $f : M_D \to \mathbb{R}$ is D-multimodular if and only if for all $a \in M_D$, and for all $0 \leqslant i < j \leqslant n$,*

$$f(a+d_i) + f(a+d_j) \geqslant f(a) + f(a+d_i+d_j). \quad (6)$$

From f, we construct a function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ by linear interpolation of f over the atoms defined by D.

Theorem 3.2 *f is D-multimodular if and only if $\tilde{f}$ is convex.*

The hard part ("only if"), in the restricted case of the L-triangulation, was done in [5]. The extension to the case of tree triangulations as well as the easy reverse part ("if" part) were presented in [3]. Using the formalism presented here, the proof in the most general case is similar to the version presented in [2] for the restricted case of the L-triangulation.

Lemma 3.3 *We consider all the n-periodic sequences $a = (a_i)_{i\in\mathbb{N}}$ with values in $\mathbb{N}$ satisfying $\sum_{i=1}^n a_i = k$. Let f be a multimodular function defined on a mesh M_D of dimension n. Then, the quantity $\sum_{i=1}^n \tilde{f}(a_i d_1 + \cdots + a_{i+n-1} d_n)$ is minimized at point $r = (\frac{k}{n} d_1 + \cdots + \frac{k}{n} d_n)$.*

Proof: Let A be the set of all integer sequences a, which are n-periodic and such that within one period, they add up to k, $\sum_{i=1}^n a_i = k$. By periodicity, and using Jensen inequality, the quantity $\min_{a\in A} \frac{1}{n} \sum_{i=1}^n \tilde{f}(a_i d_1 + \cdots + a_{i+n-1} d_n)$ is equal to $n\tilde{f}(\frac{k}{n} d_1 + \cdots + \frac{k}{n} d_n)$. ■

4 Sub-meshes

Definition 4.1 *A sub-mesh A of M_D is a convex set of $\mathbb{R}^n$ which is the union of (faces of) D-atoms.*

Since any union of D-atoms which forms a convex set is a sub-mesh by definition, typical sub-meshes are: the positive quadrant: $\{a_0 d_0 + \cdots + a_{n-1} d_{n-1}, a_i \geqslant 0\}$ and the unit cube $\{a_0 d_0 + \cdots + a_{n-1} d_{n-1}, a_i \in \{0,1\}\}$. The hyper-plane $\{a_0 d_0 + \cdots + a_{n-1} d_{n-1}, \sum_i a_i = k\}$ is a sub-mesh of dimension $n-1$ of the L-triangulation.

Lemma 4.2 *Any sub-mesh A has the following properties:*
i) The faces defining A form a multimodular triangulation of A.
ii) The vectors of this new multimodular triangulation are disjoint sums of the original vectors.

Proof: First, by convexity, A has the same dimension ,say k, as any of its faces. It should be obvious that the faces defining A form a triangulation of A. Now, let us show that this is a multimodular triangulation. Let F^1 (resp. F^2) be a face in A of a simplex S^1 (resp. S^2) with root a (resp. b) and permutation σ (resp. γ). Without loss of generality, we may assume that $a \in S^1$ and $b \in S_2$ by shifting the starting points in the definitions of atoms S^1 and S^2. Now, the vertices of F^1 (resp. F^2) are visited in an order depending on σ (resp. γ), such that

$$\begin{aligned} & p_0^1 = a \\ & p_1^1 = a + d_{\sigma(0)} + \cdots + d_{\sigma(i_1-1)} \\ & p_2^1 = p_1^1 + d_{\sigma(i_1)} + \cdots + d_{\sigma(i_2-1)} \\ & \vdots \\ & p_0^1 = p_k^1 + d_{\sigma(i_k)} + \cdots + d_{\sigma(n)-1} \end{aligned}$$

$$\begin{aligned} & p_0^2 = b \\ & p_1^2 = b + d_{\gamma(0)} + \cdots + d_{\gamma(j_1-1)} \\ & p_2^2 = p_1^2 + d_{\gamma(j_1)} + \cdots + d_{\gamma(j_2-1)} \\ & \vdots \\ & p_0^2 = p_k^2 + d_{\gamma(j_k)} + \cdots + d_{\gamma(n)} \end{aligned}$$

Since both faces are in the same space of dimension k, then for any $m < k$, we have the linear combination

$$d_{\gamma(j_m)} + \cdots + d_{\gamma(j_{m+1}-1)} = \sum_{\ell=0}^{k-1} \alpha_{m,\ell} d_{\sigma(i_\ell)} + \cdots + d_{\sigma(i_{\ell+1}-1)}.$$

This is a linear combination between the vectors $d_0, d_1, \cdots, d_n$ (with the convention that $i_0 = 0$ and $j_0 = 0$). We know that the only relation is $d_0+d_1+\cdots+d_n = 0$. Therefore, for all m, there exists a unique q such that , $\alpha_{m,q} = 1$. (all the other coefficients $\alpha_{m,\ell}$ are null.

$$d_{\gamma(j_m)} + \cdots + d_{\gamma(j_{m+1}-1)} = d_{\sigma(i_q)} + \cdots + d_{\sigma(i_{q+1})-1}.$$

This means that the vectors defining face F^2 are the same as the vectors defining face F^1 (up to a permutation). ■

Lemma 4.3 *Let p be a point in M_D, σ be a permutation of $\{0, \cdots, n\}$ and $\{i_k\}_{k=0,\cdots m}$ be an increasing sequence of integers with $i_0 = 0$ and $i_m = n$. We define the vectors $\delta_j = d_{\sigma(i_j)} + \cdots d_{\sigma(i_{j+1}-1)}$ for all $j = 0, \cdots m$. The set $p + \{a_0\delta_0 + \cdots + a_m\delta_m, a_i \in \mathbb{N}\}$ is a sub-mesh of M_D.*

Proof: First note that the set A is a convex subset of M_D of dimension m. Now we consider the triangulation of A defined by the vectors $\delta_0, \cdots, \delta_m$, each atom of this triangulation is a face of a D-atom. ■

Thus the hyper-plane $P = \{a_0 d_0 + \cdots + a_{n-1}d_{n-1}, \sum_i a_i = k\}$ is a sub-mesh of dimension $n-1$ of the L-triangulation by choosing $\delta_0 = d_1, \cdots, \delta_{n-2} = d_{n-1}$ and $\delta_{n-1} = d_n + d_0$, which yields $P = (k, 0, 0 \cdots, 0) + \{x_0\delta_0 + \cdots + x_{n-1}\delta_{n-1}, x_i \in \mathbb{N}\}$.

Lemma 4.4 *A function f multimodular on the whole space is multimodular on any sub-mesh A with respect to the induced multimodular matrix of A.*

Proof: Let a be a point in A and let u and v be two arbitrary rows for the multimodular matrix of A. By Lemma 4.2, $u = d_{i_1} + \cdots + d_{i_k}$ and $v = d_{j_1} + \cdots + d_{j_m}$ where the sets $\{d_{i_1}, \cdots, d_{i_k}\}$ and $\{d_{j_1}, \cdots, d_{j_m}\}$ are pairwise distinct. Therefore, we have

$$\begin{aligned} & f(a) + f(a+u+v) \\ & = f(a) + f(a + d_{i_1} + \cdot + d_{i_k} + d_{j_1} + \cdot + d_{j_m}) \\ & \leqslant f(a + d_{i_1} + \cdots + d_{i_k}) + f(a + d_{j_1} + \cdots + d_{j_m}) \\ & = f(a+u) + f(a+v). \end{aligned}$$

■

Corollary 4.5 *The function f is multimodular in a sub-mesh A if and only if $\tilde{f}$ is convex on A.*

A second corollary of Theorem 3.2 concerns the minimization of multimodular functions. For a function defined on A, we call x a local minimum on A if $f(x) \leqslant f(x+/-d_i)$ for all i such that $x+/-d_i$ is in A.

Corollary 4.6 *Let the function f be multimodular in A. Then a local minimum is a global minimum on A.*

Proof: If f is multimodular in A, then $\tilde{f}$ is convex in A, and is linear on the (faces of) atoms forming A. The graph of $\tilde{f}$ (*i.e.* $\{x : \exists y \text{ s.t. } x \geqslant \tilde{f}(y)\}$) is a convex polytope. Therefore, all the local minima are global and are extreme points of atoms. ■

5 Cones

Now, the convex space A of dimension n will be divided into $(n+1)!$ cones, all starting at point h, any point of the mesh of A. Consider one atom $S = \mathcal{S}(h = p_0, p_1, \cdots, p_n)$ containing h as a vertex. Let σ and τ be the permutations on $\{0, \cdots n\}$ such that $\tau(0) = 0$ and

$$\begin{aligned} p_{\tau(1)} &= h + d_{\sigma(0)} \\ p_{\tau(2)} &= p_{\tau(1)} + d_{\sigma(1)} \\ \vdots &= \vdots \\ p_{\tau(n)} &= p_{\tau(n-1)} + d_{\sigma(n-1)} \\ h &= p_{\tau(n)} + d_{\sigma(n)}. \end{aligned}$$

. Now, we define for all $1 \leqslant i \leqslant n$, $b_i = \sum_{j=1}^{i} d_{\sigma(j)}$ and $b = (b_1, \cdots, b_n)$. Therefore, $p_{\tau(i)} = h + b_i$. The vectors $(b_1, \cdots, b_n)$ will be called the *generators* of the cone. The cone associated with S, denoted $C(S)$ is made of all the points p of A such that $p = h + c^t b$ where c is a non-negative vector in $\mathbb{N}^n$. First, note that vector c^t is uniquely defined since $(b_1, \cdots, b_n)$ are independent vectors. Second, note that when we consider all the atoms containing h as a vertex, then all the $(n+1)!$ associated cones will cover A. If two neighbor atoms share a face, the two corresponding neighbor cones will also share a "face" (of dimension $n-1$). For any point p in A, p will be in one cone and we will have $p = h + c^t(p)b$, where b is uniquely defined on the support of $c(p)$. We shall denote $d(h,p) = c_1(p) + \cdots + c_n(p)$ and call it the distance from h to p. All the previous remarks show that $d(h,p)$ is well defined.

5.1 Minimization

In this section, we consider the case where $A = \mathbb{R}^n$ and $h = 0$. We also consider a function f multimodular with respect to the row vectors of D, $d_0, \cdots, d_n$. We focus on one arbitrary cone, C, defined by the permutation γ. This means that the generators of C are the vectors

$$\begin{aligned} b_0 &= d_{\gamma(0)}, \\ b_1 &= d_{\gamma(0)} + d_{\gamma(1)}, \\ &\vdots \\ b_{n-1} &= d_{\gamma(0)} + d_{\gamma(1)} + \cdots + d_{\gamma(n-1)}. \end{aligned}$$

Any point in C has non-negative coordinates $\beta_0, \cdots, \beta_{n-1}$ in the base $b_0, \cdots, b_{n-1}$. We call P

the linear application which is the passage from the base $b_0, \cdots, b_{n-1}$ to the base $d_{\gamma(0)}, d_{\gamma(1)}, \cdots, d_{\gamma(n-1)}$.

Lemma 5.1 *Let k be an integer. The set C_k of points in C such that $\beta_0 + \cdots + \beta_{n-1} = k$ is a sub-mesh of M_D.*

Proof: Set $p = kd_{\gamma(0)}$ and $\delta_0 = d_{\gamma(1)}, \cdots, \delta_{n-2} = d_{\gamma(n-1)}$, $\delta_{n-1} = d_{\gamma(0)} + d_{\gamma(n)}$. Then the sub-mesh $p + \{\sum_i a_i\delta_i, a_i \in \mathbb{N}\}$ is precisely the set C_k. ∎

The following lemma is some kind of generalization of Theorem 3.4 in [2] from the L-triangulation and the positive quadrant to any multimodular triangulation and one of its cones.

Lemma 5.2 *Let f be a m.m. function, then the quantity*

$$\frac{1}{n}\sum_{i=1}^{n} f \circ P(\beta_i \cdots, \beta_{i+n-1})$$

is minimized over the set C_k at all the points $\beta(\theta)$, $0 \leqslant \theta \leqslant 1$ of coordinates

$$\beta_i(\theta) = \lfloor i\frac{k}{n} + \theta\rfloor - \lfloor (i-1)\frac{k}{n} + \theta\rfloor.$$

Proof: The function $\frac{1}{n}\sum_{i=1}^{n} \tilde{f} \circ P(\beta_i \cdots, \beta_{i+n-1})$ is clearly convex and is minimized at point $r = (\beta_0 = \frac{k}{n}, \cdots, \beta_{n-1} = \frac{k}{n})$ over C_k (see Lemma 3.3). This function is linear on the atoms of the sub-mesh C_k. Therefore, it is also minimum at all the vertices of the face containing the point r. The vertices of this face are the points $\beta(\theta)$ with coordinates

$$\beta_i(\theta) = \lfloor (i+1)\frac{k}{n} + \theta\rfloor - \lfloor i\frac{k}{n} + \theta\rfloor,$$

when θ varies from 0 to 1. Indeed, these points are all in the sub-mesh, since all their coordinates are integer number and since P is totally-unimodular. Now, Let $f_i = 1-(i+1)\frac{k}{n}+\lfloor (i+1)\frac{k}{n}\rfloor$, all ordered in the increasing order. By construction, when θ varies from 0 to 1, then the point $\beta(\theta)$ takes only n values. The point $\beta(\theta)$ changes at all the points of the form f_i for $i = 0, \cdots, n-2$. At $\theta = f_i$ we add $-d_{i+1}$. Therefore, all the points $\beta(\theta)$ form a hyper-face. Noting that

$$r = \frac{\beta(0)}{n-1} + \frac{\beta(f_0)}{n-1} + \frac{\beta(f_1)}{n-1} + \cdots + \frac{\beta(f_{n-2})}{n-1},$$

shows that r belongs to that face. ∎

5.2 Partial order and monotonicity

Now, we define a partial order on A by choosing $h = r$, where r is the minimal point of a m.m. function f on A. First, this partial order is defined in a different manner on each cone. In a given cone C, with generating vectors $b_1, \cdots, b_n$, then we say that $x \leqslant_C y$ is $c(x) \leqslant c(y)$ component-wise. Note that in a given cone, this partial order is a lattice which is isomorphic to $\mathbb{N}^n$ with the classical component-wise order.

Theorem 5.3 *If f is a multimodular function on A, then $x \leqslant_C y$ implies that $f(x) \leqslant f(y)$. In other words, f is monotone with respect to the partial order $\leqslant_C$.*

Proof: Since x and y are comparable, this means that they are in the same cone. From now on, $b_1, \cdots, b_n$ will be the generators of this cone. First note that we can assume that $d(x, y) = 1$. If not, then we prove step by step along the path from x to y along the direction of the generators, say $x = x_1 \leqslant_C \cdots \leqslant_C x_m = y$ that $f(x) = f(x_1) \leqslant \cdots \leqslant f(x_m) = f(y)$. The proof will now proceed by induction on $d(r, x)$. First note that the property is true if $d(x, r) = 0$, since r is the argmin of f on S. Now, let assume that we have $d(x, r) \geqslant 1$. Pick a point z such that $x = z + b_i$ in cone C. (equivalently, we have $c(y) + e_i = c(x)$ and $c(y) \leqslant 0$). Note that $d(z, r) = d(x, r) - 1$ and $z \leqslant_C x$. By induction, this means $f(z) \leqslant f(x)$. Since $d(x, y) = 1$, there exist j such that $y = x + b_j$. Now we have two cases, since we may not be able to choose i such that $i = j$.
If $i = j$, then by convexity of f,

$$f(z + b_i) - f(z) \leqslant f(z + b_i + b_i) - f(z + b_i).$$

We also know by induction that $f(z + b_i) - f(z) \geqslant 0$. This means that $f(x) \leqslant f(y)$.
If $i \neq j$, then we choose yet another point, w, such that $w = z + b_j$. We can assume that $i > j$ (the case $j < i$ is similar by inverting the role played by b_i and b_j in the following). Since b_i is a sum of base vectors, it is also a sum of opposites of base vectors, since all base vectors add up to 0. Note that all these base vectors are distinct from the base vectors involved in b_j. We have: $b_j = d_{\sigma(1)} + \cdots + d_{\sigma(j)}$, and $b_i = -d_{\sigma(i+1)} - d_{\sigma(i+2)} - \cdots - d_{\sigma(n+1)}$. Therefore,

$$\begin{aligned}
& f(w) - f(z) \\
& = f(x + d_{\sigma(1)} + \cdots + d_{\sigma(j)} + d_{\sigma(i+1)} + \cdots + d_{\sigma(n+1)}) \\
& -f(x + d_{\sigma(i+1)} + \cdots + d_{\sigma(n+1)}) \\
& \leqslant f(x + d_{\sigma(1)} + \cdots + d_{\sigma(j)}) - f(x), \qquad (7) \\
& = f(y) - f(x).
\end{aligned}$$

where Inequality 7 is a direct consequence of the definition of multimodularity. Since $d(r, w) = d(r, z) + 1$, by induction we have, $f(w) - f(z) \geqslant 0$, then this implies $f(y) - f(x) \geqslant 0$. ∎

6 Application: periodic admission sequences in $G/G/1/\infty$ tandem queues

We consider queues in tandem with general stationary service times. As for the arrival sequence, let $(u_i)_{i\in\mathbb{N}}$ be a stationary process. The integer sequence $\{a_i\}_{i\in\mathbb{N}}$ is the *admission sequence* into the queues. The inter-arrival times of customers in the queue is a sequence $(\tau_i)_{i\in\mathbb{N}}$ defined by:

$$\tau_i = \sum_{j=a_1+\cdots+a_{i-1}}^{a_1+\cdots+a_i} u_j.$$

In the following, the admission sequence will be assumed to be periodic with period n. As to introduce multimodularity, we choose the L-triangulation of $\mathbb{Z}^n$. The atoms given by the L-triangulation with row vectors $d_i = -e_i + e_{i+1}$, and $d_0 = e_1, d_n = -e_n$, as in (5). The sub-set of $\mathbb{Z}^n$ that we will work with is $A = \{(a_0, \cdots, a_n), a_i \geqslant 0\ \forall i, \sum_{i=0}^n a_i = k\}$, where k is a given integer. The set A corresponds to all admission sequences with n admitted customers among k slots.

Lemma 6.1 *A is a convex union of hyper-faces of atoms in $\mathbb{Z}^n$.*

Proof: Let us consider the constraints one by one. The constraints $a_j \geqslant 0$ restrict A to N^n which is made of a convex union of atoms. Now, let us look at the constraint $\sum_{i=0}^n a_i = k$. This constraint is a convex union of faces of atoms. To finish the proof, remark that the intersection of convex union of faces of atoms is a convex union of faces of atoms. ∎

The atoms on A are defined by the vectors $d'_1 = d_1, \cdots, d'_{n-1} = d_{n-1}$ as for the L-triangulation of $\mathbb{Z}^n$, and a new vector $d'_0 = d_0 + d_n = e_n - e_1$. If f is a multimodular function, $f : \mathbb{Z}^n \to \mathbb{R}$, we will consider the restriction of f to A which is also multimodular on A with its own atoms (see Lemma 4.4). By Corollary 4.6 f has a global minimum on A. In the following, this minimum will be called r.

Theorem 6.2 *The average expected waiting time W in a SEG is a multimodular function on A.*

Proof: From the vector $a = (a_1, \cdots, a_n)$ we construct an infinite sequence $\alpha = a^\omega$. Let $W_N(a_1, \cdots, a_n) = \frac{1}{N}\sum_{k=1}^N w_k(\alpha_1, \cdots, \alpha_k)$, where w_k is the expected total sojourn time of the kth customer, and let $W(a_1, \cdots, a_n) = \lim_{N\to\infty} W_N(a_1, \cdots, a_n)$. We also denote p the largest integer such that $pn \leqslant N$. From [1], we know that w_k is multimodular with respect to the L-triangulation in $\mathbb{Z}^k$. Since the m.m. matrix for the L-triangulation in $\mathbb{Z}^k$ is a sub-matrix of the m.m. matrix for the L-triangulation in $\mathbb{Z}^N$ then we also know that W_k is multimodular in $\mathbb{Z}^N$. Therefore, the function $H(\alpha_1, \cdots, \alpha_N) = \frac{1}{N}\sum_{k=1}^N w_k(\alpha_1, \cdots, \alpha_k)$ is multimodular in $\mathbb{Z}^N$. For all $0 \leqslant i \leqslant n$, $W_N(a + d_i) = H(\alpha + d_i + d_{i+n} + \cdots + d_{i+kn})$, where $k = \lfloor \frac{N-i}{n} \rfloor$. Now, using the general characterization of multimodularity, that is $f(a + D_1 + D_2) - f(a + D_1) \leqslant f(a + D_2) - f(a)$, for D_1 and D_2 any arbitrary sum of base vectors, with the only restriction that no base vector appears in D_1 and in D_2, then it is immediate to check that $W_N(a + d_i + d_j) - W_N(a + d_i) \leqslant W_N(a + d_j) - W_N(a)$, for d_i and d_j any arbitrary distinct m.m. row vectors. Therefore, W_N is multimodular in $\mathbb{Z}^n$. The limit W is also multimodular in $\mathbb{Z}^n$. By using Corollary 4.4 , W is also multimodular on A. ∎

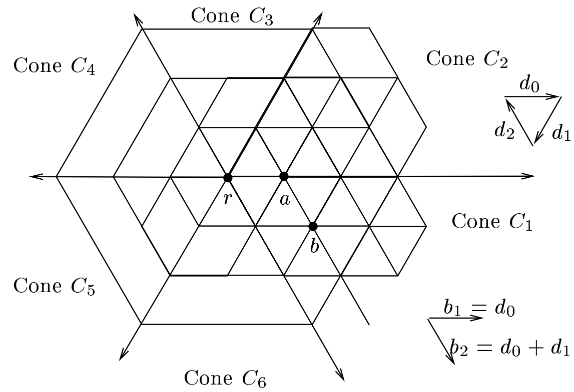

Figure 2: comparison of $b = (1, 1, 4)$ and $a = (1, 2, 3)$

For example, let us compare the expected waiting time under admission sequence $a = (1, 2, 3)$ and under admission sequence $b = (1, 1, 4)$. The corresponding space A is the hyper-plane of $\mathbb{R}^3$, defined by $\{(x, y, z), x+y+z = 6\}$. The space A is of dimension 2 with induced multimodular vectors $d_1 = (+1, -1, 0), d_2 = (0, +1, -1)$ and $d_0 = (-1, 0, +1)$ (from the L-triangulation of $\mathbb{R}^3$). The function W (expected waiting time) is minimized at point $r = (2, 2, 2)$. This is a direct consequence of a combination of Lemma 5.2 (used with the L-triangulation and the cone constructed with the n first vectors, which is the positive quadrant) and Theorem 6.2. If we consider the cone C_1 generated by $b_1 = d_0$ and $b_2 = d_0 + d_1$, then we have $a = r + b_1$ and $b = r + b_1 + b_2$. Therefore, $a \leqslant_C b$, which implies $W(a) \leqslant W(b)$ by Theorem 5.3. This example is illustrated in Figure 2.

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