Bounds for Performance Measures of Token Rings

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***Abstract*—In polling systems that have been studied in the literature, one usually asserts independent Poisson arrivals. This assumption is however unrealistic when dealing with many applications, e.g. Local Area Networks using token-ring protocols. The arrival processes there may be quite irregular, highly bursty and correlated. We use the new approach for modeling such arrival streams proposed by Cruz [8, 9], to obtain strict upper bounds on several performance measures. It is based on characterizing the inputs by bounds on the average arrival rate and the burstiness, and is especially useful in order to describe arrival streams that are filtered (policed) by Leaky Buckets. We first obtain bounds for the Gated, Exhaustive and Globally-Gated service disciplines, and then consider timed token rings (such as the FDDI). The results for the first three disciplines improve the general bounds obtained in [4, 5]. We further obtain improved exponential bounds, of the type introduced by Chang [7] and Yaron and Sidi [17] for the Globally-Gated discipline.**

1 Introduction

The analysis of polling systems has been performed in the past in isolation with assumptions of Poisson arrivals.

When polling systems are used to model Local Area Networks (LANs) or Asynchronous Transfer Mode (ATM) switches with input buffers, the Poisson assumption becomes unrealistic. Since the input traffic may be obtained as the output of other network elements (e.g. when the LAN is a backbone interconnecting other LANs), it might be quite irregular, highly bursty and correlated (interarrival times need not be independent). Moreover, burstiness and correlation may already appear directly from the input sources, for example in video applications. An analysis approach is thus needed that considers the polling system within the whole environment of the network.

More realistic traffic models have been proposed to characterized bursty traffic (MMPP - Markov Modulated Poisson Process, BMAP - Batch Markov Arrival Process, self similar processes). Unfortunately, the parameter identification, analysis, dimensioning and resource allocation become very complicated when using such characterizations. A solution that has been retained in the ATM context is to filter (police) the input traffic in order allow the implementation of efficient resource allocation algorithms. Leaky Bucket type mechanism have been normalized to implement the filters. They guarantee that the average input rates and burstiness are uniformly bounded in time. Such policing mechanisms are necessary in High Speed Networks where the product of propagation delay and bandwidth is high. In those cases, other mechanisms to regulate the flow in the network, based on feedback information, do not suffice, as huge amount of information may be transmitted by the time the feedback is received. As a result of using the above type of input policing, the input processes may now be characterized by bounds on their average rate and burstiness; this will be made precise in (1) below. The output traffic of a leaky bucket can be modeled by the Cruz-type traffic ([8, 9]).

For Cruz-type traffics, a calculus has been developed allowing study of several types of elements (queues, multiplexers, etc) within a network in [8, 9]. Several elements of networks have the property that when fed by such input processes then the output processes have also bounds on the average rate and burstiness. Such bounds were computed for queues and multiplexers, see [8, 9], and for GPS, see [11, 12]. The contribution of this paper is to study the behavior of single-server polling-type systems in this framework, and obtain bounds on quantities such as cycle times, workload and delays. The strict bound on the cycle time which we obtain for several polling regimes is quite desirable in communication applications, especially in order to guarantee some service quality for synchronous traffic (such as voice or video). Indeed, several token-ring protocols possess some distributed mechanism to enforce a strict upper bound on the cycle time. The fact that an upper bound for the cycle times is obtained for the Cruz-type arrival process suggests that previous distributed mechanisms to enforce such constraints can be replaced by shaping the input flows to a token ring using leaky buckets.

We also mention [16], which is a related study on bounds for the backlog in rings with spatial reuse with Cruz-type arrival.

Consider a polling system consisting of K queues (stations) and a server that moves cyclically between them. At each station, the amount of packets (customers) served (or the amount of work given by the server) is given by some polling regime.

Let $I(n)$ denote $(n-1)\mathrm{mod}K+1$ for all integer n. The movement of the server from the nth station that is polled till the $n+1$st station takes some time, called walking time, which we denote by D_n. We assume that there exist some nonnegative constants d_i, $i = 1, ..., K$ such that $D_n \le d_{I(n)}$. Let $d = \sum_{i=1}^K d_i$.

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We are interested in general arrival processes that can describe bursty and correlated traffic. We choose to use the methodology suggested by Cruz [8, 9]; we thus assume a general arrival process that is characterized only by the following strict bound: the amount of work, $A^j[s,t)$, that arrives to station j during any period $[s,t)$ (i.e. the total time required for the transmission of packets that arrive to station j during any period (s,t)) satisfies:

$$A^j(s,t) \leq (t-s)\rho_j + \sigma_j \tag{1}$$

where ρ_j and σ_j are some nonnegative constants. Denote $\sum_{j=1}^K \rho_j = \rho$, $\sum_{j=1}^K \sigma_j = \sigma$. (Note that (1) implies that $A^j[s,t] \leq (t-s)\rho_j + \sigma_j$ holds for the closed interval $[s,t]$ instead of the open one).

Let $\tau(n)$, $n = 1, 2, \ldots$, be the time instant when the nth station is polled, i.e. when the token (server) arrives to the nth station, $n = 1, 2, 3, \ldots$. Thus, the station polled at time $\tau(k)$ is polled again at times $\tau(k+K), \tau(k+2K), \ldots$, etc. We are interested in bounding the following quantities uniformly in time:

W_n^{out}:= the amount of work that leaves the system from the nth station that is polled;

C_n:= the duration of the cycle that starts at the moment that the token arrives to the nth station, i.e. $C_n = \tau(n+K) - \tau(n)$;

$W^j(t)$:= the total amount of work in the jth station at time t;

$Q^j(t)$:= the number of packets in the jth station at time t.

$V^j(m)$:= the sojourn time of the mth packet that arrives to station j.

We shall restrict our study to the following regimes:

1) The Gated regime, in which in every visit to a station, the server serves the workload present at the polling instant to that station. In particular, if the FIFO (First In First Out) discipline is used, this means that the packets served at the nth visited station are the ones present at that station at time $\tau(n)$. New arrivals to the station while being attended by the server will wait for the next visit.

2) The Exhaustive regime, where the server, upon switching into station i, continues serving packets in that station till it becomes empty.

3) The Globally-Gated regime, in which a global gate is closed in all stations each time that the server arrives to station 1. A cycle is defined for this regime in a slightly different way than for the previous regimes, namely, as the time between two consecutive visits of the server to station 1. The workload present in the stations just before the global gate is closed is served in the current cycle. New workload that arrives during the current cycle has to wait till the next cycle in order to be served. The Globally-Gated Mac protocol was introduced and analyzed in [6]. This protocol as well as the variant in [2] were shown to possess interesting fairness properties.

4) The FDDI timed-token ring protocol (see [13]).

The first three regimes fall into the category of policies studied in [4, 5]. We are able to improve significantly the bounds obtained there, especially under heavy traffic conditions. This should not be surprising, since the class of policies for which the bounds in [4, 5] hold are larger than those studied here. Indeed, they consist of all policies for which the amount of work that leaves the system during an arbitrary cycle is no less than the amount of work present in the beginning of the cycle. (This includes in particular policies that behave during some cycles according to the gated discipline, and then switch to the exhaustive one. For such a policy, Altman et al. show that their bounds in [4, 5] are tight). In the concluding section of this paper we address the question of the tightness of the bounds that we obtain; we were only able to establish the tightness of the bounds on the cycle times for the Globally-Gated regime.

For the Globally-Gated regime we further obtain exponential bounds of a type similar to the one introduced by Chang [7] and Yaron and Sidi [17], for arrival processes which are more bounded only in a stochastic sense (see also [10]).

Altman et al. show in [4, 5] for the policies considered there, that if $\rho < 1$ then the system is stable in the sense that waiting times, station lengths and workloads are upper bounded by some constants, uniformly for all time. We present in this paper sufficient conditions for the stability (in that sense) of the FDDI protocol, based on some sample-path properties obtained in [3].

If the work arrives in packets which queue up, and if these are served according to the FIFO regime, this implies that for arrival processes of type (1), the total number of packets in the system is also uniformly bounded in time. From the uniform bounds on the workload in the system, we can then compute the size of the buffers that guarantee that no losses will occur.

Throughout, the bounds for cycle times and workload will not depend on the order in which individual packets are served within each station (LIFO, FIFO, PS), as long as they are work conservative, as we focus on service disciplines that do not depend on these. The bounds for the queue sizes will be obtained under the assumption that packets are transmitted according to the FIFO regime.

2 Bounds for the Gated Regime

Denote

$$C := (d+\sigma)/(1-\rho), \quad \tilde{C} := \max(C_1, C_2, \ldots, C_K, C),$$

$$R(j) := \rho_j C + \sigma_j, \; \tilde{R}(j) := \rho_j \tilde{C} + \sigma_j.$$

We show below that $\tilde{C}$ is a uniform bound on the cycle times. (If the system is empty during the first K cycles then $C_i \leq d$ for $i \leq K$, and the uniform bound on the cycle duration will be $\tilde{C} = C$.) $\tilde{R}(j)$ is a uniform bound on the amount of work present in the jth queue. Note that the expected average cycle time in cyclic polling system which operate under a stationary ergodic regime with arbitrary stationary ergodic (random) input arrival streams, is known to be $C := d/(1-\rho)$ (see [1]). Therefore, the upper bound C should not be surprising. However, it is not trivial either, since the general uniform upper bound obtained by Altman et al. in [4, 5] for polling systems for which the amount of work that leaves the system during an arbitrary cycle is no less than the amount of work present in the beginning of the cycle, is $d/(1-\rho)^2 + 2\sigma/(1-\rho)$ and is shown to be tight for some policies.

Theorem 2.1 *Assume that $\rho < 1$. The following bounds hold under the Gated regime:*

$$C_n \leq \tilde{C}, \qquad n = 0, 1, 2, ... \tag{2}$$

$$W_n^{out} \leq \tilde{R}(I(n)), \qquad n = 0, 1, 2, ... \tag{3}$$

$$V^j(m) \leq \tilde{R}(j) + \tilde{C}, \qquad m = 0, 1, 2, ... \; ; \; j = 1, \ldots, K \tag{4}$$

$$W^j(t) \leq \tilde{R}(j), \qquad t \geq 0 \; ; \; j = 1, \ldots, K \tag{5}$$

Assume that the transmission time of each packet in the jth station takes at least b_j units of time. Then

$$Q^j(t) \leq \tilde{R}(j)/b_j, \qquad t \geq 0 \; ; \; j = 1, \ldots, K \tag{6}$$

In order to prove the theorem we shall need the following result:

Lemma 2.1 *Let $\Delta \geq 0$ be some constant, and let $\rho < 1$.*
(i) Assume that $C_{n-K} \leq C + \Delta$ for some n. Then $W_n^{out} \leq R(I(n)) + \rho_{I(n)}\Delta$.
(ii) Assume that $C_{n-K-m} \leq C + \Delta$ for $m = 0, 1, ..., K-1$ and some n. Then $C_l \leq C + \Delta$ for all $l \geq n - K$ and $W_l^{out} \leq R(I(l)) + \Delta$ for all $l \geq n$.

Proof: (i) follows from

$$\begin{aligned} W_n^{out} &= A^{I(n)}[\tau(n-K), \tau(n)) \leq \rho_{I(n)} C_{n-K} + \sigma_{I(n)} \\ &\leq \rho_{I(n)}(C + \Delta) + \sigma_{I(n)} = R(I(n)) + \rho_{I(n)}\Delta \end{aligned}$$

The first equality holds by the definition of the Gated policy and the first inequality is a consequence of equation (1). The second inequality is a consequence of the assumptions in (i) and the last equality uses the definition of $R(j)$. It follows from (i) and the assumptions in (ii) that $W_{n-m}^{out} \leq R(I(n-m)) + \rho_{I(n-m)}\Delta$ for $m = 0, 1, ..., K-1$. Hence

$$\begin{aligned} C_{n-K+1} &= \sum_{i=0}^{K-1} \left(W_{n-i}^{out} + D_{n-i}\right) \leq \\ &\leq \sum_{i=0}^{K-1} R(I(n-i)) + \sum_{i=0}^{K-1} \rho_{I(n-i)}\Delta + d \\ &\leq \sum_{i=0}^{K-1} R(I(n-i)) + \Delta + d \\ &= (\rho C + \sigma + d) + \Delta = C + \Delta \end{aligned}$$

The last equality follows since by definition of C, $C = \rho C + \sigma + d$. Hence by (i), $W_{n+1}^{out} \leq \rho_{I(n+1)} C + \sigma_{I(n+1)} + \Delta$. The proof then proceeds by repeating these steps with $n := n+1$, and so on. ∎

Proof of Theorem 2.1: (2) and (3) follow from Lemma 2.1 (ii) by choosing $\Delta = \tilde{C} - C$. (4) follows immediately from (2) and (3), since a packet that arrives during the interval $[\tau(n), \tau(n+1))$ has to wait for no more than a whole cycle plus W_{n+K}^{out}. Next we establish (5). Suppose it did not hold for some t and j, and at time t station j is not being served. At the next time that the token arrives to station j the amount of work leaving the station would exceed the bound (3). If station j is being served at time t and j is the n-th polled station with $\tau(n) \leq t < \tau(n+1)$ then we have:

$$\begin{aligned} W^j(t) &\leq \rho_j(t - \tau_{n-K}) + \sigma_j - (t - \tau_n) \\ &= \rho_j(t - \tau_{n-K}) + \sigma_j - \rho_j(t - \tau_n) - (1-\rho_j)(t - \tau_n) \\ &\leq \rho_j(\tau_n - \tau_{n-K}) + \sigma_j \end{aligned}$$

This establishes (5) and implies (6). ∎

3 Bounds for the Exhaustive Regime

We assume for simplicity that the system is initially empty. (The general case is easily obtained, as in the previous section). Let $\hat{\sigma}_i := \sigma_i(1-\rho_i)^{-1}$, $i = 1, ..., K$, and let

$$\hat{\sigma} = \sum_{i=1}^{K} \hat{\sigma}_i.$$

Let $\hat{C}$ and $\hat{R}(j)$ be defined as in the previous section, with $\hat{\sigma}_i$ and $\hat{\sigma}$ replacing σ_i and σ. That is $\hat{C} = (d + \hat{\sigma})/(1-\rho)$ and $\hat{R}(j) = \rho_j \hat{C} + \hat{\sigma}_j$.

Theorem 3.1 *Assume that $\rho < 1$. The following bounds hold under the Exhaustive regime:*

$$C_n \leq \hat{C}, \qquad n = 0, 1, 2, ... \tag{7}$$

$$W_n^{out} \leq \hat{R}(I(n)), \qquad n = 0, 1, 2, ... \tag{8}$$

$$V^j(m) \leq \hat{C}, \qquad m = 0, 1, 2, \ldots \; ; \; j = 1, ..., K \tag{9}$$

$$W^j(t) \leq \rho_j(1-\rho_j)\hat{C} + \sigma_j, \qquad t \geq 0 \; ; \; j = 1, ..., K \tag{10}$$

Assume that the transmission time of each packet in the jth station takes at least b_j units of time. Then

$$Q^j(t) \leq \frac{\rho_j(1-\rho_j)\hat{C} + \sigma_j}{b_j}, \qquad t \geq 0 \; ; \; j = 1, ..., K \tag{11}$$

Let J_n denote the nth intervisit time, i.e. the time duration from the moment that the token leaves the nth station that is polled, till it arrives next to that same station. Define $\eta(n)$:= the time at which the token leaves the nth station. We need the following Lemma (observe that $I(n) = I(n-K)$):

Lemma 3.1 *(i) Assume that $J_{n-K} \leq (1 - \rho_{I(n)})\hat{C}$ for some n. Then $W_n^{out} \leq \hat{R}(I(n))$.*
(ii) Assume that $J_{n-K-m} \leq (1 - \rho_{I(n-K-m)})\hat{C}$ for $m = 0, 1, ..., K-1$ and some n. Then $J_l \leq (1 - \rho(I(l))\hat{C}$ for all $l \geq n - K$ and $W_l^{out} \leq \hat{R}(I(l))$ for all $l \geq n$.

Proof: We have

$$\begin{aligned} W_n^{out} &= A^{I(n)}[\eta(n-K), \eta(n)) \\ &\le \rho_{I(n)}(J_{n-K} + W_n^{out}) + \sigma_{I(n)} \\ &\le \rho_{I(n)}((1-\rho_{I(n)})\hat{C} + W_n^{out}) + \sigma_{I(n)} \end{aligned}$$

$$\Rightarrow W_n^{out} \le \rho_{I(n)}\hat{C} + \sigma_{I(n)}/(1-\rho_{I(n)})$$

from which (i) follows. It follows from (i) and the assumptions in (ii) that $W_{n-m}^{out} \le \hat{R}(I(n-m))$ for $m = 0, 1, ..., K-1$. Hence

$$\begin{aligned} J_{n-K+1} &= \sum_{i=0}^{K-2} W_{n-i}^{out} + \sum_{i=0}^{K-1} D_{n-i} \\ &\le \sum_{i=0}^{K-2} \hat{R}(I(n-i)) + d \\ &= \sum_{i=0}^{K-2} \left(\rho_{I(n-i)}\hat{C} + \hat{\sigma}_{I(n-i)}\right) + d \\ &= (\rho - \rho_{I(n+1)})\hat{C} + \sum_{i=0}^{K-2} \frac{\sigma_{I(n-i)}}{1-\rho_{I(n-i)}} + d \\ &= (1-\rho_{I(n+1)})\hat{C} + (\rho - 1)\hat{C} \\ &\quad + \sum_{i=0}^{K-2} \frac{\sigma_{I(n-i)}}{1-\rho_{I(n-i)}} + d \\ &= (1-\rho_{I(n+1)})\hat{C} + \left(\sum_{i=0}^{K-2} \frac{\sigma_{I(n-i)}}{1-\rho_{I(n-i)}} - \hat{\sigma}\right) \\ &\le (1-\rho_{I(n+1)})\hat{C} \end{aligned}$$

The rest now follows recursively (as in Lemma 2.1). ■

Proof of Theorem 3.1: We assume without loss of generality that $\tau(0) = 0$. At that moment we know that the system was empty. We pretend that the system operated also for negative n. However, we assume that there were no arrivals before $\tau(0)$. It then follows that $J_{n-K} \le (1 - \rho_{I(n-K)})\hat{C}$ for all $n \le 0$. (8) then follow from Lemma 3.1 (ii). (7) is obtained directly from $C_n \le d + \sum_{i=0}^{K-1} \hat{R}(I(n+i))$. Next we establish (10). Fix some t and j. If t is in a period when the token is not in station j then the amount of work in station j is not more than ρ_j times the current intervisit time of station j plus σ_j and thus (10) holds (by the bound on the intervisit times from Lemma 3.1). Assume that t is in a period when the token is in station j and that it arrived to the station r seconds ago, at the end of the intervisit time J_n. Then

$$W^j(t) \le \rho_j(J_n + r) + \sigma_j - r \le \rho_j J_n + \sigma_j$$

and thus (10) holds (by the bound on the intervisit times from Lemma 3.1). This establishes (10) and implies (11). Finally, (9) follows immediately from (7), since service is Exhaustive. ■

4 Bounds for the Globally-Gated Regime

In order to analyze the Globally-Gated regime, we need to introduce slightly different notations than the ones for the two previous regimes.

Let $\tau(n)$ be the time instant when station 1 is polled for the nth time. Define

C_n:= the duration of the cycle that starts at the nth time that the token arrives to station 1, i.e. $C_n = \tau(n+1) - \tau(n)$;
W_n^{out}:= the amount of work that leaves the system during the nth cycle;
$W(t)$:= the amount of work in all the stations at time t;
$Q(t)$:= the number of packets in all stations at time t.
$V(m)$:= the sojourn time of the mth packet that arrives to the system.
$D(n)$:= the total walking time during the nth cycle. We shall assume that for all $n = 0, 1, ..., D(n) \le d$, where d is some fixed constant.

4.1 Sample path bounds

We first consider arrival processes of Cruz-type [8, 9]. We assume an arrival process that is characterized by the following strict bound: the amount of work that arrives to all stations during any period $[s, t)$ (i.e. the total time required for the transmission of packets that arrive to all stations during any period $[s, t)$) satisfies:

$$A[s, t) \le (t-s)\rho + \sigma \qquad (12)$$

where ρ and σ are some nonnegative constants.

Let C_1 be given. Denote

$$C := \max(d/(1-\rho), C_1) + \sigma, \quad R := \rho C + \sigma.$$

Our main result is the following:

Theorem 4.1 *Assume that $\rho < 1$. The following bounds hold under the Globally-Gated regime:*

$$C_n \le C, \qquad n = 1, 2, ... \qquad (13)$$

$$W_n^{out} \le R \qquad n = 1, 2, ... \qquad (14)$$

$$V(m) \le 2C, \qquad m = 1, 2, ... \qquad (15)$$

$$W(t) \le R, \qquad t \ge 0 \qquad (16)$$

Assume that the transmission time of each packet takes at least b units of time. Then

$$Q(t) \le R/b, \qquad t \ge 0 \qquad (17)$$

Proof: We shall establish (13); the other statements follow then as in the proof of Theorem 2.1. Denote

$$B_n := A[\tau_{n-1}, \tau_n) - \rho C_n, \ n = 1, 2, ...$$

$$G_{n+1} := \max(0, G_n + B_n), \ n = 1, 2, ...$$

where we set $G_1 = \max(C_1 - d/(1-\rho), 0)$. B_n denotes the excess of the amount of work that arrives during cycle

n over the work that would arrive if the input rate were equal to ρ. G_n satisfies a Lindly type equation, and can thus be expressed in an explicit form as

$$G_n = \max\left(0, B_n, B_n + B_{n-1}, ..., \sum_{i=1}^{n} B_i\right), \quad n \geq 2.$$

It follows from the definition of B_n that $G_n \leq \sigma, n \geq 2$, so that for all $n \geq 1$

$$G_n \leq \max(C_1 - d/[1-\rho], 0) + \sigma. \tag{18}$$

We shall show by induction that

$$C_n \leq \frac{d}{1-\rho} + G_n, \tag{19}$$

which together with (18) establishes (13). By definition of G_1, (19) holds for $n = 1$. Assume it holds for some n. Then

$$\begin{aligned} C_{n+1} &= A[\tau_n, \tau_{n+1}) + D(n+1) \leq \rho C_n + B_n + d \\ &\leq \rho\left(\frac{d}{1-\rho} + G_n\right) + B_n + d \leq \frac{d}{1-\rho} + G_{n+1}. \end{aligned}$$

Hence (13) is established. ■

4.2 Exponential bounds

Now we consider stochastic arrival processes of a type similar to the one introduced by Yaron and Sidi [17] and Chang [7], characterized by an exponential bound on the tail distribution. These include, in particular, Poisson, MMPP, and MAP arrival processes. Let $\mathcal{F}_s$ be an increasing family of σ-algebras, such that the arrival $A[s,t]$ is $\mathcal{F}_t$-measurable. We assume that there exist some constants $\rho > 0$, $K_0 > 0$ and $k_0 > 0$ such that for all $\sigma > 0$,

$$\begin{aligned} &P\{A[s,t) - \rho(t-s) > \sigma | \mathcal{F}_s\} \leq K_0 \exp\{-\sigma k_0\} , \\ &\forall\ 0 \leq s < t < \infty . \end{aligned} \tag{20}$$

Observe that this implies that

$$\begin{aligned} &P\{A[s,t) - \rho(t-s) > \sigma\} \leq K_0 \exp\{-\sigma k_0\} , \\ &\forall\ 0 \leq s < t < \infty , \end{aligned} \tag{21}$$

but the reverse is not true, thus our arrival processes are less general than the ones introduced in the cited papers. Observe also that our condition does not imply independent increments.

Under the previous restriction on the arrival processes, we obtain below an exponential bound that is better that the one obtained in Theorem 6 (ii) in [4, 5] for the cycle time. (This improvement is obtained, again, on the expense of a restriction to the specific Globally-Gated polling regime.) We assume that at the first polling instant of station 1 the system is empty.

Theorem 4.2 *Assume $\rho < 1$. Choose some arbitrary $\epsilon > 0$. Then the nth cycle satisfies*

$$P\left\{C_n - \frac{d+\epsilon}{1-\rho} > \sigma\right\} \leq K_1 \exp\{-\sigma k_1\} , \ n \geq 1 , \tag{22}$$

for all $\sigma \geq 0$, where k_1 is any constant satisfying

$$0 < k_1 < k_0(1-\rho) , \tag{23}$$

and K_1 is any constant satisfying

$$K_1 > \frac{K_0}{1 - \exp(-k_1\epsilon/\rho)} ; \tag{24}$$

Proof: We prove (22) inductively. To start the induction, we note that $C_{-1} \leq d$, and hence

$$P\left\{C_{-1} - \frac{d+\epsilon}{1-\rho} > \sigma\right\} = 0.$$

Assume that C_n satisfies (22). We show that C_{n+1} also satisfies (22), thus establishing the proof.

$$\begin{aligned} &P\left\{C_{n+1} - \frac{d+\epsilon}{1-\rho} > \sigma\right\} \\ &\leq P\left\{A[\tau_n, \tau_n + C_n) + D(n+1) - \frac{d+\epsilon}{1-\rho} > \sigma\right\} \\ &\leq P\left\{A[\tau_n, \tau_n + C_n) + d - \frac{d+\epsilon}{1-\rho} > \sigma\right\} && (25) \\ &= P\{A[\tau_n, \tau_n + C_n) - \rho C_n + \rho C_n > \\ &\qquad (1-\rho)\sigma + \rho\left[\frac{d+\epsilon}{1-\rho} + \sigma + \frac{\epsilon}{\rho}\right]\right\} && (26) \\ &\leq P\{A[\tau_n, \tau_n + C_n) - \rho C_n > (1-\rho)\sigma\} \\ &\quad + P\left\{\rho C_n > \rho\left[\frac{d+\epsilon}{1-\rho} + \sigma + \frac{\epsilon}{\rho}\right]\right\} && (27) \\ &\leq K_0 \exp\{-k_0\sigma(1-\rho)\} + K_1 \exp\{-k_1(\sigma + \frac{\epsilon}{\rho})\} && (28) \\ &= K_1 \exp\{-k_1\sigma\} \\ &\quad \cdot \left[\exp\{-k_1\frac{\epsilon}{\rho}\} + \frac{K_0}{K_1}\exp\{k_1\sigma - k_0(1-\rho)\sigma\}\right] && (29) \\ &\leq K_1 \exp\{-k_1\sigma\} , && (30) \end{aligned}$$

where (27) follows from the fact that for any real numbers X, x, Y, y, $X + Y > x + y$ implies either $X > x$ or $Y > y$; and (30) follows from (23) and (24). (28) is a consequence of the facts that τ_n is a stopping time with respect to $(\mathcal{F}_t)$, and that C_n is $\mathcal{F}_{\tau_n}$ measurable. A detailed proof of this inequality is given in the Appendix 7. ■

Remark : The bounds for the cycle times and workloads for the Globally-Gated regime remain unchanged if we allow arbitrary non-cyclic polling order within each cycle, provided that the total walking times per-cycle are still bounded by d. A special case of interest is when the polling order is of an elevator type, as one described in [2], for which expected waiting times were shown to be equal in all stations for the case of Poisson arrivals, even when arrival and service rates differ from station to station.

5 Bounds for the FDDI protocol and stability

The FDDI MAC protocol uses a timed token rotation protocol. Let there be K stations in a ring, numbered as $1, 2, \cdots, K$. At the ring initialization or re-initialization, the stations on the ring negotiate a value for the target token rotation time (TTRT) which is equal to the lowest bid by any of the stations. Let T^* be the value of TTRT. Each station is assigned a percentage of the TTRT for its synchronous traffic, say f_k for station k, where $1 \leq k \leq K$, $0 \leq f_k < 1$, $\sum_{k=1}^{K} f_k < 1$. Every time the token arrives at station k, the synchronous frames can be transmitted (exhaustively) for at most $f_k T^*$ units of time.

Each station has a token rotation timer (TRT), which measures the difference of successive token arrival times. When the token arrives to a station, the contents of its TRT is compared to TTRT. If the token is late, i.e., $TRT > T^*$, then the TRT is reset and restarts counting, and only the synchronous traffic can be transmitted (for at most $f_k T^*$ units of time). Otherwise, the value of TRT is copied to the token holding timer (THT), which starts counting upward at the beginning of asynchronous transmission, and the TRT is reset. In such a case, the synchronous traffic are transmitted for at most $f_k T^*$ units of time, or until all the synchronous frames queued in that station are transmitted, whichever occurs first. The asynchronous frames can be transmitted until the THT reaches the TTRT level, or until all the asynchronous frames queued in that station are transmitted, whichever occurs first. In the first case, the transmission of the last asynchronous frame can be completed before the token is passed to the next station (the so-called asynchronous overrun). (Note that when a station holds the token, then as long as the THT timer does not reach the TTRT value, the asynchronous traffic are served exhaustively). Multiple levels (up to eight) of priority for asynchronous frames may be provided within a station by specifying additional (more restrictive) time thresholds for token rotation. We confine our analysis to a single priority level below.

We shall concentrate on the problem of evaluating the performance of the asynchronous traffic. We shall thus ignore the synchronous traffic by assuming that the bounds d_i for the walking times include also the periods of transmission of the synchronous traffic.

For the FDDI, it is known that the cycle time is bounded by twice T^* [15]. The contribution of this section is to obtain bounds on the total workload in the system, uniformly over time, by making use of the special type of arrival process (1). This could be useful as it could enable to design the size of buffers in a way that guarantees no losses.

We choose an arbitrary station, say station 1. We define the nth cycle to be the period between the nth and the $n+1$st time that the token arrives to that station. Define X_n:= the workload in the beginning of the nth cycle in the whole system.
X_n^{in}:= the workload that arrives to the system during the nth cycle.
X_n^{out}:= the workload that leaves the system during the nth cycle (i.e. the total transmission time of asynchronous traffic during that cycle).
$X(t)$:= the workload in the system at time t.
We assume that the arrival processes and the definitions of ρ, σ and d are as in section 4.1.

Theorem 5.1 *Assume*

$$\rho < 1, \qquad T^* > \frac{\sigma + 2d}{1-\rho}. \tag{31}$$

(i) Assume that $X_0 < 4T^* + \rho d$. Then $X_n < 4T^* + 2\rho d + \sigma$ for all n.
(ii) Assume that $X(s) < 4T^* + \rho d$ during the first cycle (that start from queue 1). Then $X(t) < 4T^* + 2\rho d + \sigma$ for all t.

Remarks: (i) Note that the condition on T^* and ρ in (31) is quite reasonable. Indeed, for the case of Poisson arrival and constant walking times d, it is shown in [3] that necessary conditions for stability are

$$\rho \leq 1, \qquad T^* \geq \frac{d}{1-\rho} \max(1 + \max_{i=1,\ldots,K} \rho_i) - e$$

where e is an upper bound over the asynchronous overrun, and ρ is the expected average rate at which work arrives. Furthermore, $\rho \leq 1$ is a necessary condition for stability of any polling system with stationary ergodic arrival process where work arrives with average rate ρ; moreover, under fairly general conditions, a stable polling system with ergodic input has an average cycle duration of $d/(1-\rho)$ (see e.g. [1, 14]). It is thus reasonable to require that the average cycle duration is sufficiently less than T^*, as is the case in (31).
(ii) Theorem 5.1 is also a statement on a sufficient stability condition; if (31) holds then the system is stable in the strong sense of uniformly bounded workload.
Proof of Theorem 5.1: The proof of (i) is based on the following sample path bound from the proof of Lemma A.2 of [3] (that strongly makes use of the FDDI timer mechanisms):

$$\text{If} \quad X_n > 4T^* \quad \text{then} \quad X_n^{out} + X_{n+1}^{out} \geq T^* - 2d \tag{32}$$

Assume now that for some n, $X_n \leq 4T^*$. Then

$$\begin{aligned} X_{n+2} &= X_n - X_n^{out} - X_{n+1}^{out} + X_n^{in} + X_{n+1}^{in} \\ &\leq X_n - X_n^{out} - X_{n+1}^{out} + \rho(X_n^{out} + X_{n+1}^{out} + 2d) + \sigma \quad (33) \\ &< 4T^* + 2\rho d + \sigma \quad (34) \end{aligned}$$

If for some n, $4T^* < X_n \leq 4T^* + 2\rho d + \sigma$ then it follows from (32) and (33) that

$$\begin{aligned} X_{n+2} &\leq 4T^* + 2\rho d + \sigma - (1-\rho)(T^* - 2d) + 2d\rho + \sigma \\ &< 4T^* + 2\rho d + \sigma \end{aligned}$$

where the last inequality follows from (31). Combining this with (34) and the assumption in (i) we see that for all n,

$X_{2n} < 4T^* + 2\rho d + \sigma$. Now,

$$\begin{aligned} X_1 &\leq X_0 - X_0^{out} + X_0^{in} \\ &\leq X_0 - X_0^{out} + \rho(X_0^{out} + d) + \sigma \\ &\leq X_0 + \rho d + \sigma \\ &\leq 4T^* + 2\rho d + \sigma \end{aligned} \tag{35}$$

Hence X_1 is bounded by (34) too. It then follows that this bound also holds for X_{1+2n} for all n, by the arguments above.

To prove (ii), it suffices to consider t greater than the time at which cycle 0 ends. We first assume that t is greater than the time at which cycle 1 ends. Let n be defined such that at time t, cycle number $n+1$ has been completed, and cycle $n+2$ has not yet ended. We assume first that at time t the server is serving queue 1. Denote
Y^{in} := the workload that arrives to the system since the beginning of the $n+3rd$ cycle, till time t.
Y^{out} := the workload that leaves the system since the beginning of the $n+3rd$ cycle, till time t. Assume now that $X_n \leq 4T^*$. Then

$$\begin{aligned} X(t) = X_n - X_n^{out} - X_{n+1}^{out} - Y^{out} + X_n^{in} + X_{n+1}^{in} + Y^{in} \\ \leq X_n - X_n^{out} - X_{n+1}^{out} - Y^{out} \\ +\rho(X_n^{out} + X_{n+1}^{out} + Y^{out} + 2d) + \sigma \quad (36) \\ < 4T^* + 2\rho d + \sigma \quad (37) \end{aligned}$$

If $4T^* \leq X_n \leq 4T^* + 2\rho d + \sigma$ then it follows from (32) and (36) that

$$\begin{aligned} X(t) &\leq 4T^* + 2\rho d + \sigma - (1-\rho)(T^* - 2d) + 2d\rho + \sigma \\ &< 4T^* + 2\rho d + \sigma \end{aligned}$$

where the last inequality follows from (31). Combining this with (37) we get $X(t) < 4T^* + 2\rho d + \sigma$.

A similar argument as above, using (35), shows that (ii) holds also when t is during a serving time of queue 1, during cycle 1.

Assume now that t occurs during a busy (serving) period of a queue other than 1. Since $X(s) < 4T^* + \rho d$ during the first cycle, the arguments above (for t in a busy period of queue 1) can be repeated to any other queue, and hence (ii) holds for any busy period. Finally, since the total work in the system at any polling instant of a queue is greater than or equal to the total work in the system at any time during the last walking time of the server, it follows that for any t, (ii) is satisfied. ■

The bounds above easily extend to the case that $X_0 \geq 4T^* + \rho d$. In that case, it can easily be seen be using the arguments above that we get for all n, $X_n \leq X_0 + 2\rho d + \sigma$. Moreover, there exists some N_0 such that for all $n > N_0$, $X_n \leq 4T^* + 2\rho d + \sigma$.

6 Concluding remarks

We have presented in the previous sections Cruz-type sample-path bounds that are uniform in time, for the performance measures of several polling regimes. The strict bound on the cycle time which we obtain for the first three regimes is quite desirable in communication applications, especially in order to guarantee some service quality for synchronous traffic (such as voice or video). Indeed, several token-ring protocols possess some distributed mechanism to enforce a strict upper bound on the cycle time; an example is the *FDDI* protocol. The fact that an upper bound for the cycle times is obtained for the Cruz-type arrival process suggests that previous distributed mechanisms to enforce such constraints can be replaced by shaping the input flows to a token ring using leaky buckets.

A natural question is the tightness of the bounds presented above. When the burstiness constraint σ is zero and the total walking time per cycle d and input rates ρ are constants then simple calculations show that the duration C_n of the nth cycle converges to $d/(1-\rho)$, as n tends to infinity. Thus, the bounds for the Gated, Exhaustive and Globally-Gated regime are tight if $\sigma = 0$, and are fairly tight for $\sigma > 0$ if σ is sufficiently smaller than d (for example, our bound for the Gated regime is $(d+\sigma)/(1-\rho)$, and it is clear that we cannot expect a tight upper bound to be less than $d/(1-\rho) + \sigma$). It is also quite obvious that the bounds for the cycle times are quite tight when ρ is very small.

It turns out that the bound that we obtained for the cycle time for the Globally-Gated regime is tight for any ρ, σ, d. To see that, consider the following scenario. If during the first n cycles the input rate is constant and equals to ρ, then the duration of the nth cycle will be within an ϵ-neighborhood of $d/(1-\rho)$, if n is chosen sufficiently large. If during the nth cycle, a batch of σ arrives (in addition to the constant arrival rate of ρ), then the duration of the $n+1$st cycle will be within an ϵ-neighborhood of $d/(1-\rho) + \sigma$. This establishes the tightness of the bound for the cycle time duration for the Globally-Gated regime.

The bounds for the Gated and Exhaustive regimes are in general not tight. We note, however, that we cannot expect to obtain there the same bound as the one for the Globally-Gated regime. To see that, consider a polling system with a gated regime with two queues. Let d be very short (almost zero), and $\rho_1 = 0$, $\rho_2 = \rho$. Assume that a batch of size σ arrives to station one just before it is polled, and the system is initially empty. Moreover, station 2 is fed by an input with constant rate ρ. Then the duration of the cycle that starts when station one is polled is approximately $\sigma(1+\rho)$, which is larger than the bound σ for the Globally-Gated regime (but still way apart from our bounds). We were not able yet to obtain tighter bounds for the the gated and exhaustive regimes.

7 Appendix

We present below the proof of inequality (28), i.e. we prove that

$$\begin{aligned} P(A[\tau_n, \tau_n + C_n] - \rho C_n > (1-\rho)\sigma) \\ < K_0 \exp\{-k_0 \sigma(1-\rho)\} \text{ a.s.} \end{aligned} \tag{38}$$

Observe that τ_n is a stopping time with respect to $(\mathcal{F}_t)$. We will prove (38) for any stopping time T. Let us first suppose that T is a discrete stopping time taking values in a set V. Denote by $C_{res}(t)$ the residual cycle time, i.e. the time interval between the instant t and the next time after t that a cycle ends. In particular, we have $C_{res}(\tau_n) = \tau_{n+1} - \tau_n = C_n$. We assume that C_{res} is right continuous, and assume that it is $\mathcal{F}_t$-measurable. (This is a reasonable assumption, since in the Globally-Gated regime, the duration of a current cycle is only determined by the arrival process during the previous cycle, as well as the total walking duration in the current cycle).

$$
\begin{aligned}
&P(A[T, T + C_{res}(T)] - \rho C_{res}(T) > (1-\rho)\sigma) \\
&= \sum_{v\in V} E\big[\mathbb{1}\{T = v\}\mathbb{1}\{A[v, v + C_{res}(v)] \\
&\qquad -\rho C_{res}(v) > (1-\rho)\sigma\}\big] \\
&= \sum_{v\in V} E\big[E\big(\mathbb{1}\{T = v\}\mathbb{1}\{A[v, v + C_{res}(v)] \\
&\qquad -\rho C_{res}(v) > (1-\rho)\sigma\}|\mathcal{F}_v\big)\big] \qquad (39) \\
&= \sum_{v\in V} E\big[\mathbb{1}\{T = v\}P\big(A[v, v + C_{res}(v)] \\
&\qquad -\rho C_{res}(v) > (1-\rho)\sigma|\mathcal{F}_v\big)\big] \qquad (40) \\
&\leq K_0 \exp\{-k_0\sigma(1-\rho)\} \text{ a.s. .} \qquad (41)
\end{aligned}
$$

If T is not discrete, let us approximate T by the sequence

$$
T(k) = n2^{-k} \text{ on } \{(n-1)2^{-k} < T \leq n2^{-k}\} \qquad (42)
$$

Since the arrival process and C_{res} are right continuous, it follows that

$$
\begin{aligned}
&\lim_{k\to\infty} \mathbb{1}\{A[T(k), T(k) + C_{res}(T(k))] \\
&\qquad -\rho C_{res}(T(k)) > (1-\rho)\sigma\} \\
&= \mathbb{1}\{A[T, T + C_{res}(T)] - \rho C_{res}(T) > (1-\rho)\sigma\}.
\end{aligned}
$$

The result follows as a consequence of the dominated convergence theorem.

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