Routing into two parallel links: Game-Theoretic Distributed Algorithms

Eitan Altman[*], Tamer Başar[†]
Tania Jiménez[‡], and Nahum Shimkin[§]

Abstract

We study a class of noncooperative networks where N users have to send traffic to a destination node over two links with given capacities in such a way that a Nash equilibrium is achieved. Under a linear cost structure for the individual users, we obtain several dynamic policy adjustment schemes for the on-line computation of the Nash equilibrium, and study their global or local convergence properties. These policy adjustment schemes require minimum information on the part of each user regarding the cost/utility functions of the others.

Keywords. Networks; routing; nonzero-sum games; noncooperative equilibria; greedy algorithms

1 Introduction

We consider in this paper a routing problem in networks, in which two parallel links are shared between a number of players. In the context of telecommunication networks, the players could stand for users, who have to decide on what fraction of their traffic to send on each link of the network. In road traffic networks, we consider a player to be a transportation

[*]INRIA B.P. 93, 06902 Sophia Antipolis Cedex, France. Tel: (33) 4 92 38 77 86; Fax: (33) 4 92 38 77 65; Email: altman@sophia.inria.fr

[†]Coordinated Science Laboratory, University of Illinois, 1308 West Main Street, Urbana, IL 61801-2307, USA. Tel: (1) 217 333-3607; Fax: (1) 217 244-1653; Email: tbasar@decision.csl.uiuc.edu. Research supported in part by Grants NSF INT 98-04950, NSF ANI 98-13710, and MURI AF-DC-5-36128.

[‡]C.E.S.I.M.O., Universidad de los Andes, Mérida, Venezuela, and INRIA B.P. 93, 06902 Sophia Antipolis Cedex, France. Email: tjimenez@sophia.inria.fr Tel: (33) 4 92 38 79 74; Fax: (33) 4 92 38 77 65;

[§]Electrical Engineering Department, Technion, Haifa 32000, Israel. Tel: (+972) 4 829 4734; Fax: +972 4 906990; Email: shimkin@ee.technion.ac.il.

company that has to ship flow (using a large number of vehicles) [12] (as opposed to the case where a single driver is a player; see [12] for the relationship between the two cases). Each player minimizes his own cost which is assumed to be linear in the congestion at the links.

A natural framework to analyze this class of problems is that of noncooperative game theory, and an appropriate solution concept is that of Nash equilibrium [4]: a routing policy for the users is a Nash equilibrium if no user can gain by unilaterally deviating from his own policy.

There exists rich literature on the analysis of equilibria in networks, particularly for the case of the infinitesimal user; see, e.g., [9, 12]. More recently, the issue of *competitive routing* has been studied, where the network is shared by several users with each one having a non-negligible amount of flow. Our starting point here is the work reported in [12, 22, 19, 16, 17] on competitive routing, which has presented the basic optimality concept in routing within a noncooperative framework, and using the Nash equilibrium solution concept. In these cited papers, conditions have been obtained for the existence and uniqueness of a Nash equilibrium. This has enabled, in particular, the design of network management policies that induce efficient equilibria [19]. This framework has also been extended to the context of repeated games in [20] (in which cooperation can be induced by using policies that include threats of punishments for deviating users).

Nash equilibrium is operationally meaningful if all users employ from the beginning their equilibrium policies; in that case it is self-enforcing and is optimal, in the sense that no user has any incentive to deviate from it. The main difficulty with this notion of equilibrium from a practical point of view, however, is that in realistic situations there is no justification for the assumption that the system is initially in equilibrium, nor for the one that there is coordination among the users in reaching an equilibrium. Moreover, the users are generally unable to compute the equilibrium individually, since it is a function of the unknown utilities and other parameters private to each player. Instead, a natural assumption on the behavior of users is that they are likely to behave in a greedy way: each user would update occasionally its own decisions so as to optimize its individual performance, without coordination with other users. Accordingly, a central objective of this paper is to analyze different types of greedy behavior on the part of the users, and to relate these to the Nash equilibrium.

We conclude this introduction with some background on routing games in telecommunications. In recent years, we have been facing an accelerated rate of development of high speed telecommunication networks, and a huge growth in the amount of traffic, number of applications, and number of subscribers. New types of applications typically have different requirements on the quality of services that they expect the network to provide (in terms of delays, loss rates, jitter and throughputs). These require network resources that have to be shared between the applications. A central issue in designing telecommunication networks is thus to control the use of the network's resources (memory, bandwidth, etc). When the network is responsible for the control of the use of the resources, then this is typically done in a decentralized way: local decisions, based on local information are typically taken at different switches in the network [1]. A centralized control is impossible to implement due to the large dimensionality of the networks, the high transmission rates, and the large number of connections. On the other hand, applications which have the flexibility to adapt to the varying network conditions often use decentralized control protocols in order to optimize their own performances. This is, for example, the case when transmitting data over the Internet, where users themselves determine their transmission rates (typically using the TCP/IP protocol [15]).

The need for decentralized distributed individual controls in telecommunication networks has stimulated a substantial amount research using game theoretic methods, in both routing as well as flow control [2, 3, 5, 6, 7, 8, 10, 11, 14, 16, 17, 19, 18, 21, 22, 23, 25]. Within the design of flow control, decentralized and distributed greedy protocols have been analyzed. The study of decentralized greedy routing is, however, new, and heretofore only the special case of two links and two players has been studied [22].

In the next section we introduce the model adopted in the paper, and establish existence of a unique Nash equilibrium. In Section 3, we obtain an explicit expression for the equilibrium under a linear cost on rate of flow for M parallel links. In Section 4 we develop and study stability of various update algorithms restricting to two links. The concluding remarks of the last section end the paper.

2 The model

Consider M parallel links between two points — source and destination. There are N users who have to send traffic from the source to the destination. User i requires a total rate of transmission Λ^i, and the total traffic rate sent by all users is $\Lambda := \sum_{i=1}^{N} \Lambda^i$. Each user has to decide which part of his traffic to send on each given link. Introduce the quantities:
λ_m^i:= the rate of traffic that user i sends over link m,
λ_m:= the total rate of traffic sent over link m, i.e., $\sum_i \lambda_m^i$
$\lambda_m^{-i} := \lambda_m - \lambda_m^i$.

We formulate this problem as a noncooperative network, i.e., a network in which the rate at which traffic is sent is determined by selfish users, each having its own utility (or cost). As in [19, 22], we assume that the cost per unit of traffic of player i sent over link m is a function of λ_m, which we denote by $f_m(\lambda_m)$. This cost represents typically the expected delay, or a function of the expected delay over the link. Thus the overall cost for user i (per unit of time) of sending at a rate of λ^i_m over link m is $\lambda^i_m f_m(\lambda_m)$, and the total cost (over all links) for user i is given by:

$$J_i(\lambda) := \sum_m \lambda^i_m f_m(\lambda_m). \tag{1}$$

Then, a multi-policy $\lambda := \{\lambda^m_i\}_{i,m}$ is said to be feasible if it satisfies the following three conditions:

- (R1) Positivity: $\lambda^i_m \geq 0, \ \forall i, m$;
- (R2) Rate constraints: $\sum_{m=1}^M \lambda^i_m = \Lambda^i, \ \forall i$.

The optimality concept adopted is that of Nash equilibrium, i.e., we seek a feasible multi-policy $\lambda = \{\lambda^i_m\}_{m,i}$ such that no player (user) i can benefit by deviating unilaterally from his policy $\lambda^i = \{\lambda^i_m\}_i$ to another feasible multi-policy. In other words, for every player i,

$$\begin{aligned} J^i(\lambda) &= J^i(\lambda^1, \ldots, \lambda^{i-1}, \lambda^i, \lambda^{i+1}, \ldots, \lambda^I) \\ &= \min_{\tilde{\lambda}^i} J^i(\lambda^1, \ldots, \lambda^{i-1}, \tilde{\lambda}^i, \lambda^{i+1}, \ldots, \lambda^I) \end{aligned} \tag{2}$$

where the minimum is taken over all policies $\tilde{\lambda}^i$ which lead to a feasible multi-policy.

We recall the following result from [22] for the unconstrained case.

Lemma 1 *Assume that $f_m : \mathbb{R} \to [0, \infty]$, $m = 1, ..., M$, are continuous and convex increasing. Then the game has a unique Nash equilibrium.*

In [19, 22], a further special case was considered, where the cost functions have the structure:

$$f_m(\lambda_m) = \frac{1}{C_m - \lambda_m},$$

which corresponds to the expected delay of an M/M/1 queue with exponential service rate with parameter $1/C_m$, and a Poisson arrival process of packets with parameter λ_m. For this special structure, it is possible to compute explicitly the corresponding Nash equilibrium.

In this paper we specialize to the following particular (but different) cost function, which will allow us to compute and characterize the Nash equilibrium:

$$f_m(\lambda_m) = a_m \lambda_m .$$

In the context of telecommunication systems, this cost would represent the expected waiting time of a packet in a light traffic regime. Indeed, under a large variety of statistical conditions, the expected waiting time of a packet in link m is proportional to $\lambda_m/(C_m - \lambda_m)$. For example, in an M/G/1 queue, the expected value of the waiting time W_m is given by

$$EW_m = \frac{\beta_m^{(2)}}{1 - \lambda_m \beta_m} \lambda_m ,$$

where β_m and $\beta_m^{(2)}$ are, respectively, the first and second moments of the service time in the queue; C_m can then be identified with β_m^{-1}, since the number of packets that the server can handle per unit time on the average is β_m^{-1}. Now, under the light traffic regime, i.e., $\lambda_m << C_m$, the above expression for EW_m can be approximated by $a_m \lambda_m$, where $a_m = \beta_m^{(2)}$.

A special application for this model arises in road traffic. The setting is appropriate if a user is some company that ships goods using many vehicles. The company then has to decide what fraction of its traffic to route on each link.

3 Computation of Nash Equilibrium

Theorem 1 *There exists a unique equilibrium given by*

$$\lambda_l^i = \Big(\frac{1}{a_l} / \sum_{m=1}^{M} \frac{1}{a_m} \Big) \Lambda^i , \qquad \forall i = 1, ..., N, \forall l = 1, ..., M. \tag{3}$$

Proof: In order to avoid using the rate constraints for user i, we can assume that this user has only $N-1$ decision variables: $(\lambda_1^i, ..., \lambda_{N-1}^i)$; the value of λ_M^i is then determined by $\lambda_M^i = \Lambda^i - \sum_{k=1}^{M-1} \lambda_k^i$. In terms of these decision variables, the cost (1) for user i can be rewritten as:

$$J_i(\lambda) = \sum_{k=1}^{M-1} \lambda_k^i a_k \lambda_k + \left(\Lambda^i - \sum_{k=1}^{M-1} \lambda_k^i \right) a_M \lambda_M . \tag{4}$$

In the computation of the Nash equilibrium, we first ignore the positivity constraints (R1), by assuming the existence of a Nash equilibrium solution with strictly positive flows, and then show that such a solution indeed exists. By the uniqueness results of [22], it follows that this is in fact the unique equilibrium point.

With the positivity constraint thus neglected, the best response of user i can be obtained by taking the partial derivative

$$\frac{\partial J_i(\lambda)}{\partial \lambda_l^i} = \lambda_l a_l + \lambda_l^i a_l - a_M \lambda_M - a_M \lambda_M^i \tag{5}$$

and equating it to 0. This yields, using the definitions at the beginning of this section, the following expression for the minimizer $(\lambda_l^i)^*$:

$$\frac{1}{2(a_l + a_M)} \left[-(a_l + a_M)\lambda_l^{-i} + a_M \left(\Lambda - \sum_{k=1,k\neq l}^{M-1} \lambda_k + \Lambda^i - \sum_{k=1,k\neq l}^{M-1} \lambda_k^i \right) \right] \tag{6}$$

The sum $\lambda_l^* \stackrel{\text{def}}{=} \sum_{i=1}^N (\lambda_i^l)^*$ of optimal responses for all players on link l is then given by

$$\begin{aligned} & \frac{1}{2(a_l + a_M)} \left[-(a_l + a_M)\lambda_l(N-1) + a_M \left(N\Lambda - N \sum_{k=1,k\neq l}^{M-1} \lambda_k + \Lambda - \sum_{k=1,k\neq l}^{M-1} \lambda_k \right) \right] \\ = \; & \frac{1}{2(a_l + a_M)} [-(a_l + a_M)\lambda_l(N-1) + a_M(N+1)(\Lambda - \sum_{k=1,k\neq l}^{M-1} \lambda_k)] \end{aligned} \tag{7}$$

Since $\lambda_l^* = \lambda_l$ is a necessary condition for $\{\lambda_l^i\}_{i,l}$ to be in equilibrium, we obtain the following necessary condition

$$\lambda_l = \frac{a_M}{a_l + a_M}(\Lambda - \sum_{k=1,k\neq l}^{M-1} \lambda_k) = \frac{a_M}{a_l + a_M}(\lambda_M + \lambda_l). \tag{8}$$

Hence, a necessary condition for equilibrium is

$$\lambda_l = \frac{a_M}{a_l}\lambda_M,$$

or equivalently,

$$\lambda_l = \frac{k}{a_l},$$

where k is a constant. This constant can be obtained from the rate constraint, i.e.,

$$\Lambda = \sum_{l=1}^{M} \lambda_l = k \sum_{l=1}^{M} \frac{1}{a_l},$$

which leads to

$$\lambda_l = (\frac{1}{a_l} / \sum_{l=1}^{M} \frac{1}{a_l})\Lambda. \tag{9}$$

By substituting this in (5) and equating the resulting expression to 0, we obtain $\lambda_l^i a_l = \lambda_M^i a_M$. Since $\sum_{l=1}^{M} \lambda_l^i = \Lambda^i$, this finally leads to

$$\lambda_l^i = \left(\frac{1}{a_l} \Big/ \sum_{l=1}^{M} \frac{1}{a_l} \Lambda^i \right). \tag{10}$$

◇

Remark 1 We observe from the theorem above that the total amount of traffic at equilibrium,

$$\lambda_l = \Big(\frac{1}{a_l} / \sum_{i=1}^{M} \frac{1}{a_i} \Big) \Lambda ,$$

is *not a function of the number of users.* In particular, this is also the socially optimal solution, i.e., the routing policy that is obtained when there is only one user whose total requirement for transmission rate equals Λ (this can be seen from our analysis by setting $\Lambda^j = 0$ for all users $j \neq i$). This implies that the Nash equilibrium is *efficient*, and is also a Pareto optimal solution; even if there were cooperation, the users would still use the same equilibrium policy.

Remark 2 Up to this point, we have considered the situation where there are several users shipping flows. We may consider, however, yet another problem which is encountered in road traffic: the problem of individual drivers' optimal decisions. Link l would be chosen by a driver if the total cost $f_l(\lambda_l)$ it would experience on that link is the minimal among all l's. If several links have minimum delay, the driver may randomize between them. Let α_l be the probability that a driver assigns to link l, and assume that all drivers choose these probabilities according to

$$\alpha_l = \frac{1}{a_l} / \sum_{m=1}^{M} \frac{1}{a_m} .$$

Then the total flow on link l is

$$\frac{1}{a_l} / \sum_{m=1}^{M} \frac{1}{a_m} \Lambda,$$

and the average cost $f_l(\lambda_l)$ over each link l,

$$f_l(\lambda_l) = a_m \lambda_m = \Lambda / \sum_{m=1}^{M} 1/a_m,$$

does not depend on l. Hence all users turn out to assign positive probability only to links that have minimum costs. This characterizes the so-called Wardrop equilibrium [24], which is often used in road traffic. We thus conclude that the Nash equilibrium identified in Theorem 1 is also a Wardrop (individual) equilibrium.

An intuitive way to relate our analysis to that of individual equilibria is to consider that each of the N users represents in fact a large class of n_i micro-users (drivers) $i = 1, ..., N$. According to Theorem 1 all micro users should route the same proportion α_l of their traffic to link l. The limit, as $n_i \to \infty$, is known to converge to the Wardrop equilibrium [12] and can be interpreted as the policy of an "infinitesimal" user.

4 Convergence to equilibrium

The equilibrium obtained in the previous section is defined as a multi-policy for which no user would gain by deviating from. This implies that once an equilibrium point is reached, the users will continue using the same policy, and the system will remain in equilibrium. A crucial question now is what happens if the system is not originally in equilibrium. Toward addressing this question, we shall make the behavioral assumption that users occasionally update their policies in a greedy way, i.e., they use their best response to the current policy of other users. We study in this section the convergence of such schemes.

We focus on the scenario characterized by two links, $M = 2$. The best response of user i is computed through (6). Let $\lambda^i_m(E)$ be the equilibrium value given in Theorem 1. Since we are interested in just two links, and since the sum $\lambda^i_1 + \lambda^i_2 = \Lambda^i$ is fixed for each i, it will suffice to focus on the best response of λ^i_m with $m = 1$, with the subscript $m = 1$ suppressed in the following. Define

$$\overline{\lambda}^i \stackrel{\text{def}}{=} \lambda^i - \lambda^i(E), \qquad (\overline{\lambda}^i)^* \stackrel{\text{def}}{=} (\lambda^i)^* - \lambda^i(E).$$

In terms of this notation, we obtain from (6) and Theorem 1:

$$(\overline{\lambda}^i)^* = \sum_{k=1}^{I} (\nabla A)_{ik} \overline{\lambda}^k, \tag{11}$$

where ∇A is the matrix whose entries are given by

$$(\nabla A)_{ik} = \left(\frac{\partial \lambda^i}{\partial \lambda^k}\right) = \begin{cases} -1/2 & \text{if } i \neq k \\ 0 & \text{if } i = k. \end{cases} \tag{12}$$

Remark 3 *The optimal response requires from each user to know only the sum of flows on each link, and not their individual values — which makes implementation easier. Indeed, it suffices for a user to know the cost he is paying on each link and his own flows, to be able to compute the sum of flows of other users on each link.*

4.1 Round-Robin response

We now consider a round-robin adjustment scheme: the users update their policies (by using the optimal response (6) or (11)) one after the other in the order $1, 2, \ldots, N$, $1, 2, \ldots, N$, etc.

Denote by $\overline{\lambda}^i(n)$ the value of $\overline{\lambda}^i$ at the nth time it is updated. We note that it is a function of all $\overline{\lambda}^j$'s for $j \neq i$, where for $j < i$ these have already been updated n times, and for $j > i$ these have only been updated $n-1$ times.

More precisely, we can express the updates in the form

$$\begin{aligned}
\overline{\lambda}^1(n+1) &= g_1(\overline{\lambda}^2(n), \overline{\lambda}^3(n) ..., \overline{\lambda}^N(n)), \\
\overline{\lambda}^2(n+1) &= g_2(\overline{\lambda}^1(n+1), \overline{\lambda}^3(n), ..., \overline{\lambda}^N(n)), \\
&\vdots \\
\overline{\lambda}^N(n+1) &= g_M(\overline{\lambda}^1(n+1), ..., \overline{\lambda}^{N-1}(n+1)).
\end{aligned}$$

Thus, using (11) we have

$$\begin{aligned}
\overline{\lambda}(n+1) &= -\frac{1}{2}\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}\overline{\lambda}(n) - \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}\overline{\lambda}(n+1) \\
&:= -\frac{1}{2}\mathcal{A}\overline{\lambda}(n) - \frac{1}{2}\mathcal{B}\overline{\lambda}(n+1). \qquad (13)
\end{aligned}$$

This provides an implicit recursive updating formula for the policies at the end of the $n+1st$ update cycle. We proceed to obtain an explicit formula. Let $\mathcal{I}$ denote the identity matrix. (13) is then equivalent to

$$(2\mathcal{I} + \mathcal{B})\overline{\lambda}(n+1) = -\mathcal{A}\overline{\lambda}(n).$$

Define $\mathcal{C} \stackrel{\text{def}}{=} (2\mathcal{I} + \mathcal{B})$. Then, the explicit recursive update formula is

$$\overline{\lambda}(n+1) = -\mathcal{C}^{-1}\mathcal{A}\overline{\lambda}(n). \qquad (14)$$

Theorem 2 *The Round Robin update algorithm converges to the unique Nash equilibrium given in Theorem 1.*

Proof: From (14) we see that a sufficient condition for convergence of the Round Robin scheme is that all eigenvalues of $-\mathcal{C}^{-1}\mathcal{A}$ be in the interior of the unit disk; the latter implies in fact that $(-\mathcal{C}^{-1}\mathcal{A})^n$ converges to 0 ([13] p. 138). The eigenvalues of $-\mathcal{C}^{-1}\mathcal{A}$ are the zeros z of

$$\det(\mathcal{C}^{-1}\mathcal{A} + \mathcal{I}z) = 0 \tag{15}$$

or equivalently, of $\det \mathcal{M}(z) = 0$ where

$$\mathcal{M}(z) \stackrel{\text{def}}{=} \mathcal{A} + 2\mathcal{I}z + \mathcal{B}\mathcal{I}z = \begin{bmatrix} 2z & 1 & 1 & \cdots & 1 \\ z & 2z & 1 & \cdots & 1 \\ z & z & 2z & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z & z & z & \cdots & 2z \end{bmatrix}$$

Simplification of this matrix (by multiplying rows by constants and adding rows), yields:

$$\mathcal{M}'(z) \stackrel{\text{def}}{=} \begin{bmatrix} z & (1-2z) & 0 & \cdots & 0 \\ 0 & z & (1-2z) & \cdots & 0 \\ 0 & 0 & z & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -(1-2z) & 0 & 0 & \cdots & -1 \end{bmatrix}$$

Details of this simplification, for a generic row i is the following:

- (i) For $i < N$, we have replaced row i by the difference between row i and row $i+1$ in matrix $\mathcal{M}(z)$.
- (ii) For $i = N$, we have first multiplied this row by $(1 - 1/z)$ and then added to it row 1.

It is easy to see that $z = 1$ is not a solution of $\det \mathcal{M}(z) = 0$, but it is a solution of $\det \mathcal{M}'(z) = 0$. On the other hand, $z = 0$ is a solution of $\det \mathcal{M}(z) = 0$, but not a solution of $\det \mathcal{M}'(z) = 0$. The reason for this is that in step (ii) we have multiplied row N by $(z-1)/z$, and by doing this we have canceled a zero at $z = 0$ from the characteristic polynomial of the determinant of $\mathcal{M}(z)$, and inserted one at $z = 1$. For all other rows and for all other values of z, the multiplying factor is nonzero, and hence the remaining set of solutions of $\det \mathcal{M}'(z) = 0$ is the same as the remaining set of solutions of $\det \mathcal{M}(z) = 0$.

Thus it remains to show that the zeros of

$$\det(\mathcal{M}'(z)) = -z^{N-1} + (1-2z)^N(-1)^N$$

are all in the interior of the unit disk.

Consider first $|z| > 1$. Then, since $|1-2z| > |z|$,

$$|\det(\mathcal{M}'(z))| \geq |(1-2z)^N(-1)^N| - |z^{N-1}| > |z|^N - |z|^{N-1} = (|z|-1)|z|^{N-1} > 0.$$

It remains to check the case $|z| = 1$ with $z \neq 1$. Toward this end, write z as $z = z_x + iz_y$ where z_x and z_y are respectively the real and imaginary parts of z. We have (since $z_x < 1$)

$$|1-2z| = \sqrt{(1-2z_x)^2 + 2{z_y}^2} = \sqrt{5-4z_x} > 1$$

Hence

$$|\det(\mathcal{M}'(z))| \geq |(1-2z)^N(-1)^N| - |z^{N-1}| = |(1-2z)|^N - 1 > 0.$$

We conclude that all the zeros of $\mathcal{M}(z)$ are in the interior of the unit disk, which implies that the Round Robin update scheme converges to the unique Nash equilibrium. ◇

4.2 Pairwise simultaneous adjustment

We consider in this subsection a Round Robin scheme in which the users update their policies in pairs, i.e., player 1 and 2 update, then players 3 and 4 update, and so on. For simplicity we assume that the number of players is even. We call this scheme *Round Robin in blocks of two*.

Theorem 3 *The Round Robin in blocks of two converges to the unique Nash equilibrium given in Theorem 1.*

Proof: With the notation of the previous subsection, we obtain the implicit update rule as

$$\lambda(n+1) = -\frac{1}{2}\mathcal{A}\lambda(n) - \frac{1}{2}\mathcal{B}\lambda(n+1)$$

where

$$\mathcal{A} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \text{ and } \mathcal{B} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 0 \end{bmatrix}.$$

The explicit update formula is again (14) with the new definitions for $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{C}$ be as defined below, again with the new definitions of $\mathcal{A}$ and $\mathcal{B}$. As before, the eigenvalues are the zeros of $\det(\mathcal{C}^{-1}\mathcal{A}+\mathcal{I}z)=0$, which is equivalent to the zeros of $\det(\mathcal{M}(z))=0$ where

$$\mathcal{M}(z) \stackrel{\text{def}}{=} \mathcal{A}+2\mathcal{I}z+\mathcal{B}z = \begin{bmatrix} 2z & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2z & 1 & 1 & \cdots & 1 & 1 \\ z & z & 2z & 1 & \cdots & 1 & 1 \\ z & z & 1 & 2z & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ z & z & z & z & \cdots & 2z & 1 \\ z & z & z & z & \cdots & 1 & 2z \end{bmatrix} \tag{16}$$

We can simplify this matrix to (by manipulation of its rows and columns, as before):

$$\mathcal{M}'(z) \stackrel{\text{def}}{=} \begin{bmatrix} 2z-1 & 1-2z & 0 & 0 & \cdots & 0 & 0 \\ 1-z & z & 1-2z & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2z-1 & 1-2z & \cdots & 0 & 0 \\ 0 & 0 & 1-z & z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2z-1 & 1-2z \\ z & z & z & z & \cdots & 1 & 2z \end{bmatrix}$$

The characteristic polynomial $P_n(z)$ of this matrix is given by

$$(x-1)^n[(x-1)^{n-2}(x^2+1)+2[(x-1)^{n-3}+(x-1)^{n-4}+\cdots+(x-1)+1]] \tag{17}$$

where $x=2z$, and $n=N/2$. For $\lambda \neq 1$, it can be simplified to:

$$\frac{(2z-1)^n}{1-z}[1-z(2z-1)^n] \tag{18}$$

This step is obtained by induction. Indeed, it holds for $n=1$, for which we obtain $P_1(z) = (2z-1)(2z+1)$. Assume that it holds for some n.

Note that the characteristic polynomial for the matrix without the first column and the last row is $(1-x)^{2n}$. $P_{n+1}(z)$ can be written as $P_{n+1}=(x-1)[Minor^1+Minor^2]$, where by the induction hypothesis we have:

$$\begin{aligned} Minor^1 &= z(P_n+(1-x)^{2n}), \\ Minor^2 &= (1-z)P_n+z[(1-x)^{2n}]. \end{aligned}$$

Thus,

$$\begin{aligned}
P_{n+1} &= (x-1)[2z(1-x)^{2n} + \frac{(x-1)^n}{1-z}(1-z(x-1)^n) \\
&= \frac{(x-1)^{n+1}}{1-z}[2z(x-1)^n(1-z)+1-z(x-1)^n] \\
&= \frac{(x-1)^{n+1}}{1-z}[(x-1)^n(z-2z^2)+1] \\
&= \frac{(x-1)^{n+1}}{1-z}[1-z(x-1)^{n+1}]
\end{aligned}$$

This establishes by induction the form (18) of the characteristic polynomial $P_n(z)$ for $z \neq 1$.

Now, we prove that the roots of the polynomial $P_n(x)$ are in the interior of the unit circle:

If $|z| > 1$, then $|x| > 2$, so that $|x-1| > 1$ and $|x-1|^n > 1$. Hence $|1 - z(x-1)^n| > 0$, which implies that $P_n(z) \neq 0$.

By a simple verification in (17), $z = 1$ is not a zero.

If $|z| = 1, z \neq 1$, we have (with the notation of the previous subsection)

$$|z| = \sqrt{z_x^2 + z_y^2} \Rightarrow |x-1| = \sqrt{5-4z_x} > 1 \Rightarrow |1-z(x-1)^n| > 0.$$

This implies that z is not a zero of $P_n(z)$.

We conclude that all the zeros of the characteristic polynomial $P_n(x)$ are in the interior of the unit disk, and hence the update scheme converges to the unique Nash equilibrium. $\diamond$

4.3 Random Greedy algorithms

Consider a sequence of i.i.d. random vectors $\{X_n\}$, $n = 1, 2, ...$, where $X_n = (X_1^1, \ldots, X_n^N)$, with X_n^i taking the values 0 and 1. (Note that for a fixed n, we allow the different components of X_n to be dependent.)

We define the Random Greedy algorithm as follows: player i updates at time n if and only if $X_n^i = 1$.

Consider the symmetric case where $p \stackrel{\text{def}}{=} EX_n^i$ does not depend on i or n.

Theorem 4 *Consider the Random Greedy algorithm. If $p \geq 4/(N+1)$, then the algorithm does not converge.*

Proof: It follows from (11) that

$$\overline{\lambda}^i(n+1) = -\frac{X_n^i}{2} \sum_{k \neq i} \overline{\lambda}^k(n) + (1 - X_n^i)\overline{\lambda}^i(n).$$

Letting $L^i \stackrel{\text{def}}{=} E\overline{\lambda}^i$, we obtain

$$L^i(n+1) = -\frac{p}{2} \sum_{k \neq i} L^k(n) + (1-p)L^i(n).$$

Let

$$L(n) \stackrel{\text{def}}{=} \sum_{i=1}^{N} L^i \, .$$

Then

$$L(n+1) = \left[-\frac{p}{2}(N-1) + (1-p)\right] L(n) = \left[-\frac{p}{2}(N+1) + 1\right] L(n).$$

If the term in the square brackets is smaller than or equal to -1, then $L(n)$ does not converge. This condition is equivalent to $p \geq 4/(N+1)$. $\diamond$

The above theorem implies, in particular, that the parallel update algorithm, in which all users update simultaneously ($p = 1$), does not converge.

Consider now the following Simultaneous **Random Polling** algorithm. There is a fixed integer $K < N$, such that at each update time n, K players are chosen at random to update. As a corollary to Theorem 4, we see that if $K > 3$ then we do not have convergence.

Next, we consider the above Random Polling where $K = 1$. We shall show that this algorithm converges to the Nash equilibrium.

Theorem 5 *Consider the the Random Polling. If $K = 1$, then the algorithm converges to the Nash equilibrium with probability 1.*

Proof: We shall show that for any realization, $\lim_{n\to\infty} \overline{\lambda}(n) = 0$ with probability one. Let $\mathcal{A}(i)$ denote the matrix that corresponds to the update of player i, i.e. the matrix given by

$\mathcal{A}(i)_{(j,j)} = 1$ and $\mathcal{A}(i)_{i,j} = -1/2$ for all $j \neq i$, and all other elements are 0. For example,

$$\mathcal{A}(1) = \begin{bmatrix} 0 & -1/2 & -1/2 & \cdots & -1/2 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Let $\psi(n)$ be the (random) user that updates at step n. Then $\overline{\lambda}(n)$ has the form

$$\overline{\lambda}(n) = \mathcal{A}(\psi(n))\mathcal{A}(\psi(n-1))...\mathcal{A}(\psi(1))\overline{\lambda}(0).$$

In order to prove the Theorem, we define the following function $Z : \mathbb{R}^N \to \mathbb{R}_+$:

$$Z(x) \stackrel{\text{def}}{=} \max_{Y \subset \{1,...,N\}} \left| \sum_{j \in Y} x_j \right|,$$

where $x = (x_1, ..., x_N) \in \mathbb{R}^N$.

We shall first show that for any $i = 1, ..., N$ and any $x \in \mathbb{R}^N$,

$$Z(\mathcal{A}(i)x) \leq Z(x). \tag{19}$$

Consider some $Y \subset \{1, ..., N\}$. Assume that $i \notin Y$. Then $[\mathcal{A}(i)x]_j = x_j$ for all $j \neq i$. Hence

$$\left| \sum_{j \in Y} [\mathcal{A}(i)x]_j \right| = \left| \sum_{j \in Y} x_j \right|$$

which implies

$$\left| \sum_{j \in Y} [\mathcal{A}(i)x]_j \right| \leq \max_{Y' \subset \{1,...,N\}} \left| \sum_{j \in Y'} x_j \right|. \tag{20}$$

Next, assume that $i \in Y$. Since

$$[\mathcal{A}(i)x]_i = -\frac{1}{2} \sum_{j \neq i} x_j,$$

we have

$$\begin{aligned} \sum_{j \in Y} [\mathcal{A}(i)x]_j &= \sum_{j \in Y, j \neq i} [\mathcal{A}(i)x]_j + [\mathcal{A}(i)x]_i \\ &= \sum_{j \in Y, j \neq i} x_j - \frac{1}{2} \sum_{j \neq i} x_j \\ &= \frac{1}{2} \sum_{j \in Y, j \neq i} x_j - \frac{1}{2} \sum_{j \notin Y} x_j \end{aligned}$$

This again implies (20), which establishes (19).

Since $\psi(n)$ are i.i.d. and for every j, the probability that $\psi(n) = j$ is strictly positive, it follows that in any realization of the sequence $\psi(n)$, the round robin sequence $(1, 2, ..., N)$ appears infinitely often with probability 1. Moreover, for any integer V, the sequence $(1, 2, ..., N)^V$ appears infinitely often with probability 1. Denote $B = \prod_{j=1}^{N} \mathcal{A}(j)$. We have established in Theorem 2 that $B^n \to 0$. This implies, in particular, that the L_1 norm of B^n converges to 0. It is then easy to show that for any $\epsilon > 0$ there exists some integer V such that

$$Z(B^V x) \leq \epsilon Z(x),$$

for any $x \in \mathbb{R}^N$. This, together with (19), implies that with probability one, $\lim_{n\to\infty} Z(\overline{\lambda}(n)) = 0$, and hence $\lim_{n\to\infty} \overline{\lambda}(n) = 0$. $\diamond$

Remark 4 We see from the proof of the above Theorem that convergence to the Nash can be obtained, in fact, under more general probabilistic assumptions. In fact, the convergence is obtained along each sample path for which the sequence B^V appears infinitely often.

4.4 Parallel update with a forgetting factor

We have seen above that if $N > 3$ and all players update simultaneously at each step $(K = N)$, then there is no convergence to the Nash equilibrium. The instability seems to be due to the fact that all players change their strategies in the same manner leading to growing oscillations. We consider in this subsection a possible dampening of these oscillations by allowing only a partial update; we consider a natural scenario where a player does not completely change his strategy from one step to the next, but rather uses a new strategy obtained as a convex combination of the calculated optimal response and the previously used action. In other words, the strategy vector $\overline{\lambda}_{n+1}$ at the $n+1$th step is given by

$$\overline{\lambda}_{n+1} = (\gamma \nabla A + (1-\gamma) I) \overline{\lambda}_n,$$

where ∇A is as defined by (12), I is the identity matrix, and $0 < \gamma < 1$ is some forgetting factor.

Theorem 6 *Consider the parallel update algorithm with forgetting factor γ. Then the system is locally stable if and only if $0 < \gamma < 4/(N+1)$.*

Proof: We have local stability if and only if all eigenvalues of $\gamma\nabla A + (1-\gamma)I$ are in the interior of the unit disk, or equivalently, if and only if the zeros of the determinant of

$$Q(z) = \gamma\nabla A + (1-\gamma-z)I = \begin{bmatrix} 1-\gamma-z & -\gamma/2 & -\gamma/2 & \cdots & -\gamma/2 \\ -\gamma/2 & 1-\gamma-z & -\gamma/2 & \cdots & -\gamma/2 \\ -\gamma/2 & -\gamma/2 & 1-\gamma-z & \cdots & -\gamma/2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma/2 & -\gamma/2 & -\gamma/2 & \cdots & 1-\gamma-z \end{bmatrix}$$

are within the interior of the unit disk. The zeros of $Q(z)$ are the same as those obtained by replacing row i, $i = 2, ..., N$, by the difference between row i and row $i-1$, which yields the matrix:

$$Q'(z) = \begin{bmatrix} 1-\gamma-z & -\gamma/2 & -\gamma/2 & \cdots & -\gamma/2 \\ -w & w & 0 & \cdots & 0 \\ 0 & -w & w & \cdots & 0 \\ 0 & 0 & -w & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & w \end{bmatrix} \quad \text{where } w = 1 - \frac{\gamma}{2} - z.$$

The determinant of $Q'(z)$ is easily computed and is given by

$$det[Q'(z)] = \left(1-\gamma-z-\frac{(N-1)\gamma}{2}\right)(1-\frac{\gamma}{2}-z)^{N-1}.$$

Its zeros are $(1-\gamma/2)$ and $(1-(N+1)\gamma/2)$, and this complete the proof. $\diamond$

5 Conclusion

We have investigated in this paper the problem of static competitive routing to two parallel links with linear link holding costs. We have focused on the stability of the equilibrium, i.e., on the question of whether equilibrium is actually reached if players start originally at some other arbitrary point. In practice, even if every user starts the operation at equilibrium, users may join or leave, so it is of importance to know whether in such situations we may expect convergence to the new equilibrium.

Our analysis suggests that in cases (which might be quite realistic in practice) where users update their actions independently and in a completely greedy way, the unique equilibrium is not stable. On the other hand if updated actions use a convex combination of previous actions as well as the new greedy best response, then stability can be obtained (by giving an appropriate weight to previous actions, which increases in the number of users).

We plan to study in the future the stability of equilibria in more complex topologies, in particular for those for which the uniqueness of equilibrium is known. Furthermore, we intend to consider other cost functions (whose linearized version contains a bias term).

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