A Case where a Paradox Like Braess's Occurs in the Nash Equilibrium but Does Not Occur in the Wardrop Equilibrium — A Situation of Load Balancing in Distributed Computer Systems —

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Abstract

The Braess paradox which was originally identified and observed in road traffic context, shows that it may happen that by adding capacity to the network, the performance of all users degrades. This paradox has long been known in a framework called the Wardrop equilibrium, in which there are infinitely many individuals (such as car drivers) and in which the decision of one single individual has a negligible effect on the performance of the other individuals. Another framework in which such a paradox may occur is that of the Nash equilibrium in which there are a finite number of players, and in which the decision of each player has non-negligible effect on the other players. It is natural to expect the same type of paradox in the Nash equilibrium context (for a large number of players), whenever it occurs for the Wardrop equilibrium. In this paper, we present a case where a paradox similar to that of the Braess appears in a Nash equilibrium but does not appear in a Wardrop equilibrium in the same environment. We consider the model of load balancing in distributed computer systems. We further establish the uniqueness of the Nash equilibrium for this problem.

Keywords: Braess paradox, Nash equilibrium, Wardrop equilibrium, performance optimization, distributed computer system, load balancing.

1 Introduction

In many systems including communication networks in distributed computer systems, transportation flow networks, etc., we have several distinct objectives for performance optimization. Among them, we have three typical objectives or optima:

(1) the system optimum, overall optimum, or social optimum, where a certain overall and single measure like the total cost or the overall average response time is to be optimized. We call it the *overall optimum* here.
(2) the individual optimum, Wardrop optimum, or user optimum by some people; where each of infinitely many individuals, users, or jobs cannot receive any benefit by changing its own decision. We call it the *individual optimum* here.
(3) the Nash noncooperative optimum, class optimum, or user optimum by some other people, where each of a finite number of users, classes, or players cannot receive any benefit by changing its decision. We call it the *class optimum* here.
Actually, (3) is reduced to (1) when the number of players reduces to 1 and approaches (2) when the players are symmetric and the number of players becomes infinitely many [4].

We can think that the total processing capacity of a system will increase when the capacity of a part of the system increases, and so we expect improvements in performance objectives accordingly in that case. The Braess paradox tells us that this is not always the case; i.e., increased capacity of a part of the system may sometimes lead to the degradation in the benefits of all users in a Wardrop equilibrium [1, 2, 3]. As it is known that the Nash equilibrium converges to the Wardrop equilibrium as the number of users becomes large [4], it is thus natural to expect the same type of paradox in the Nash equilibrium context (for a large number of players), whenever it occurs for the Wardrop equilibrium. Indeed, Korilis et al. found examples wherein the Braess-like paradox appears in a Nash equilibrium with symmetric user classes where all user classes are identical [10, 11].

On the other hand, it has been observed that the increased capacity of a part of a system may lead to somewhat awkward behavior in terms of a system wide measure [7, 6]. In particular, Kameda et al. found a seemingly anomalous case where in a Nash equilibrium each of two processing nodes (servers) forwards

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the same type of jobs mutually to be processed by the other node, thus incurring additional communication delays [6].

In this paper, we consider load balancing problems for distributed computer systems and present examples wherein the increased capacity of a part of a system would degrade the benefits of all classes in a Nash equilibrium whereas it should not degrade the benefits of all classes at the same time in a Wardrop equilibrium in the same environment. Our models have asymmetric classes; i.e., classes are not identical. Several paradoxes that include the one described in this paper have been reported in [5]. In this paper we focus on a single model and provide an extensive analysis to establish the uniqueness of the overall, the class, and the individual optima. We note that very little is known on the uniqueness of class optimal solutions for networks with *asymmetric users*. To the best of our knowledge, for asymmetric users, uniqueness has only been established for the case of parallel links. The model we present here is thus the first one having different classes with different sources and destinations, for which the uniqueness of the class optimal solution is established.

2 The Model and Assumptions

We consider a model consisting of two nodes (hosts) and a communication means that connects both nodes. Nodes are numbered 1 and 2. Each node consists of a single exponential server with service rate μ_i ($i = 1, 2$). Node i has the external Poisson arrival with rate ϕ_i, out of which the rate x_{ii} of jobs are processed at node i. The rate x_{ij} ($i \neq j$) of jobs are forwarded through the communication means to the other node j to be processed there, and the results of those jobs are returned back through the communication means to node i. Then we have $x_{ii} + x_{ij} = \phi_i$ ($i \neq j$), $x_{ij} \geq 0$, $i, j = 1, 2$. We denote the vector (x_{11}, x_{12}, x_{21}, x_{22}) by $\boldsymbol{x}$. We denote the set of $\boldsymbol{x}$'s that satisfy the constraints by $\boldsymbol{C}$ and let $\Phi = \phi_1 + \phi_2$. Within these constraints, a set of values of x_{ij} ($i, j = 1, 2$) are chosen to achieve optimization. Thus the load on node i is $x_{ii} + x_{ji}$ ($i \neq j$) and is denoted by β_i. Then, the expected processing (including queueing) time of a job that is processed at node i, is $1/(\mu_i - \beta_i)$ for $\beta_i < \mu_i$ (otherwise it is infinite).

As to the communication means, we consider the following case. It consists of a single-channel communication line that is used commonly in forwarding and sending back of jobs that arrive at both nodes. We assume that the expected time length of forwarding and sending back a job is to be

$$G(\lambda) = 1/\{(1/t) - (x_{12} + x_{21})\}$$

if $x_{12} + x_{21} < 1$, and is otherwise inifinite, where $\lambda = x_{12} + x_{21}$ is the network traffic. That is, we assume the communication channel is modelled by a processor sharing server with service rate $1/t$; *i.e.*, the mean communication (without queueing) time is t, and, thus, the capacity of the communication line is $1/t$.

Figure 1: The system model.

We refer to the length of time between the instant when a job arrives at a node and the instant when a job leaves the node, where it has arrived, after all processing and communication, if any, are over as *the response time for a job* arriving at the node. Thus the expected response time of a job that arrives at node i is

$$T_i(\boldsymbol{x}) = (1/\phi_i) \sum_k x_{ik} T_{ik}(\boldsymbol{x}),$$

where $T_{ii}(\boldsymbol{x}) = 1/(\mu_i - \beta_i)$ if $\beta_i < \mu_i$ (and it is otherwise inifinite), and for $j \neq i$,

$$T_{ij}(\boldsymbol{x}) = 1/(\mu_j - \beta_j) + 1/(\{1/t - (x_{ij} + x_{ji})\},$$

(The above expressions hold, again, only for positive values of denominators, and are otherwise infinite.) Then, the overall expected response time of a job that arrives at the system is

$$T(\boldsymbol{x}) = (1/\Phi) \textstyle\sum_i \phi_i T_i.$$

We have three optima, the overall, the individual, and the node.

(1) The overall optimum is given by such $\bar{\boldsymbol{x}}$ as satisfies the following,

$$T(\bar{\boldsymbol{x}}) = \min T(\boldsymbol{x}) \quad \text{with respect to} \quad \boldsymbol{x} \in \boldsymbol{C}.$$

The solution $\bar{\boldsymbol{x}}$ is characterized by the Kuhn-Tucker condition or the following variational inequality,

$$\boldsymbol{t}(\bar{\boldsymbol{x}})(\boldsymbol{x} - \bar{\boldsymbol{x}}) \geq 0 \qquad \text{for all} \quad \boldsymbol{x} \in \boldsymbol{C}$$

where $\boldsymbol{t}(\boldsymbol{x}) = (\frac{\partial}{\partial x_{11}}\Phi T, \frac{\partial}{\partial x_{12}}\Phi T, \frac{\partial}{\partial x_{21}}\Phi T, \frac{\partial}{\partial x_{22}}\Phi T)$ as seen from [7, 8].

(2) The individual optimum is given by such $\hat{\boldsymbol{x}}$ as satisfies the following for all i,

$$T_i(\hat{\boldsymbol{x}}) = \min_{i \neq j}\{T_{ii}(\hat{\boldsymbol{x}}), T_{ij}(\hat{\boldsymbol{x}})\} \quad \text{such that} \quad \hat{\boldsymbol{x}} \in \boldsymbol{C}$$

The solution $\hat{\boldsymbol{x}}$ is characterized by the following variational inequality,

$$\boldsymbol{T}(\hat{\boldsymbol{x}})(\boldsymbol{x} - \hat{\boldsymbol{x}}) \geq 0 \quad \text{for all} \quad \boldsymbol{x} \in \boldsymbol{C}$$

where $\boldsymbol{T}(\boldsymbol{x}) = (T_{11}(\boldsymbol{x}), T_{12}(\boldsymbol{x}), T_{21}(\boldsymbol{x}), T_{22}(\boldsymbol{x}))$ as seen from [7, 8].

(3) The node optimum is given by such $\tilde{\boldsymbol{x}}$ as satisfies the following for all i,

$$T_i(\tilde{\boldsymbol{x}}) = \min T_i(\boldsymbol{x}),$$

with respect to $x_{ii},\ x_{ij}\ (j \neq i)$, such that

$$\boldsymbol{x} \in \boldsymbol{C}.$$

The solution $\tilde{\boldsymbol{x}}$ is characterized by the following variational inequality

$$\tilde{\boldsymbol{t}}(\tilde{\boldsymbol{x}})(\boldsymbol{x} - \tilde{\boldsymbol{x}}) \geq 0 \quad \text{for all} \quad \boldsymbol{x} \in \boldsymbol{C}$$

where $\tilde{\boldsymbol{t}}(\boldsymbol{x}) = (\frac{\partial}{\partial x_{11}}\phi_1 T_1, \frac{\partial}{\partial x_{12}}\phi_1 T_1, \frac{\partial}{\partial x_{21}}\phi_2 T_2, \frac{\partial}{\partial x_{22}}\phi_2 T_2)$ as shown similary as above.

We discuss in Section 5 the existence and the uniqueness of the overall, individual and node (class) optima.

Remark 2.1 Note that, there should be no mutual forwarding in overall and individual optima. That is, in overall and individual optima, either one of x_{ij} $(i \neq j)$ must be zero due to [12]. And thus, when one of x_{ij} $(i \neq j)$ say x_{ij} is non zero, $T_i(\boldsymbol{x})$ decreases and $T_j(\boldsymbol{x})$ increases with the increase of t as shown in Theorems 2.5 and 2.7 of [7].

3 The Numerical Experiments

We examined the cases of the following parameter values of the model given in Section 2.

$$\begin{array}{ll} \phi_1 = 100(\text{jobs/sec}), & \phi_2 = 20(\text{jobs/sec}) \\ \mu_1 = 120(\text{jobs/sec}), & \mu_2 = 25(\text{jobs/sec}) \end{array}$$

The value of the communication (without queueing) time parameter t is varied from 0 (*sec*) till 1 (*sec*) in steps of 0.005 (*sec*), *i.e.*, t = 0, 0.005, 0.010, 0.015, $\cdots$, 1. The algorithms used to obtain the overall and individual optima are based on the algorithms given in [7, 9]. The algorithm for the node optimum is obtained similary as above.

4 Results and Discussion

Figures 2 and 3 show the values T_1 and T_2 of node optima (Figure 2) and of individual optima (Figure 3) with various values of t. We can observe the Braess-like paradox for node optima in the following way. In Figure 2, for $0.08 \leq t \leq 0.12$, we see that there are sets of two values of t for which T_1 and T_2 with the smaller value of t are greater than T_1 and T_2 with the larger value of t, which looks paradoxical. For example, consider the case with $t = 0.08$. Then

$$\begin{array}{ll} T_{11} = 0.058..., & T_{12} = 0.299..., \\ T_{21} = 0.228..., & T_{22} = 0.128..., \\ x_{11} = 98.060..., & x_{12} = 1.939..., \\ x_{21} = 4.707..., & x_{22} = 15.292.... \end{array}$$

Therefore, by noting that $T_1 = (x_{11}T_{11} + x_{12}T_{12})/\phi_1$, $T_2 = (x_{21}T_{21} + x_{22}T_{22})/\phi_2$,

$$T_1 = 0.0627..., \quad T_2 = 0.1522....$$

Furthermore, consider the case with $t = 0.11$. Then

$$\begin{array}{ll} T_{11} = 0.060..., & T_{12} = 0.294..., \\ T_{21} = 0.236..., & T_{22} = 0.118..., \\ x_{11} = 100.0, & x_{12} = 0.0, \\ x_{21} = 3.405..., & x_{22} = 16.594.... \end{array}$$

Therefore $T_1 = 0.0602..., T_2 = 0.1389....$

Figure 2: The mean response times, T_1 and T_2, of a job that arrives at nodes 1 and 2 in node optima, respectively.

As we saw in the last few statements given in Section 2, for individual optima, such a Braess-like paradox as given here would not occur. Figure 3 shows the values of individually optimal T_1 and T_2 with various values of t. We see that, although T_1 is nonincreasing in t, T_2 is nondecreasing in t, which is not paradoxical.

Figure 3: The mean response times, T_1 and T_2, of a job that arrives at nodes 1 and 2 in individual optima, respectively.

5 The Existence and the Uniqueness of the Optima

The model given here has the unique values for its overall, individual, and node optima. The existence and the uniqueness for overall and individual optima have been given in a more general setting [7, 8]. The existence and the uniqueness of the node optimum for this model have not been shown yet, and we show it by extending the result of [13]. Thus we present the proof in the setting as general as possible here in Subsection 5.2.

5.1 The Existence and the Uniqueness of the Overall and Individual Optima

We show in this subsection the existence and the uniqueness of the overall and individual optima. Optimization in the models of this paper is regarded as optimal static routing in open state-independent BCMP queueing networks as defined in [7, 8], where the existence of the optima is straightforward. Then we can apply the theorems 3.4 and 4.4 of [8] (6.4 and 6.9 of [7]) and show that the utilization factors ρ_1, ρ_2 and ρ_c of nodes 1, 2, and communication channel c are unique in both overall and individual optima. Since we have two independent variables x_{11} and x_{12} and two independent equations.

$$x_{11} + \phi_2 - x_{22} = \rho_1\mu_1,$$
$$\phi_1 - x_{11} + \phi_2 - x_{22} = \rho_c/t,$$

we see that $\boldsymbol{x}$ is unique in both overall and individual optima in the model.

5.2 The Existence and the Uniqueness of the Node Optimum

We show in this subsection the existence and the uniqueness of the node optimum or Nash equilibrium.We shall consider a more general cost, and the more general topology. More precisely, let there be K sets of classes: $\mathcal{K}_i$ $i = 1, ..., K$ where $\mathcal{K}_i$ contains k_i classes of jobs. The set $\mathcal{K}_i$ corresponds to the classes that send jobs originally to node i. Each of these classes can decide to route part of his flow to node j, $j \neq i$ through a communication line. We consider the problem of the existence and the uniqueness of the equilibrium in which each one of the classes minimizes its own average delay. Denote $\mathcal{K} = \cup_{i=1}^{K}\mathcal{K}_i$. We extend the notation from the previous section in the natural way as follows:

- $\phi^{(r)}$ the input arrival rate of class r.
- $x_{ii}^{(r)}$ the flow originating at node i by class r that is processed at node i.
- $x_{ij}^{(r)}$ the flow originating at node i by class r that is processed at node j.
- $\mathbf{x}$ the set of all flows, i.e. $\mathbf{x} = (x_{ij}^{(r)}, r \in \mathcal{K}_i, i = 1, ..., K, j = 1, ..., K)$.
- $\mathbf{x}_{ij}$ the set of flows through link (i,j), i.e. $\mathbf{x}_{ij} = (x_{ij}^{(r)}, r \in \mathcal{K}_i))$.
- β_i the total load on node i, i.e. $\beta_i = \sum_{j=1}^{K}\sum_{r\in\mathcal{K}_j} x_{ji}^{(r)}$.
- λ_{ij} the total flow forwarded from node i to node j.
- λ the total network flow, i.e. $\lambda = \sum_{i=1}^{K}\sum_{j\neq i}\lambda_{ij}$.

We assume that there is a single communication bus through which all network traffic flow. We define $\mathcal{L}_A$ to be the set of system elements. $\mathcal{L}_A$ contains the set of nodes as well as the communication bus $\mathcal{L}_A = \{1, ..., K, c\}$ (where c stands for the communication bus). Associated with each system element l and each class r, there is a cost $J_l^{(r)}(\mathbf{x})$. We allow for a more general cost $J^{(r)}(\mathbf{x})$ for each class r. We make the following assumptions on the cost, which ensure the existence of an equilibrium.

G1 $J^{(r)}$ is the sum of the local processing cost and the communication cost for class r where the latter are only functions of their local flow rate; i.e. we have

$$J^{(r)}(\mathbf{x}) = \sum_{i=1}^{K} J_i^{(r)}(\mathbf{x}) + J_c^{(r)}(\mathbf{x})$$

.

G2 $J_l^{(r)}$ are continuous functions whose range is the nonnegative quadrant and their image is $[0, \infty]$.

G3 $J_l^{(r)}$ are convex functions in the rate sent by class r over the system element l. For example, if l is a processor and $r \in \mathcal{K}_l$ then $J_l^{(r)}$ is assumed to be convex in $x_{ll}^{(r)}$.

G4 Whenever finite, $J_l^{(r)}$ is continuously differentiable in the flow sent by class r to system element l. We denote by $K_l^{(r)}(\mathbf{x})$ the partial derivative of $J_l^{(r)}(\mathbf{x})$ with respect to the flow sent by class r to system element l.

G5 If not all classes have finite cost and one of the classes has infinite cost then it can change its own flow to make this cost finite.

G5 ensures that any equilibrium has finite costs for all players.

Lemma 5.1 *Under conditions G1-G5 there exists an equilibrium.*

Proof: The proof follows from Theorem 1 in [14] (see also [13]). ∎

We introduce the following further assumptions on the cost that will be used to establish uniqueness.

(Π_1) $K_l^{(r)}$ is a function of two arguments: (i) the total flow on the system element l, and (ii) the flow that class r sends to node element l. $K_l^{(r)}$ is strictly increasing in each of its two arguments.

We note that the costs in the previous section indeed satisfy conditions (Π_1) and G1-G4.

For a fixed assignment of the other classes, class r is faced with a constrained minimization problem. Its associated Lagrangian is given by

$$\Lambda^{(r)}(\mathbf{x}) = J^{(r)}(\mathbf{x}) - \alpha^{(r)}\left(\sum_{j=1}^{K} x_{ij}^{(r)} - \phi^{(r)}\right).$$

$\mathbf{x}^* = \{x_{ij}^{(r)}, i, j, r\}$ is thus an equilibrium if and only if it satisfies the Kuhn Tucker conditions (see [15] pp. 158-165): There exist some real numbers $\alpha^{(r)}, r \in \mathcal{K}$ such that for $r \in \mathcal{K}_i$, $i = 1, 2$, $j \neq i$:

$$\begin{gathered} K_i^{(r)}(\mathbf{x}^*) \geq \alpha^{(r)} \text{ and} \\ K_i^{(r)}(\mathbf{x}^*) = \alpha^{(r)} \text{ if } x_{ii}^{(r)} > 0, \end{gathered} \tag{1}$$

$$\begin{gathered} K_c^{(r)}(\mathbf{x}^*) + K_j^{(r)}(\mathbf{x}^*) \geq \alpha^{(r)} \text{ and} \\ K_c^{(r)}(\mathbf{x}^*) + K_j^{(r)}(\mathbf{x}^*) = \alpha^{(r)} \text{ if } x_{ij}^{(r)} > 0; \end{gathered} \tag{2}$$

$$x_{ii}^{(r)}, x_{ij}^{(r)} \geq 0, \qquad \sum_{j=1}^{K} x_{ij}^{(r)} = \phi^{(r)}. \tag{3}$$

We shall consider in the rest of this subsection only the case of two nodes (i.e. $K = 2$) and a single bidirectional communicating link between them. (We allow for several classes to arrive at each one of the two nodes.)

Theorem 5.1 *The node optimization has a unique solution under assumptions* (Π_1) *and G1-G5.*

Proof: Let $\hat{\mathbf{x}}$ and $\mathbf{x}$ be two equilibria such that

$$\hat{\lambda} \geq \lambda. \tag{4}$$

Let i be such that $\hat{\beta}_i \geq \beta_i$. We shall show that for all $r \in \mathcal{K}_j$, $j \neq i$,

$$\hat{x}_{jj}^{(r)} \geq x_{jj}^{(r)}, \quad \text{or equivalently, } \hat{x}_{ji}^{(r)} \leq x_{ji}^{(r)} \tag{5}$$

(the equivalence follows from the constraint (3) on the sum of the flows.) (5) holds trivially if $x_{jj}^{(r)} = 0$, and so we have to check only the case $x_{jj}^{(r)} > 0$. To do so, fix some $r \in \mathcal{K}_j$ and consider the following two subcases. Assume that
(a) $\hat{\alpha}^{(r)} \geq \alpha^{(r)}$. Note that $\hat{\beta}_i \geq \beta_i$ is equivalent to $\hat{\beta}_j \leq \beta_j$ (the equivalence follows from the constraint (3)). Hence

$$\begin{gathered} K_j^{(r)}(\hat{x}_{jj}^{(r)}, \hat{\beta}_j) \geq \hat{\alpha}^{(r)} \geq \alpha^{(r)} \\ = K_j^{(r)}(x_{jj}^{(r)}, \beta_j) \geq K_j^{(r)}(x_{jj}^{(r)}, \hat{\beta}_j). \end{gathered} \tag{6}$$

The equality as well as the first inequality follow from the Kuhn Tucker conditions, whereas the last inequality follows from the monotonicity assumption (Π_1) . Using again the monotonicity assumption (Π_1) , this time for the first argument, we conclude from the fact $K_j^{(r)}(\hat{x}_{jj}^{(r)}, \hat{\beta}_j) \geq K_j^{(r)}(x_{jj}^{(r)}, \hat{\beta}_j)$ that (5) holds.

Thus we try instead of (a):
(b) $\hat{\alpha}^{(r)} \leq \alpha^{(r)}$. (5) holds trivially if $\hat{x}_{ji}^{(r)} = 0$. So it remains to check the case $\hat{x}_{ji}^{(r)} > 0$. We then have for $r \in \mathcal{K}_j$ ($j \neq i$)

$$\begin{aligned} & K_c^{(r)}(x_{ji}^{(r)}, \lambda) + K_i^{(r)}(x_{ji}^{(r)}, \beta_i) \geq \alpha^{(r)} \\ \geq \;& \hat{\alpha}^{(r)} = K_c^{(r)}(\hat{x}_{ji}^{(r)}, \hat{\lambda}) + K_i^{(r)}(\hat{x}_{ji}^{(r)}, \hat{\beta}_i) \\ \geq \;& K_c^{(r)}(\hat{x}_{ji}^{(r)}, \lambda) + K_i^{(r)}(\hat{x}_{ji}^{(r)}, \beta_i). \end{aligned}$$

Here, the first inequality and the equality follow from the Kuhn Tucker conditions, whereas the last inequality follows from the monotonicity of $K_l^{(r)}$ (property (Π_1)). Using again the monotonicity, we conclude that (5) holds in case (b) as well.

We conclude that

$$\sum_{r \in \mathcal{K}_j} \hat{x}_{ji}^{(r)} \leq \sum_{r \in \mathcal{K}_j} x_{ji}^{(r)} \quad (j \neq i). \tag{7}$$

Combining this with $\hat{\beta}_i \geq \beta_i$, we conclude that

$$\sum_{r \in \mathcal{K}_i} \hat{x}_{ii}^{(r)} \geq \sum_{r \in \mathcal{K}_i} x_{ii}^{(r)}.$$

However, since for $i \in \mathcal{K}_i$, $\hat{x}_{ii}^{(r)} + \hat{x}_{ij}^{(r)} = \phi^{(r)}$, it follows that

$$\sum_{r \in \mathcal{K}_i} \hat{x}_{ij}^{(r)} \leq \sum_{r \in \mathcal{K}_i} x_{ij}^{(r)}. \tag{8}$$

Combining this with (7) we conclude that $\hat{\lambda} \leq \lambda$. This contradicts our assumption (4), unless we have equality in (4).

This implies in particular that (5) holds (as we derived above). Now, if for some $r \in \mathcal{K}_j$ (5) holds with strict inequality then (7) and (8) would hold with strict inequality, which would imply that $\hat{\lambda} < \lambda$. But since we established that (4) holds with equality, we conclude that (5) holds with equality. By a symmetric argument we establish (5) also for $r \in \mathcal{K}_i$. We conclude that $\mathbf{x} = \hat{\mathbf{x}}$. ■

An interesting property that can be obtained from the above proof is that not only the Nash equilibrium is unique, but also:

Lemma 5.2 *There are unique Lagrange multipliers for the node optimization (that satisfy the Kuhn-Tucker conditions (1)-(3)) under assumptions* (Π_1) *, G1-G5.*

Proof: Assume that there are two sets of Lagrange multipliers, α and $\hat{\alpha}$ corresponding to the Nash equilibria $\mathbf{x}$ and $\hat{\mathbf{x}}$ (where $\mathbf{x} = \hat{\mathbf{x}}$ due to the uniqueness). Assume that there are some r, j for which $\hat{\alpha}^{(r)} > \alpha^{(r)}$, $r \in \mathcal{K}_j$. It follows that (6) holds with the first inequality being a strict one. Since K_j is strictly monotone in both arguments, it follows from (6) that $\hat{x}_{jj}^{(r)} > x_{jj}^{(r)}$. This contradicts, however, the uniqueness of the Nash equilibrium. ■

6 Concluding Remarks

In this paper, we showed the existence of some paradoxical and awkward behavior of the Nash equilibrium in a model of load balancing for distributed computer systems. We secured uniqueness of the overall and Wardrop optima. We extended the previous results and gave proofs of the uniqueness of the Nash equilibrium for the model. We saw that the Nash noncooperative and independent decision may sometimes lead to the degradation of performance for each member. Please note that this class optimum is with respect to a case of nonsymmetric user classes, whereas symmetric user classes are treated in [10, 11]. Such a paradoxical behavior does not occur for the overall and Wardrop optimum in the same setting of the model. That may imply that the Nash equilibrium may have more complicated characteristics than the overall optimum and further the Wardrop equilibrium.

References

[1] J. E. Cohen and F. P. Kelly, "A Paradox of Congestion in a Queuing Network," *J. Appl. Prob. 27*, pp.730-734, 1990.

[2] J. E. Cohen and C. Jeffries, "Congestion Resulting from Increased Capacity in Single-Server Queueing Networks," *IEEE/ACM Trans. on Networking 5*, 2, April pp.1220-1225, 1997.

[3] B. Calvert, W. Solomon and I. Ziedins, "Braess's Paradox in a Queueing Network with State-Dependent Routing," *J. Appl. Prob. 34,* pp.134-154, 1997.

[4] A. Haurie and P. Marcottet, "On the Relationship between Nash-Cournot and Wardrop Equilibria," *Networks 15*, 1985, pp.295-308.

[5] H. Kameda, E. Altman and T. Kozawa, *Braess-like Paradoxes of Nash Equilibria for Load Balancing in Distributed Computer Systems* , ISE Technical Report ISE-TR-98-157, Institute of information Sciences and Electronics, University of Tsukuba, 1998

[6] H. Kameda, T. Kozawa and J. Li, "Anomalous Relations among Various Performance Objectives in Distributed Computer Systems," *Proc. 1st World Congress on Systems Simulation*, IEEE, 1997, pp.459-465.

[7] H. Kameda, J. Li, C. Kim and Y. Zhang, *Optimal Load Balancing in Distributed Computer Systems*, Springer, 1997.

[8] H. Kameda and Y. Zhang, "Uniqueness of the Solution for Optimal Static Routing in Open BCMP Queueing Networks," *Mathematical and Computer Modelling 22*, 10-12, pp.119-130, 1995.

[9] C. Kim and H. Kameda, "An Algorithm for Optimal Static Load Balancing in Distributed Computer Systems," *IEEE Trans. Compt. 41*, 3, 1990, pp.381-384.

[10] Y. A. Korilis, A. A. Lazar, and A. Orda, A., "Architecting noncooperative networks," *IEEE Journal on Selected Areas in Communications 13*, 7, pp. 1241-1251, September 1995.

[11] Y. A. Korilis, A. A. Lazar and A. Orda, "Avoiding the Braess Paradox in Noncooperative Networks," *Proc. IEEE Conference on Decision & Control*, San Diego, 1997, pp.864-878.

[12] J. Li and H. Kameda, "Load Balancing Problems for Multiclass Jobs in Distributed/Parallel Computer Systems," *IEEE Trans. Compt. 47*, 3, 1998, pp.322-1998.

[13] A. Orda, R. Rom and N. Shimkin, "Competitive Routing in Multiuser Communication Networks," *IEEE/ACM Trans. on Networking 1*, 1993, pp.614-627.

[14] J. B. Rosen, "Existence and Uniqueness of Equilibrium Points for Concave N-person Games," *Econometrica 33*, pp. 153-163, 1965.

[15] J. F. Shapiro, *Mathematical Programming, Structures and Algorithms*, J. Wiley and Sons, 1979.

[16] A. N. Tantawi and D. Towsley, "Optimal Static Load Balancing in Distributed Computer Systems," *J. ACM 32*, 2, pp.445-465, Apr, 1985.