Regular Ordering and Applications in Control Policies

Eitan Altman
(altman@sophia.inria.fr)
INRIA B.P. 93 06902 Sophia Antipolis Cedex France
Bruno Gaujal*
(gaujal@loria.fr)
LORIA B.P. 102 54602 Villers-les-Nancy, France

Arie Hordijk
(hordijk@wi.LeidenUniv.nl)
Leiden University, POB 9512, 2300RA Leiden, The Netherlands.

December 22, 1999

Abstract

In this paper we introduce a notion of regularity of periodic sequences based on the distances between its entries. We define the notion of regular preserving functions using Schur convexity and we show that they constitute an essential class of multimodular functions. This deepens some optimization results in queuing control problems. In particular, we show that the maximal traveling time in stochastic event graphs is regular preserving, as well as the transmission throughput of a channel with redundancy.

Keywords Schur convexity, Multimodularity, Balanced sequences, Admission Control, (max,plus) algebra.

1 Introduction

An integer sequence is said to be balanced if the partial sums on any two windows of the same length differs by at most one. This gives a formal definition to the notion of "even distribution" within the sequence. Such sequences may also be called "most regular" sequences (see the section 3 for a justification of the term). They have been investigated in the past [16, 13] and lead to fruitful results in several fields. In a recent paper, [3], it is shown under general assumptions that balanced sequences minimize several cost functions in discrete event systems. Most of these results rely on the following theorem. *If $f : \mathbb{Z}^n \to \mathbb{R}$ is a multimodular function, then f is minimized by a balanced sequence.*

Although these results bring a lot of understanding, they fail to provide any insight in comparing the cost of two given sequences s and s' when none is balanced. These kinds of comparisons are of even bigger importance when one knows that the balanced sequence may not exist in some cases (see [2]).

On the other hand, majorization and Schur convex functions seem to be the right notions to be used when one wants to compare evenly distributed sequences [15]. However, majorization does not take into account the initial order in the sequences to be compared; this notion is left invariant by the group of all permutations over the sequence.

For application purposes, it is desirable to take the initial ordering into account. For many systems, the input sequence $(1, 1, 1, 0, 0, 0)$ certainly yields a different behavior than the input

*The work of this author was partially supported by the special year on networking at Dimacs, Piscataway, NJ.

sequence $(1,0,1,0,1,0)$, which looks more balanced. However, these two sequences cannot be compared by using the classical majorization technique.
To overcome this difficulty, we introduce the notion of distance sequences (similar to partial sums, already used in [10]) that takes into some account the initial positions in the sequence. This narrows the gap between the notion of regular monotonicity and multimodularity.
This also allows us to show in some particular cases that if s is "more regular" than s', then $f(s) \leqslant f(s')$ for some of the cost functions mentioned earlier. We call these functions *regular preserving functions.*
Usually, the performance measure in queuing networks in the average waiting time. However, in communication models where real time constraints are important, such as voice traffic for instance, the maximal waiting time (or sojourn time) are likely top be more relevant. In Section 4, we show that in a rather general network of FIFO stochastic event graphs, the maximal waiting time (or sojourn time) is regular preserving. Hence, for finding good allocation patterns, a mathematical programming problem with object function as in corollary 3.8 is adequate. This procedure for minimizing the average waiting time has been used by Combé and Boxma [12] and in [11] for a different performance measure. By a counter example, we show that the average waiting time is not regular preserving.
The rest of the paper is organized as follows. In section 2, we introduce the notion of balanced sequences and distance sequences. In section 3, we introduce the definitions of majorization, Schur convexity and show some preliminary properties. In Section 3.2 we define multimodularity. We also show that a broad class of regular preserving functions are multimodular up to an adequate transformation. In section 4, we introduce a model of a controlled queue with periodic inter-arrival sequences and stationary service times. We show that the maximum waiting time is regular preserving w.r.t. the inter-arrival time as well as w.r.t the service times. Section 4.6 extends the result to a routing problem between several queues. Finally we show in Section 5, that the transmission rate on a link with redundancy is also regular preserving.

2 Distance Sequences

We use the notation $\mathbb{N} = \{0,1,2,\cdots\}$ of the set of all non-negative integers and we consider the set $\mathcal{P}(T,n)$ of all non-negative integer sequences $s = (s_0, s_1, \cdots, s_{T-1})$ of size T which sum exactly to n. Let $s \in \mathcal{P}(T,n)$. We call $(p_s(0), p_s(1), \cdots, p_s(n))$ the positions in s of all its partial sums.

$$p_s(0) = 0 \text{ and } p_s(i) \triangleq \inf\{j | \sum_{k=0}^{j} s_k \geqslant i\}, i \geqslant 1.$$

We define the vector of distances of order $i \geqslant 1$, $d^i(s)$, by:

$$d^i_j(s) = p_s(j+i) - p_s(j), \tag{1}$$

where all indexes are defined modulo n. When the sequence s is in $\{0,1\}^n$, then the distances are never null and measure the "gap" in the sequence s between the ones (hence the name distances). This is illustrated in Figure 1.
The two following lemmas are straightforward consequences of the definition of the distances.

Lemma 2.1. *The distance sequences satisfy the following properties:*

$$d^1_j + d^1_{j+1} + \cdots + d^1_{j+k} = d^{k+1}_j, \tag{2}$$
$$d^i_1 + d^i_2 + \cdots + d^i_n = iT, \tag{3}$$
$$d^i_j + d^{n-i}_{j+i} = T. \tag{4}$$

Proof. The proof is a straightforward consequence of the definition of the distances. □

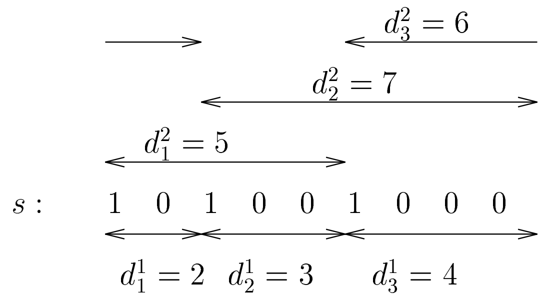

Figure 1: Distances of order one and two for a sequence in $\mathcal{P}(9,3)$

Lemma 2.2. *Let s and s' be two sequences in $\mathcal{P}(T,n)$. If s' is a* shift *of s or a* mirror *of s (i.e. $\forall j, s_j = s'_{j+k}$ or $s_j = s'_{-j+k}$), then $\forall i, \{d^i(s)\} = \{d^i(s')\}$.*

Proof. The proof also follows directly from the definition of the distances. □

Remark 2.3. The converse of Lemma 2.2 is false in general, as shown by the following counter example[1]; choose s and s' such that $d^1(s) = (1,1,2,0,2,1,1,0)$ and $d^1(s') = (1,1,0,2,0,1,1,2)$. s' is not a mirror nor a shift of s. However, for all order i, we have $\{d^i(s)\} = \{d^i(s')\}$. We will prove in the following (see Remark 2.10) that the converse of Lemma 2.2 is true in the special case of balanced sequences.

Lemma 2.4. *If $s \in \mathcal{P}(T,n)$, then $d^1(s) \in \mathcal{P}(n,T)$ and $d^1(d^1(s))$ is a shift of s.*

Proof. By using Equation (3), then $d^1(s) \in \mathcal{P}(n,T)$. Now, let s' be a shift of s such that $p_{s'}(1) = 0$ and such that $d^1(s') = d^1(s)$. We will show that $d^1(d^1(s')) = s'$. Since $p_{s'}(i+1) = \inf\{j | \sum_{k=0}^{j} s'_k \geqslant i+1\}$, then $p_{s'}(i+1) - p_{s'}(i) = \#\{j | \sum_{k=0}^{j} s'_k = i\}$. Now we compute

$$\begin{aligned} d_i^1(d^1(s')) &= \#\{j | \sum_{k=0}^{j} d_k^1(s') = i\} \\ &= \#\{j | p_{s'}(j) - p_{s'}(0) = i\} \\ &= \#\{j | p_{s'}(j) = i\} = s'_i. \end{aligned}$$

□

We can construct the different distances by computing in the set $\mathbb{Z}[X]/(X^n - 1)$. Indeed, from the set $d^i(s)$, we can define the polynomial

$$P_s^i(X) \triangleq \sum_{j=1}^{n} d_j^i(s) X^{j-1}.$$

Note that by definition, we have $P_s^i(X) = (X^{i-1} + \cdots + X + 1) P_s^1(X) \bmod (X^n - 1)$.

Lemma 2.5. *Let $i > 1$. The original sequence s can be retrieved from $d^i(s)$ if and only if i and n are relatively prime.*

Proof. There is a one to one correspondence between s and $d^1(s)$. Therefore, we rather show that the knowledge of $d^i(s)$ enables one to compute $d^1(s)$ if and only if i and n are relatively prime. If i and n are relatively prime, then the polynomials $X^n - 1$ and $X^{i-1} + \cdots + X + 1$ are relatively prime. Using the Bezout equality, there are two polynomials $u(X)$ and $v(X)$ in $\mathbb{Z}[X]$ and a non-zero integer k such that $k = (X^{i-1} + \cdots + X + 1)u(X) + v(X)(X^n - 1)$. Therefore,

[1] This counter example was provided to the authors by Jerôme Galtier

by multiplying by $P_s^1(X)$, we get $kP_s^1(X) = P_s^i(X)u(X) \bmod (X^n - 1)$. The knowledge of $P_s^1(X)$ induces the knowledge of $d^1(s)$, which in turn gives all distances by using Equation (2).
On the other hand, if i and n are not relatively prime, then $\gcd((X^{i-1}+\cdots+X+1), X^n-1) = k(X)$, and if we denote by $q(X) \triangleq (X^n-1)/k(X)$, then $q(X) \in \mathbb{Z}[X]$ and $q(X)(X^{i-1}+\cdots+X+1) = 0 \bmod (X^n-1)$. We have $P_s^i(X) = P_s^1(X)(X^{i-1}+\cdots+X+1) \bmod (X^n-1)$ as well as $P_s^i(X) = (P_s^1(X)+q(X))(X^{i-1}+\cdots+X+1) \bmod (X^n-1)$. Therefore, two different distance sequences of order one are compatible with the distance sequence of order i. □

2.1 Balanced Sequences

Definition 2.6. *A sequence s is balanced if the sums of the elements of s over any two intervals of the same length differs by at most one. In other words, for all j, k and n,*

$$\left|\sum_{i=j}^{j+n} s_i - \sum_{i=k}^{k+n} s_i\right| \leqslant 1.$$

Theorem 2.7 ([16]). *If a sequence s in $\mathcal{P}(T,n)$ is balanced, then there exists $\theta \in [0,1]$ such that for all $j \geqslant 0$,*

$$s_j = \lfloor (j+1)\frac{n}{T} + \theta \rfloor - \lfloor j\frac{n}{T} + \theta \rfloor.$$

Proof. The proof comes from the results in [16] adapted to the case of a finite sequence. □

If s is a balanced sequence, then the associated distance sequences have the following property.

Lemma 2.8. *If s is a balanced sequence in $\mathcal{P}(T,n)$, then for all j, i, $d_j^i \in \{\lfloor \frac{iT}{n} \rfloor, \lceil \frac{iT}{n} \rceil\}$.*

Proof. By definition, $d_j^i = p(i+j) - p(j)$. By definition of p_j, we have: $p(j) = \inf\{k | \lfloor (k+1)\frac{n}{T} + \theta \rfloor \geqslant j\}$. Therefore, $\lfloor (p(j)+1)\frac{n}{T} + \theta \rfloor \geqslant j$ and $\lfloor p(j)\frac{n}{T} + \theta \rfloor < j$. This implies that

$$j - \frac{n}{T} \leqslant p(j)\frac{n}{T} + \theta < j.$$

Similarly, we have:

$$j + i - \frac{n}{T} \leqslant p(j+i)\frac{n}{T} + \theta < j + i.$$

As for d_j^i, we get by subtracting the two previous equations

$$\frac{iT}{n} - 1 < d_j^i < \frac{iT}{n} + 1.$$

□

The following lemma in the converse of Lemma 2.8.

Lemma 2.9. *Let s be a sequence such that for all i, all the distances of order i, d_j^i, differ by at most one. Then, s is balanced.*

Proof. Assume that s is not balanced. There exist two intervals of the same length, l. w_1 and w_2 such that the sum in w_1, denoted n_1 is larger than the sum in w_2, denoted n_2, plus two: $n_1 \geqslant n_2 + 2$. This means that there exists j_1 and j_2 such that $d_{j_1}^{n_1} \leqslant l$ and $d_{j_2}^{n_1} \geqslant l + 2$. □

Remark 2.10. Note that if for a sequence s, all the distances of order i differ by at most one, for all i, and if $\{d^i(s)\} = \{d^i(s')\}$ for all i, then s and s' and balanced by lemma 2.9. On the other hand, all balanced sequences are shift of each other (see Theorem 2.7) (It is interesting to see that in this case, all mirror transformations are also shifts). This is an answer to the question asked in Remark 2.3 in the case where the distance sequences are all formed by two consecutive numbers.

Lemma 2.11. *A sequence $s \in \mathcal{P}(T,n)$ is balanced if and only if $d^1(s)$ is balanced.*

Proof. The "only if" assertion: we consider the sequence $d^1(s)$. Let $w_1 = \{i, \cdots, i+k\}$ and $w_2 = \{j, \cdots, j+k\}$ be two windows of length k. Then the sums over w_1 of the sequence $d^1(s)$ is of the form $d^1_i(s) + \cdots + d^1_{i+k}(s) = d^{k-1}_i(s)$ and over w_2 is $d^1_j(s) + \cdots + d^1_{j+k}(s) = d^{k-1}_j(s)$. By lemma 2.8, then these two quantities differ by at most one.
The "if" assertion" $d^1(s)$ is balanced means that there exists θ such that for all j, $d^1_j(s) = \lfloor (j+1)\frac{T}{n} + \theta \rfloor - \lfloor j\frac{T}{n} + \theta \rfloor$. Therefore, $d^i_j = \lfloor (j+i)\frac{T}{n} + \theta \rfloor - \lfloor j\frac{T}{n} + \theta \rfloor$. This means that

$$\frac{iT}{n} - 1 < d^i_j < \frac{iT}{n} + 1.$$

Therefore, for any order i, the distances of order i differ by at most one. By applying Lemma 2.9, s is balanced. $\square$

Remark 2.12. Note however, that the distances of order higher than one in a balanced sequence may not be balanced. For example, if $d^1(s) = (2,3,3,2,3,3,3)$ (which is balanced), then $d^2(s) = (5,6,5,5,6,6,5)$ which is not balanced because it contains two consecutive 5 (which sum up to 10) and also two consecutive 6 (which sum up to 12).

3 Regularity and Schur Convexity

If $x \in \mathbb{R}^n$, then we denote by $(x_{[1]}, x_{[2]}, \cdots, x_{[n]})$ the components of x arranged in decreasing order. For $x, y \in \mathbb{R}^n$, we say that x (strictly) majorizes y denoted $x \succeq y$ ($x \succ y$) if for all $1 \leqslant m \leqslant n-1$, then

$$\sum_{i=1}^{m} x_{[i]} \geqslant \sum_{i=1}^{m} y_{[i]} \quad \text{and} \quad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

and with strict inequality for at least one m.
A function $f : \mathbb{R}^n \to \mathbb{R}$ is (strictly) *Schur convex* if $x \succeq y \Rightarrow f(x) \geqslant f(y)$ ($x \succ y \Rightarrow f(x) > f(y)$). For more on the theory of majorization, see [15].

3.1 Regularity

Definition 3.1. *If $s, s' \in \mathcal{P}(T,n)$, then s' is more regular than s (denoted $s' \trianglelefteq s$) if for all order i, $d^i(s') \preceq d^i(s)$.*

Lemma 3.2. *The balanced sequences are the minimal sequences in $\mathcal{P}(T,n)$ for the regularity order. If s is balanced, then for all $s' \in \mathcal{P}(T,n), s \trianglelefteq s'$.*

Proof. This result is a consequence of Lemma 2.9. Indeed, if n divides iT, then for a balanced sequence σ, $d^i_j(\sigma) = \frac{iT}{n}$, which is minimal. If n does not divide iT, then for the balanced sequence, $d^i_j(\sigma) \in \{\lfloor \frac{iT}{n} \rfloor, \lceil \frac{iT}{n} \rceil\}$ and is also minimal. $\square$

Balanced sequences in $\mathcal{P}(T,n)$ may also be called the "most regular" sequences in $\mathcal{P}(T,n)$ in the following.

Definition 3.3 (regular-preserving). *A function $f : \mathbb{N}^T \to \mathbb{R}$ is regular-preserving (ri) if $x \trianglerighteq y \Rightarrow f(x) \geqslant f(y)$.*

Lemma 3.4. *A function $f : \mathbb{N}^T \to \mathbb{R}$ which depends only on the distance sequences $d^1, \cdots, d^n$ is regular-preserving if and only if it can be written under the form:*

$$f(s) = F(H_1(d^1(s)), H_2(d^2(s)), \cdots, H_n(d^n(s))),$$

where $F : \mathbb{R}^n \to \mathbb{R}$ is increasing in all coordinates and $H_i : \mathbb{N}^n \to \mathbb{R}, i = 1 \cdots, n$ are Schur convex.

Proof. The only if assertion is a direct consequence of Lemma 3.2. As for the if assertion, then f is necessarily Schur convex in all d^i. Now, the combination of all these arguments for all i shows that f is of the form given in the lemma. $\square$

Note that if f is ri, then on the set $\mathcal{P}(T,n)$, it reaches its minimal value for the balanced sequences.

Remark 3.5. (Regularity without the distance sequences)
The definition of regularity, as well as regular preserving functions, can be done in a similar fashion by using the partial sum sequences of the original sequence s instead of the partial sums of its order one distance sequence.
All the results given in this paper are still true in this new framework.
The passage from one framework to the other is simply done through Equations (1) and (2.4).

3.2 Relations with Multimodularity

Let $e_i \in \mathbb{N}^T$ denote the vector with all components equal to 0 except the *i-th* component equal to 1. Define $b_i = e_{i-1} - e_i$ for $1 \leqslant i < T$, $b_T = e_{T-1} - e_0$ and $\mathcal{F} = \{b_1, b_2, \cdots, b_T\}$.
Since we deal with sequences which sums are fixed to n, we use an adapted version of the definition of multimodularity.

Definition 3.6. *A function f on $\mathcal{P}(T,n)$ is multimodular with respect to $\mathcal{F}$ if for all $s \in P(T,n)$, $v, w \in \mathcal{F}$, $v \neq w$,*

$$f(s+v) + f(s+w) \geqslant f(s) + f(s+v+w), \tag{5}$$

whenever $s+w, s+v, s+v+w \in \mathbb{N}^T$.

This is not the classical definition given by Hajek in [14], which considers arbitrary sequences in $\mathbb{N}^T$. However, both definitions are closely related. One can show that the projection of a multimodular function (in the sense of Hajek) on $\mathcal{P}(T,n)$ is multimodular according to Definition 3.6. (see [3] as well as [7] for a deeper insight on the definition given here.)

Theorem 3.7. *Let $f(s)$ be a function that only depends on $d^i(s)$, which is also denoted by $H_i(d^i(s))$. If $H_i(d^i(s))$ can be put under the from $\sum_{j=1}^n g(d^i_j(s))$, where g is a convex function, then it is is multimodular on $\mathcal{P}(T,n)$.*

Proof. According to the definition of multimodularity, we need to check the inequality (5) for all $s \in P(T,n)$.
First note that for all $1 \leqslant k \leqslant T$, there exists j_k such that:

$$d^i_j(s+b_k) = \begin{cases} d^i_j(s) - 1 & \text{if } j = j_k, \\ d^i_j(s) + 1 & \text{if } j = j_k + i, \\ d^i_j(s) & \text{otherwise.} \end{cases} \tag{6}$$

Now, for any $1 \leqslant k, l \leqslant p_{n+i}$ such that $j_k \neq j_l \pm i$, then b_l and b_k modify different distances. Therefore,

$$f(s+b_k) + f(s+b_l) - f(s) - f(s+b_l+b_k) = 0.$$

For all $1 \leqslant k, l \leqslant T$ such that $j_k = j_l - i$, then,

$$\begin{aligned}
&f(s+b_k) + f(s+b_l) - f(s) - f(s+b_l+b_k) = \\
&\big(g(d^i_{j_k+2i}(s)) + g(d^i_{j_k}(s)) + g(d^i_{j_k}(s)-1) + g(d^i_{j_k+i}(s)+1) + g(d^i_{j_k+i}(s)-1) + g(d^i_{j_k+2i}(s)+1)\big) \\
&-\big(g(d^i_{j_k}(s)) + g(d^i_{j_k+2i}(s)) + g(d^i_{j_k+i}(s)) + g(d^i_{j_k}(s)-1) + g(d^i_{j_k+2i}(s)+1) + g(d^i_{j_k+i}(s))\big) = \\
&-2g(d^i_{j_k+i}(s)) + g(d^i_{j_k+i}(s)+1) + g(d^i_{j_k+i}(s)-1)
\end{aligned}$$

This is positive since g is convex.

$\square$

The following corollary covers the case when several distance orders are used.

Corollary 3.8. *If for all i, $H_i(d^i(s)) = \sum_{j=1}^n g_i(d^i_j(s))$, where g_i are convex functions, then define the function*

$$f(s) \stackrel{\triangle}{=} \sum_{i=1}^{n} c_i H_i(d^i(s)),$$

where c_i are non negative constants. Then f is multimodular.

Proof. For all i, the function $f_i(s) \stackrel{\triangle}{=} H_i(d^i(s))$ is multimodular. A positive linear combination of multimodular functions is multimodular. □

Remark 3.9. Note that the functions $f_i(s)$ also form a "characteristic set" of regular preserving in the same way sums of convex functions play an important role among Schur convex functions ([15].

Remark 3.10. Functions of this form have been used in [10] as a criterion to test regularity of a sequence. In that paper, the minimization problem of functions of this form is written as under quadratic assignment formulation. This makes computations possible for small n and T and can be used for optimization purposes.

In order to prove that a balanced sequence minimizes a function f of the previous form (that is, such that $f(s)$ is regular preserving and multimodular), one can choose two different methods. First, one can try to use the fact that $f(s)$ is multimodular in $\mathbb{N}^T$. By referring to a general (complicated) theorem proved in [14, 3], $f(s)$ will be minimized by a balanced sequence and then use the fact that if s is balanced, then s is a balanced sequence in $\mathcal{P}(T,n)$.
Another way to prove the same result, it by using the fact that $f(s)$ is regular preserving directly. This second proof is easier and has a clear advantage on the previous approach since it provides a more precise result. It says that if $s \trianglerighteq s'$, then $f(s) \geqslant f(s')$.
This fact will be illustrated in the following section, where we show that a given cost function is ri. The minimization will then derive directly from Lemma 3.2 as well as the comparison between several policies.

Remark 3.11. Finally, one may remark that regular preserving functions constitute an essential class among multimodular functions. Indeed, as shown in Section 4.2, if a given sequence in $P(T,n)$ minimizes all possible regular preserving functions, then this sequence in balanced.

4 Application 1: Maximal Waiting Times in Networks

In this part, we show that the maximum waiting time in queuing system with general stationary inter-arrival and service times is a regular preserving function.

4.1 The D/D/1 Model

First, we consider a slotted D/D/1 queue where the arrival sequence is such that n customers arrive within T time slots (each of unit length) according to a sequence in $\mathcal{P}(T,n)$ and is periodic with period T. The i-th customer arrives at time $p_i(s)$ and brings a workload of σ in the system. That is, a customer requires σ time slots for its service. Note that by definition, the total number of customers that arrive at time i s_i. Note that the arrival sequence s is periodic with a period $(s_0, \cdots, s_{T-1}) \in \mathcal{P}(T,n)$. This is denoted as $s \in \mathcal{P}(T,n)^\omega$.
We will also denote by $C \stackrel{\triangle}{=} n\sigma$, the total load brought by n customers.
We finally require that the queue is initially empty and that the customers are served in a FIFO order (this last assumption can be relaxed to a non-idling server assumption in some cases).

Now, let us consider the waiting time W_i of the ith customer that enters the system. We have

$$W_i(s,\sigma) = \max_{j=1}^{i}\{\sum_{k=j}^{i-1}(\sigma - d_k^1(s)) + \sigma\}.$$

We define $M(s,\sigma) \triangleq \max_i W_i(s,\sigma)$.

Lemma 4.1. *The function $M(s)$ is regular preserving in s.*

Proof. First, let us remark that if $C > T$, then $W_i \to \infty$ when $i \to \infty$. Therefore $M(s,\sigma) = \infty$. In the rest of the proof, we will only consider the case where $C \leqslant T$. In that case, note that after a transient period of length n, the waiting times are periodic: $W_{i+n} = W_n$, if $i \geqslant n$. Also note that since the system is initially empty, the maximum waiting time is not reached during the initial transient period. So we have

$$\begin{aligned}
M(s,\sigma) &= \max_{i=n+1}^{2n}\max_{j=i-n}^{i}\{\sum_{k=j}^{i-1}(\sigma - d_k^1(s)) + \sigma)\}, && (7)\\
&= \max_{i=n+1}^{2n}\max_{j=i-n}^{i}\{(\sum_{k=j}^{i}\sigma) - d_j^{i-j}(s)\},\\
&= \max_{i=n+1}^{2n}\max_{j=i-n}^{i}\{(\sum_{k=j}^{i}\sigma) + d_i^{n-i+j}(s) - T\}, && (8)\\
&= \max_{i=1}^{n}\max_{j=1}^{n}\{(\sum_{k=j+i}^{n+i}\sigma) + d_i^j(s) - T\},\\
&= \max_{j=1}^{n}\max_{i=1}^{n}\{(n-j+1)\sigma + d_i^j(s) - T\}. && (9)
\end{aligned}$$

Equation (7) follows from the fact that $C \leqslant T$. Equation (8) comes from Equation (4). Changing the order of the two max operators yields Equation (9).
Now, the function $d^j \to \max_{i=1}^n\{(n-j+1)\sigma + d_i^j - T\}$ is Schur convex for each j and the function M is increasing in d^j for all j, and thus satisfies the conditions of Lemma 3.4. The Lemma then implies that M is regular preserving. ☐

Theorem 4.2. *Among all integer periodic inter-arrival time sequences that contain a total number of customers equal to n, into the system in every period, the balanced sequence in $\mathcal{P}(T,n)$ minimizes the maximum waiting time in the system. Moreover, for two sequences $s, s' \in \mathcal{P}(T,n)$, $s \trianglerighteq s' \Rightarrow M(s,\sigma) \geqslant M(s',\sigma)$.*

Remark 4.3. The function $M(s)$ is multimodular. This is a direct consequence of Corollary 3.8.

4.2 Characterization using regular preserving functions

The function M when $C = T$ plays an important role. Indeed, we have the following result giving yet another characterization of balanced sequences.

Theorem 4.4. *If $C = T$, and given $s \in \mathcal{P}(T,n)$, if $M(s) \leqslant M(s')$ for all $s' \in \mathcal{P}(T,n)$, then s is balanced.*

Proof. When $C = T$, then $\sigma = T/n$. Equation 9 becomes

$$M(s) = \max_{i=1}^{n}\max_{j=1}^{n}\{-jT/n + d_i^j(s)\} + T/n.$$

Note that jT/n is the average value of $d_i^j(s)$. Using Lemmas 2.8 and 2.9, s is balanced is equivalent to the fact that $M(s) < 1 + T/n$. ☐

4.3 The average waiting time is not regular preserving

In this section, we will show that unlike for the maximal waiting time, the average waiting time in a stable D/D/1 queue is not regular preserving through an example.
Consider in $\mathcal{P}(16,5)$ the sequences s and s' with the respective one-order distances,

$$\begin{aligned} d^1(s) &= (4,4,3,1,4), \\ d^1(s') &= (4,4,2,2,4). \end{aligned}$$

It is immediate to check that $s \trianglerighteq s'$. However, if one computes the average waiting time in both cases, when the load brought by each customer is equal to 3, one gets:

$$\overline{W}(s) = 18 < 19 = \overline{W}(s').$$

Remark 4.5. The function $\overline{W}$ is multimodular [1], which means it is minimized by a balanced sequence in $\mathcal{P}(16,5)$ ($\overline{W}(u) = 15$ if $d^1(u) = (3,3,3,3,4)$.

4.4 The G/G/1 Model

Here, we generalize the previous model. The slotted interarrival opportunities process is replaced by a stochastic process with stationary interarrival times: $\{\tau_i\}_{i\in\mathbb{N}}$. The load carried by the customers forms a stationary sequence: $\{\sigma_k\}_{k\in\mathbb{N}}$, independent of the arrival times. We denote by $\delta_i(s)$ the time elapsed between the arrival of customer i and customer $i+1$ for sequence s,

$$\delta_i(s) = \sum_{j=p_s(i)+1}^{p_s(i+1)} \tau_j, foralli = 0, \cdots, m-1$$

Using Lindley's equation, the k-th waiting time satisfies

$$W_i(s) = \max_{j=0}^{i} \{\sum_{k=j}^{i-1} (\sigma_k - \delta_k(s)) + \sigma_i\}.$$

If we consider the maximal waiting time within on period, starting with customer $m+1$, we define $\Phi^n_m(s) \triangleq \max_{i=m+1}^{n+m} W_i(s)$.
In a similar fashion as with the D/D/1 case, we define

$$\Phi^n_{r,m}(s) \triangleq \max_{i=m+1}^{n+m} \max_{j=i-r+1}^{i} \{\sum_{k=j}^{i-1} (\sigma_k - \delta_k(s)) + \sigma_i\}.$$

Intuitively, $\Phi^n_{r,m}$ is constructed by going back r customers in time, while Φ^n_m goes back to time 0. Note that if the queue has emptied during the last r arrivals, then both quantities coincide.
We assume that the queuing system is stable, that is $TE(\tau_1) > nE(\sigma_1)$ (this generalizes the previous assumption, $C \leqslant T$ to the stochastic case). Under this stability condition, the queue empties almost surely in finite time. Therefore, taking r large enough will ensure that $\Phi^n_{r,m}$ and Φ^n_m will coincide. More precisely,

$$\left|\Phi^n_{an,an}(s) - \Phi^n_{an}(s)\right| \longrightarrow_{a\to\infty} 0, \quad a.s. \tag{10}$$

Now, we have,

$$\Phi^n_{an,an}(s) = \max_{i=an+1}^{n+an} \max_{j=i-an+1}^{i} \left\{ \sum_{k=j}^{i-1} (\sigma_k - \delta_k(s)) + \sigma_i \right\}, \tag{11}$$

$$= \max_{i=an+1}^{n+an} \max_{j=i-an+1}^{i} \left\{ \sum_{k=j}^{i} \sigma_k + \sum_{k=i-an}^{j-1} \delta_k(s) - \sum_{k=i-an}^{i-1} \delta_k(s) \right\}, \tag{12}$$

$$= \max_{i=1}^{n} \max_{j=1}^{an} \left\{ \sum_{k=j+i}^{i+an} \sigma_k + \sum_{k=i}^{j+i-1} \delta_k(s) - \sum_{k=i}^{i+an-1} \delta_k(s) \right\}. \tag{13}$$

Lemma 4.6. *Let us assume that s and s' are two sequences in $\mathcal{P}(T,n)$ such that $s \trianglelefteq s'$. Then, $\Phi^n_{an,an}(s) \leqslant_{st} \Phi^n_{an,an}(s')$.*

Proof. For a given pair (i,j) in $\{1,\cdots,n\} \times \{1,\cdots an\}$, we know by definition of the regular ordering that there exits $i' \in \{1,\cdots,n\}$ such that $d^j_i(s) \leqslant d^j_{i'}(s')$. Indeed, $d^j_i(s) = bT + d^r_i(s)$ where $bn + r = j, \quad r < n$.
Now, we define a coupling of the service sequence as well as of the arrival sequence such that $\sigma'_{k+i'} = \sigma_{k+i}$ and $\tau'_{k+p_{s'}(i')} = \tau_{k+p_s(i)}$ for all $k \in \mathbb{N}$. Note that the coupled sequences have the same probability distributions as the original ones because of stationarity and independence between them and because the sequences σ and τ are independent of s and s'.
Using this coupling, and considering equation 13, we have directly

$$\sum_{p=j+i'}^{i'+an} \sigma'_p = \sum_{p=j+i}^{i+an} \sigma_p.$$

Since during a periods, the total number of arrivals with s and s' is the same and using the coupling, we have

$$\begin{aligned}
\sum_{k=i}^{i+an-1} \delta_k(s) &= \sum_{p=p_s(i)+1}^{p_s(i+an)} \tau_p \\
&= \sum_{p=p_s(i)+1}^{p_s(i)+Ta} \tau_p \\
&= \sum_{p=p_{s'}(i')+1}^{p_{s'}(i')+Ta} \tau'_p \\
&= \sum_{k=i'}^{i'+an-1} \delta'_k(s).
\end{aligned}$$

Finally, using the fact that $d^j_i(s) \leqslant d^j_{i'}(s')$, we also get

$$
\begin{aligned}
\sum_{k=i}^{j+i-1} \delta_k(s) &= \sum_{p=p_s(i)+1}^{p_s(i+j)} \tau_p \\
&= \sum_{p=p_s(i)+1}^{p_s(i)+d_i^j(s)} \tau_p \\
&\leqslant \sum_{p=p_{s'}(i')+1}^{p_{s'}(i')+d_{i'}^j(s')} \tau'_k \\
&= \sum_{k=i'}^{j+i'-1} \delta'_k(s').
\end{aligned}
$$

Using these three relations yields

$$
\max_{j=1}^{an} \left\{ \sum_{k=j+i}^{i+an} \sigma_k + \sum_{k=i}^{j+i-1} \delta_k(s) - \sum_{k=i}^{i+an-1} \delta_k(s) \right\} \leqslant \max_{j=1}^{an} \left\{ \sum_{k=j+i'}^{i'+an} \sigma'_k + \sum_{k=i'}^{j+i'-1} \delta'_k(s') - \sum_{k=i'}^{i'+an-1} \delta'_k(s') \right\}. \tag{14}
$$

This is true for all i, and therefore also for the maximum over all possible i from 1 to an. Using the fact that under the coupling, the sequences σ' and τ' are in a "typical" situation, we get $\Phi^n_{an,an}(s) \leqslant_{st} \Phi^n_{an,an}(s')$. □

Now, we can look at the stationary version of the queue. Using a backward coupling argument and Lemma 4.6, we obtain the following theorem.

Theorem 4.7. *Let $\mathbf{W}_i$ be the stationary waiting time in a stable G/G/1 queue. Then for two sequences $s, s' \in \mathcal{P}(T,n)^\omega$ with $s \trianglelefteq s'$, then $\max_{i=1}^n \mathbf{W}_i(s) \leqslant_{ST} \max_{i=1}^n \mathbf{W}_i(s')$.*

4.5 The Event Graph Model

We further generalize the previous (G/G/1) model in several ways.

- *Networks assumptions.* We replace the single queue by a network of queues forming a FIFO stochastic event graph with a single entry made of Q servers (transitions) and P buffers (places). (see [6] for more details on this class of networks. In particular, it includes queues in tandem with general blocking).

- *Stochastic assumptions.* The loads carried by the customers form a stationary vector sequence: $\{\sigma_k^j\}_{k\in\mathbb{N}, j\in Q}$. We assume this is independent of the arrival sequence. However no independence assumption among the vector sequence is required.

- *Initial state assumptions.* We assume that initially, the event graph is empty, in the sense that there is not any customer in any buffer. (this is a generalization of the empty queue assumption). Without loss of generality, we can further assume that initially, each buffer contains at most one customer.

To any stochastic event graph, we can associate two matrices, $A(k,\ell)$, $k \in \{0,1\}$, of size $Q \times Q$, where the entry (i,j) in matrix $A(k,\ell)$ is $\sigma^j_{\ell-k}$ if there exists a place between transitions q_j and q_i with k initial tokens, and $-\infty$ otherwise.
In the following, we will be using the $(max,+)$ notation, where $\oplus$ stands for max and $\otimes$ for $+$. Matrix operations are defined accordingly. For more details on (max,+) notations and its use in event graphs, see for example [6].

We define $A(0,\ell)^* \stackrel{\triangle}{=} \bigoplus_{i=0}^{Q} A(0,\ell)^i$. The matrix exists whenever the network cannot reach a deadlock state (see [6]).
In that case, we can define $A(\ell) = A(0,\ell)^* \otimes A(1,\ell)$. Let $X_q(n)$ be the epoch when the n-th customer starts service in server q. Then the vector $X(n) = (X_1(n), \cdots, X_Q(n))$ satisfies the following recurrence equation:

$$X(n) = A(n) \otimes X(n-1).$$

Let W_n be a vector , with its q-th component equal to: $(W_n)_q \stackrel{\triangle}{=} X_q(n) - X_0(n)$. $(W_n)_q$ can be seen as the *traveling time* for customer n between its entrance in the system and its passage in server q. In the example of queues in tandem, let q be the last queue. Then $(W_n)_q$ is the total sojourn time of the n-th customer in the system.

Lemma 4.8 ([1]). *W_n satisfies the following equation:*

$$W_{n+1} = A(n) \otimes D(-\delta_n) \otimes W_n \oplus B(n+1), \tag{15}$$

where $D(h)$ is the diagonal matrix with h on the diagonal and $-\infty$ everywhere else, and $B(n)$ is a vector which describes the input connection.

Equation 15 can be seen as a vectorial form of the Lindley's equation. Its solution is similar to the solution of the scalar case.

$$W_n(s) = B(n) \oplus \bigoplus_{j=0}^{n-1} C_j(s), \quad \text{with} \quad C_j(s) = \bigotimes_{k=j}^{n-1} \big(A(k) \otimes D(-\delta_k(s))\big) \otimes B(n-j-1).$$

If we define as previously, $\Phi_m^n = \bigoplus_{i=m+1}^{m+n} \bigoplus_{j=0}^{i} C_j(s)$ and

$$\Phi_{r,m}^n(s) = \bigoplus_{i=m+1}^{m+n} \bigoplus_{j=i-r+1}^{i} \left(\bigotimes_{k=j}^{i-1} \big(A(k) \otimes D(-\delta_k(s))\big) \otimes B(i-j-1) \right).,$$

we get by a similar technique as in the G/G/1 case (with a vectorial notation), and provided the event graph is stable (see [5]),

$$\left|\Phi_{an,an}^n(s) - \Phi_{an}^n(s)\right| \longrightarrow_{a\to\infty} 0, \quad a.s. \tag{16}$$

The quantity $\Phi_{an,an}^n(s)$ can be transformed as in the scalar case using the fact that diagonal matrices commute with everything.

$$\begin{aligned}
\Phi_{an,an}^n(s) &= \bigoplus_{i=an+1}^{an+n} \bigoplus_{j=i-an+1}^{i} \left(\bigotimes_{k=j}^{i-1} \big(A(k) \otimes B(i-j-1) \otimes D(-\delta_k(s))\big) \right) \\
&= \bigoplus_{i=an+1}^{an+n} \bigoplus_{j=i-an+1}^{i} \left(\bigotimes_{k=j}^{i-1} \big(A(k) \otimes B(i-j-1)\big) \otimes \bigotimes_{k=j}^{i-1} D(-\delta_k(s)) \right) \\
&= \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{an} \left(\bigotimes_{k=j+i}^{i+an-1} \big(A(k) \otimes B(an-j-1)\big) \otimes \bigotimes_{k=j+i}^{i+an-1} D(-\delta_k(s)) \right) \\
&= \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{an} \left(\bigotimes_{k=j+i}^{i+an-1} \big(A(k) \otimes B(an-j-1)\big) \otimes \bigotimes_{k=i}^{i+an-1} D(-\delta_k(s)) \otimes \bigotimes_{k=i}^{i+j-1} D(\delta_k(s)) \right).
\end{aligned}$$

Lemma 4.9. *Let us assume that s and s' are two sequences in $\mathcal{P}(T,n)^{\omega}$ such that $s \trianglelefteq s'$. Then, $\Phi^n_{an,an}(s) \leqslant_{st} \Phi^n_{an,an}(s')$.*

Proof. The proof is established in a similar fashion as with the single queue case, using a coupling argument and the definition of the regular ordering.
For a given pair (i,j) in $\{1,\cdots,n\} \times \{1,\cdots an\}$, we know by definition of the regular ordering that there exits $i' \in \{1,\cdots,n\}$ such that $d^j_i(s) \leqslant d^j_{i'}(s')$.
Now, we define a coupling of the service sequence such that for each queue q, $\sigma'^q_{k+i'} = \sigma^q_{k+i}$. As for the interarrival sequence, we set $\tau'_{k+p_{s'}(i')} = \tau_{k+p_s(i)}$ for all $k \in \mathbb{N}$.
Note that the coupled sequences have the same probability distributions as the original ones because of stationarity and independence between them and because the sequences σ and τ are independent of s and s'.
Using this coupling, and considering equation 17, we have directly be definition of $A(k)$,

$$\bigotimes_{k=j+i}^{i+an-1} \left(A(k) \otimes B(an-j-1)\right) = \bigotimes_{k=j+i'}^{i'+an-1} \left(A(k) \otimes B(an-j-1)\right).$$

During a consecutive periods, the total number of arrivals with s and s' is the same. Therefore, we have using the fact that $D(-\delta_k(s))$ are diagonal matrices,

$$\begin{aligned}
\bigotimes_{k=i}^{i+an-1} D(-\delta_k(s)) &= \bigotimes_{p=p_s(i)+1}^{p_s(i+an)} D(-\tau_p) \\
&= \bigotimes_{p=p_s(i)+1}^{p_s(i)+Ta} D(-\tau_p) \\
&= \bigotimes_{p=p_{s'}(i')+1}^{p_{s'}(i')+Ta} D(-\tau'_p) \\
&= \bigotimes_{k=i'}^{i'+an-1} D(-\delta'_k(s)).
\end{aligned}$$

As for the last term,

$$\begin{aligned}
\bigotimes_{k=i}^{j+i-1} D(\delta_k(s)) &= \bigotimes_{p=p_s(i)+1}^{p_s(i+j)} D(\tau_p) \\
&= \bigotimes_{p=p_s(i)+1}^{p_s(i)+d^j_i(s)} D(\tau_p) \\
&\leqslant \bigotimes_{p=p_{s'}(i')+1}^{p_{s'}(i')+d^j_{i'}(s')} D(\tau'_k) \\
&= \bigotimes_{k=i'}^{j+i'-1} D(\delta'_k(s')).
\end{aligned}$$

These three relations are true for all i and j, therefore, $\Phi^n_{an,an}(s) \leqslant_{st} \Phi^n_{an,an}(s')$. □

Finally, we obtain the following extension of Theorem 4.7.

Theorem 4.10. *In a stochastic event graph, the maximum traveling time $M(s)$ over one period of the admission sequence is a regular preserving function (in the stochastic sense).*

4.6 Routing Problem

Now, we consider the following network of queues. Customers enter a system composed by K networks. The routing of the customers to the different networks is controlled by a sequence of vectors $\{S_i\}$, with $S_i \in \{0,1\}^K$ and $1 \leqslant i \leqslant T$. If $S_i^j = 1$, then this means that the i-th customer is routed to queue j. Note that S is a feasible admission sequence as long as, for all i, $\sum_j S_i^j = 1$. Customers arrive in the system at each time unit and each customer brings a constant load of σ. Figure 2 displays the model we are considering.

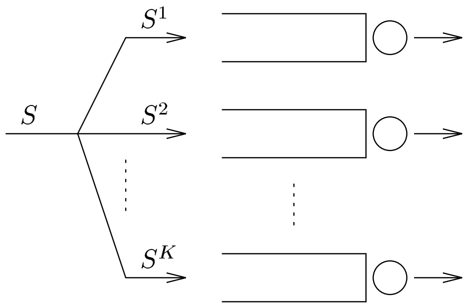

Figure 2: Routing of customers in a multiple queue system

4.6.1 The Constrained Case

We restrict the routing policy to eb periodic and to verify the following constraint.

$$\forall 1 \leqslant j \leqslant K, \quad \sum_i S_i^j = n_j, \tag{17}$$

where n_j are fixed and verify $\sum_j n_j = T$. In other words, if we focus on the admissions in queue j, the constraints say that $S^j \in \mathcal{P}(T, n_j)^\omega$. We say that $S \in \mathcal{A}(n_1, \cdots, n_K)$.
Now, we define $I(S)$ to be the maximum waiting of any customer in the system, for $S \in \mathcal{A}(n_1, \cdots, n_K)$. $I(S) \stackrel{\triangle}{=} \max_{j=1}^K M_j(S^j)$,, where M_j is the maximum waiting time in queue j. We say that $(n_1, \cdots, n_K)$ is *balanceable* if there exists $S \in \mathcal{A}(n_1, \cdots, n_K)$ such that for all j, S^j is balanced. See [2] for a more detailed discussion on balanceable allocations. From Lemmas 3.2 and 4.1, we get,

Lemma 4.11. *If $(n_1, \cdots, n_K)$ is balanceable then $I(S)$ is minimized by a routing policy S such that each S^j is balanced.*

Proof. This result is a direct consequence of Lemmas 3.2 and 4.1. □

This result is similar to some extend to the results proved in [2] where multimodularity of the waiting times in each queue is used.
However, we have the following new result which follows directly from the previous lemma and which holds for all $(n_1, \cdots, n_K)$ (*i.e.* not necessarily balanceable).

Theorem 4.12. *If $S, U \in \mathcal{A}(n_1, \cdots, n_K)$, such that for all j, $S^j \trianglerighteq U^j$, then $I(S) \geqslant I(U)$.*

4.6.2 The Unconstrained Case

Now, we do not impose that the number of customers sent to each queue should be fixed.
We want to consider all admission sequences S in the set

$$\mathcal{A} = \bigcup_{\sum_j n_j = T} \mathcal{A}(n_1, \cdots, n_K).$$

Lemma 4.13. *The function $I(S)$ is minimized on $\mathcal{A}$ for some balanced $S \in \mathcal{A}(n_1, \cdots, n_K)$, with $n_j \in \{\lfloor T/K \rfloor, \lceil T/K \rceil\}$, for all j.*

Proof. First, note that if $n_j \in \{\lfloor T/K \rfloor, \lceil T/K \rceil\}$, for all j, then $(n_1, \cdots, n_K)$ is balanceable. (see [2] for a proof).We denote such a sequence b. Second, note that if not all $n_j \in \{\lfloor T/K \rfloor, \lceil T/K \rceil\}$, then at least one of then is larger than $\lceil T/K \rceil$. Finally, note that if s is a balanced sequence in $\mathcal{P}(T,n)^\omega$ and if $s' \in \mathcal{P}(T,n')^\omega$, with $n' \geqslant n$, then $M(s',\sigma) \geqslant M(s,\sigma)$.
Now, the proof goes like this. By Lemma 4.12 and I is minimized on $\mathcal{A}(b)$ by a balanced sequence $S(b)$. Moreover, there is some j such that $I(S(b)) = M_j(S^j(b))$, with $n_j = \lceil T/K \rceil$. For any U which is not in $\mathcal{A}(b)$, then there exists k such that $n_k \geqslant \lceil T/K \rceil$.
Now, $I(U) \geqslant M_k(U^k) \geqslant M_j(S^j(b)) = I(S(b))$. This finishes the proof □

4.7 Computational Problems

For the constrained problem, we can compute the best sequence using a procedure similar to the one used in [10] with quadratic programming techniques.
As for the unconstrained case, from Lemma 4.13, it is easy to see that when $T/K \in \mathbb{N}$, then the best routing policy (which minimizes the function I) is the round robin policy. Similar results can be found in [2].
However, when $T/K \notin \mathbb{N}$, then little was know before. Here, we can come up with the optimal routing sequence using the following procedure.
First, we compute the optimal sequence $b = (n_1, \cdots, n_k)$ using Lemma 4.13. This sequence is unique up to a permutation. Note that by symmetry of the cost function, all those permutation will perform the same. Moreover, since all balanced sequences with the same number of appearance in the routing sequence are shifts of each other, then for all j such that $n_j = \lceil T/K \rceil$, all the waiting times $M_j(S^j(b))$ are equal.
The construction of $S^j(b)$ for any j can be done by any known method to construct balanced sequence, as for example, by using the Beaty construction (see [8]) or the iterative formula given in Theorem 2.7.

5 Application 2: Assignment to one-buffer queues with redundancy

In this section, we present an example of optimal control of admission into a single buffer queue with redundancy.
– Transmission opportunities occur at time $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots..$ At each transmission opportunity, a controller can decide to actually transmit a packet, or not. If transmission occurs, we assume that it is instantaneous. Define $\zeta_n = \mathcal{T}_n - \mathcal{T}_{n-1}$.
– At a down link node in the network a packet that has been transmitted is buffered in a single buffer till its service there is completed. The duration of the service is exponentially distributed and it has expectation μ^{-1}. We assume that the delay between the transmitter and the down link node is zero (the results of this paper do not change if we take this time to be any other constant period).
– If a packet is transmitted before the previous packet has completed its service time then it is lost.
A model that can be used to solve this problem was introduced and studied in [4]. Here, we further focus on the following problem arising from higher layer network considerations. We consider the problem of redundancy of packet transmission: each packet is retransmitted at K consecutive transmission opportunities, which we call a frame, so as to decrease the probability of losses. Thus, only if all the K packets in a frame are lost then the frame, or equivalently the original information packet, is considered to be lost.
Our goal, roughly speaking, is to obtain a control with two objectives: on one hand it maximizes the average throughput, and on the other it minimizes the loss probabilities of frames. Since the

throughput is the acceptance rate minus the losses, the problem can be formulated as maximization of the average number of accepted packets, on one hand, and minimizing the loss probabilities of frames, on the other.

We now formulate more precisely our problem. We define a control policy as a sequence $\{s_i\}$, $i \in \mathbb{N}$, where $s_i = 1$ if a transmission is scheduled at the ith transmission opportunity, and it is 0 otherwise. We consider here the case where the controller has no information on the state of the buffer (nor on which packet is the beginning of a frame).

Define $D(k) = \sum_{i=0}^{k} s_i$. The actual kth transmission occurs at times $p_s(k)$ (see Definition 1). We define the process X_k to be the number of packets in the buffer just prior to time $p_s(k)$. Define $G(s)$ as

$$G(s) = \limsup_{N\to\infty} \frac{1}{N}\left(\lambda D(N) + \sum_{j=1}^{n} E^a \min_{k=0,\ldots,K-1} X_{N-k}\right)$$

where λ is a negative constant. We consider the problem of minimizing G over all sequences s.

Note that $D(N)/N$ is the acceptance rate, whereas the second term is the average number of frames lost among the first N ones. The cost thus contains a term responsible for maximizing the throughput and a term for minimizing the loss rates. The parameter λ can be viewed as a Lagrange multiplier related to a constrained problem; indeed, in [4] a similar problem (for the case $K = 1$) is studied where the loss rate is to be minimized subject to a constraint on the rate of acceptance of packets.

Next, we focus on the cost $G(s)$. We shall restrict to policies s that are periodic. (Recent techniques in [9] can be used to show that restricting δ to periodic policies is without loss of optimality.) Our main result, stated below, is that $G(s)$ is regular-preserving. (This allows us to conclude, in particular, that there exists an optimal sequence which is balanced.)

Theorem 5.1. *Assume that ζ_k are stationary and that the service times are i.i.d. exponentially distributed, and independent of the sequence ζ_k. Let $s \in \mathcal{P}(T,n)^\omega$. Then*
(i) G can be rewritten as

$$G(s) = \frac{\lambda n}{T} + \frac{1}{n}\sum_{i=1}^{n} E\exp(-\mu d_i^k).$$

where d_i^k is defined in (1).
(ii) $G(s)$ is multimodular for $n \geqslant K$ and is regular-preserving.

Proof. (i) Let Σ_n denote the length of the time interval that started when the $(n-K+1)$th packet was transmitted, and ended when the nth packet is transmitted. $f_n(s) := \min_{k=0,\ldots,K-1} X_{n-k}$ equals 0 if and only if there has been no service completion during Σ_n, and is otherwise 1. Thus, the expectation of $f_n(s)$ equals the probability that an exponential random variable with parameter μ is greater than or equal to Σ_n. Using the periodicity of s, this yields (i).

(ii) This follows directly from Theorem 3.7. Note that since $g'(x) = \exp(-\mu x)$ is convex in x, $g(x) = E\exp(-\mu x)$ is convex too. □

References

[1] E. Altman, B. Gaujal, and A. Hordijk. Admission control in stochastic event graphs. Technical Report 3179, INRIA, 1997.

[2] E. Altman, B. Gaujal, and A. Hordijk. Balanced sequences and optimal routing. Technical Report 3180, INRIA, 1997.

[3] E. Altman, B. Gaujal, and A. Hordijk. Multimodularity, convexity and optimization properties. Technical Report 3181, INRIA, 1997.

[4] E. Altman, B. Gaujal, A. Hordijk, and G. Koole. Optimal admission, routing and service assignment control: the case of single buffer queues. In *CDC*, Tampa Bay, Fl., dec 1998. IEEE.

[5] F. Baccelli. Ergodic theory of stochastic Petri networks. *Annals of Probability*, 20(1):375–396, 1992.

[6] F. Baccelli, G. Gohen, G.J. Olsder, and J.-P. Quadrat. *Synchronization and Linearity.* Wiley, 1992.

[7] Marcelo Bartroli and Shaler Stidham. Multimodular triangulations. Technical Report T.R. N. UNC/OR/TR-88/7, University of North Carolina at Chapell Hill, 1988.

[8] J. Berstel. *Mots*, chapter Tracé de droites, fractions continues et morphismes itérés. Hermes, Paris, 1990.

[9] S. Bhulai. Optimal routing problems and multimodularity. Master's thesis, Vrije Universiteit, Amsterdam, 1998.

[10] S. Borst and K.G. Kamakrishnan. Optimization of template-driven scheduling mechanisms. private communication, 1997.

[11] Sem Borst and K.G. Kamakrishnan. Optimization of template-driven scheduling mechanisms. *J. of Discrete Event Dynamics Systems*, 1998.

[12] M. B. Combé and O. Boxma. Optimzation of static traffic allocation policies. *Theoretical Computer Science*, 125:17–43, 1994.

[13] R. Graham. Covering the positive integers by disjoint sets of the form $\{[n\alpha+\beta] : n = 1, 2, \cdots\}$. *J. Combi. Theory*, 15:354–358, 1973.

[14] B. Hajek. Extremal splitting of point processes. *Mathematics of Operation Research*, 10:543–556, 1985.

[15] A.W. Marshall and I. Olkin. *Inequalities: Theory of Majorization and Its Applications*, volume 143 of *Mathematics in Science and Engineering.* Academic Press, 1979.

[16] M. Morse and G. A. Hedlund. Symbolic dynamics II- Sturmian trajectories. *American Journal Math.*, 62:287–306, 1940.