Continuum Equilibria for Routing in Dense Static Ad-hoc Networks

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Abstract We consider massively dense ad-hoc networks and study their continuum limits as the node density increases and as the graph providing the available routes becomes a continuous area with location and congestion dependent costs. We study both the global optimal solution as well as the non-cooperative routing problem among a large population of users where each user seeks a path from its source to its destination so as to minimize its individual cost. We seek for a (continuum version of the) Wardrop equilibrium. We first show how to derive meaningful cost models as a function of the scaling properties of the capacity of the network and of the density of nodes. We present various solution methodologies for the problem: (1) the viscosity solution of the Hamilton-Bellman-Jacobi equation, for the global optimization problem, (2) a method based on Green Theorem for the least cost problem of an individual, and (3) a solution of the Wardrop equilibrium problem using a transformation into an equivalent global optimization problem.

Keywords: Routing, Ad-hoc networks, equilibrium.

1 Introduction

In the design and analysis of wireless networks, researchers frequently stumble on the scalability problem that can be summarized in the following sentence: "As the number of nodes in the network increases, problems become harder to solve" [26]. The sentence takes its meaning from several issues. Some examples are the following:

- In Routing: As the network size increases, routes consists of an increasing number of nodes, and so they are increasingly susceptible to node mobility and channel fading [22].
- In Transmission Scheduling: The determination of the maximum number of non-conflicting transmissions in a graph is a NP-complete problem [29].
- In Capacity of Wireless Networks: As the number of nodes increases, the determination of the precise capacity becomes an intractable problem.

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Nevertheless when the system is sufficiently large, one may hope that a macroscopic model will give a better description of the network and that one could predict its properties from microscopic considerations. Indeed we are going to sacrifice some details, but this macroscopic view will preserve sufficient information to allow a meaningful network optimization solution and the derivation of insightful results in a wide range of settings.

The physics-inspired paradigms used for the study of large ad-hoc networks go way beyond those related to statistical-mechanics in which macroscopic properties are derived from microscopic structure. Starting from the pioneering work by Jacquet (see [17]) in that area, a number of research groups have worked on massively dense ad-hoc networks using tools from geometrical optics [17][1] as well as electrostatics (see e.g. [26, 25, 13], and the survey [27] and references therein). We shall describe these in the next sections.

The physical paradigms allow the authors to minimize various metrics related to the routing. In contrast, Hyytia and Virtamo propose in [15] an approach based on load balancing arguing that if shortest path (or cost minimization) arguments were used, then some parts of the network would carry more traffic than others and may use more energy than others. This would result in a shorter lifetime of the network since some parts would be out of energy earlier than others and earlier than any part in a load balanced network.

The term "massively dense" ad-hoc networks is used to indicate not only that the number of nodes is large, but also that the network is highly connected. By the term "dense" we further understand that for every point in the plain there is a node close to it with high probability; by "close" we mean that its distance is much smaller than the transmission range. In this paper and in previous work (cited in the next paragraphs) one actually studies the limitting properties of massively dense ad-hoc networks, as the density of nodes tends to infinity.

The development of the original theory of routing in massively dense networks among the community of ad-hoc networks has emerged in a complete independent way of the existing theory of routing in massively dense networks which had been developed within the community of road traffic engineers. Indeed, this approach had already been introduced in 1952 by Wardrop [30] and by Beckmann [4] and is still an active research area among that community, see [6, 7, 14, 16, 32] and references therein. We combine in this paper various approaches from this area as well as from optimal control theory in order to formulate models for routing in massively dense networks. We further propose simple novel approach to that problem using a classical device of 2-D. singular optimal control [19] based on Green's formula to obtain a simple characterization of least cost paths of individual packets. We end the paper by a numerical example for computing an equilibrium.

We consider in this paper static networks (say sensor networks) characterized by communications through horizontally and vertically oriented directional antennas. The use of directional antennas allows one to save energy and to use it in an efficient way which may result in a longer life time of the network.

The structure of this paper is as follows. We begin by presenting models for costs relevant to optimization models in routing or to node assignment. We then formulate the global optimization problem and the individual optimization one with a focus on the directional antennas scenario. We provide several approaches to obtain both qualitative characterization as well as quantitative solutions to the problems.

[1] We note that this approach is restricted to costs that do not depend on the congestion

2 Determining routing costs in dense ad-hoc networks

In optimizing a routing protocol in ad-hoc networks, or in optimizing the placement of nodes, one of the starting points is the determination of the cost function. To determine it, we need a detailed specification of the network which includes the following:

- A model for the placement of nodes in the network.
- A forward rule that nodes will use to select the next hop of a packet.
- A model for the cost incurred in one hop, i.e. for transmitting a packet to an intermediate node.

Below we present several ways of choosing cost functions.

2.1 Costs derived from capacity scaling

Many models have been proposed in the literature that show how the transport capacity scales with the number of nodes n or with their density λ. Assume that we use a protocol that provides a transport capacity of the order of $f(\lambda)$ at some region in which the density of nodes is λ. A typical cost (see e.g. [25]) at a neighborhood of $\mathbf{x}$ is the density of nodes required there to carry a given flow. Assuming that a flow[2] $\mathbf{T}(\mathbf{x})$ is assigned through a neighborhood of $\mathbf{x}$, the cost is taken to be

$$c(\mathbf{x}, \mathbf{T}(\mathbf{x})) = f^{-1}(|\mathbf{T}(\mathbf{x})|) \tag{1}$$

where $|\cdot|$ represents the norm of a vector.

Examples for f:

- Using a network theoretic approach based on multi-hop communication, Gupta and Kumar prove in [12] that the throughput of the system that can be transported by the network when the nodes are optimally located is $\Omega(\sqrt{\lambda})$, and when the nodes are randomly located this throughput becomes $\Omega(\frac{\sqrt{\lambda}}{\sqrt{\log \lambda}})$. Using percolation theory, the authors of [9] have shown that in the randomly located set the same $\Omega(\sqrt{\lambda})$ can be achieved.
- Baccelli, Blaszczyszyn and Mühlethaler introduce in [2] an access scheme, MSR (Multi-hop Spatial Reuse Aloha), reaching the Gupta and Kumar bound $O(\sqrt{\lambda})$ which does not require prior knowledge of the node density.
- A protocol introduced by Tse and Glosglauser [10] has a capacity that scales as $O(\lambda)$. However, it does not fall directly within the class of massively dense ad-hoc networks and indeed, it relies on mobility and on relaying for handling disconnectivity.

[2] We denote with bold font the vectors.

We conclude that for the model of Gupta and Kumar with either the optimal location or the random location approaches, as well as for the MSR protocol with a Poisson distribution of nodes, we obtain a quadratic cost of the form

$$c(\mathbf{T}(\mathbf{x})) = k|\mathbf{T}(\mathbf{x})|^2 \tag{2}$$

This follows from (1) as $f(x)$ behaves like $\sqrt{x}$ so its inverse is quadratic.

2.2 Congestion independent routing

A metric often used in the Internet for determining routing is the number of hops, which routing protocol try to minimize. The number of hops is proportional to the expected delay along the path in the context of ad-hoc networks, in case that the queuing delay is negligible with respect to the transmission delay over each hop. This criterion is insensitive to interference or congestion. We assume that it depends only on the transmission range. We describe various cost criteria that can be formulated with this approach.

- If the range were constant then the cost density $c(\mathbf{x})$ is constant so that the cost of a path is its length in meters. The routing then follows a shortest path selection.
- Let us assume that the range $R(\mathbf{x})$ depends on local radio conditions at a point $\mathbf{x}$ (for example, if it is influenced by weather conditions) but not on interference. The latter is justified when dedicated orthogonal channels (e.g. in time or frequency) can be allocated to traffic flows that would otherwise interfere with each other. Then determining the routing becomes a path cost minimization problem. We further assume, as in Gupta and Kumar, that the range is scaled to go to 0 as the total density λ of nodes grows to infinity. More precisely, let us consider a scaling of the range such that the following limit exists:
$$r(\mathbf{x}) := \lim_{\lambda\to\infty} \frac{R^\lambda(\mathbf{x})}{\lambda}$$
Then in the dense limit, the fraction of nodes that participate in forwarding packets along a path is $1/r(\mathbf{x})$ and the path cost is the integral of this density along the path.
- The influence of varying radio conditions on the range can be eliminated using power control that can equalize the hop distance.

2.3 Costs related to energy consumption

In the absence of capacity constraints, the cost can represent energy consumption. In a general multi-hop ad-hoc network, the hop distance can be optimized so as to minimize the energy consumption. Even within a single cell of 802.11 IEEE wireless LAN one can improve the energy consumption by using multiple hops, as it has been shown not to be efficient in terms of energy consumption to use a single hop [20].

Alternatively, the cost can take into account the scaling of the nodes (as we had in Subsection 2.1) that is obtained when there are energy constraints. As an example, assuming random deployment of nodes, where each node has data to send to another randomly selected node, the capacity (in bits per Joule) has the form $f(\lambda) = \Omega\left((\lambda/\log\lambda)^{(q-1)/2}\right)$ where q is the path-loss, see [21]. The cost is then obtained using (1).

3 Preliminary

In the work of Toumpis et al. ([26, 25, 13, 28, 27]), the authors address the problem of the optimal deployment of Wireless Sensor Networks by a parallel with Electrostatic.

Consider in the two dimensional plane $X_1 \times X_2$, the continuous **information density function** $\rho(\mathbf{x})$, measured in bps/m^2, such that at locations $\mathbf{x}$ where $\rho(\mathbf{x}) > 0$ there is a distributed data source such that the rate with which information is created in an infinitesimal area of size $d\Omega$ centered at $\mathbf{x}$ is $\rho(\mathbf{x})d\Omega$. Similarly, at locations $\mathbf{x}$ where $\rho(\mathbf{x}) < 0$ there is a distributed data sink such that the rate with which information is absorbed by an infinitesimal area of size $d\Omega$, centered at point $\mathbf{x}$, is equal to $-\rho(\mathbf{x})d\Omega$.

The total rate at which sinks must absorb data is the same as the total rate which the data is created at the sources, i.e.

$$\int_{X_1 \times X_2} \rho(\mathbf{x})\mathrm{d}S = 0.$$

Next we present the flow conservation condition (see e.g. [25, 6] for more details). For information to be conserved over a domain Ω_0 of arbitrary shape on the $X_1 \times X_2$ plane, (but with smooth boundary) it is necessary that the rate with which information is created in the area is equal to the rate with which information is leaving the area, i.e

$$\int_{\Omega_0} \rho(\mathbf{x})\mathrm{d}\mathbf{x} = \oint_{\partial\Omega_0} [\mathbf{T}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})]\mathrm{d}\ell$$

The integral on the left is the surface integral of $\rho(\mathbf{x})$ over Ω_0. The integral on the right is the path integral of the inner product $\mathbf{T} \cdot \mathbf{n}$ over the curve $\partial\Omega_0$. The vector $\mathbf{n}(\mathbf{x})$ is the unit normal vector to $\partial\Omega_0$ at the boundary point $\mathbf{x} \in \partial\Omega_0$ and pointing outwards. The function $\mathbf{T}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$ measured in bps/m^2 is equal at the rate with which information is leaving the domain Ω_0 per unit length of boundary at the boundary point $\mathbf{x}$.

This holding for *any* (smooth) domain Ω_0, it follows that necessarily

$$\nabla \cdot \mathbf{T}(\mathbf{x}) := \frac{\partial T_1(\mathbf{x})}{\partial x_1} + \frac{\partial T_2(\mathbf{x})}{\partial x_2} = \rho(\mathbf{x}), \tag{3}$$

where "$\nabla\cdot$" is the divergence operator.

Extension to multi-class The work on massively dense ad-hoc networks considered a single class of traffic. In the geometrical optics approach it corresponded to demand from a point $\mathbf{a}$ to a point $\mathbf{b}$. In the electrostatic case it corresponded to a set of origins and a set of destinations where traffic from any origin point could go to any destination point. The analogy to positive and negative charges in electrostatics may limit the perspectives of multi-class problems where traffic from distinct origin sets has to be routed to distinct destination sets.

The model based on geometrical optics can directly be extended to include multiple classes as there are no elements in the model that suggest coupling between classes. This is due in particular to the fact that the cost density has been assumed to depend only on the density of the mobiles and not on the density of the flows.

In contrast, the cost in the model based on electrostatics is assumed to depend both on the location as well as on the local flow density. It thus models more complex interactions that would occur if we considered the case of ν traffic classes. Extending the relation (3) to the multi-class case, we have traffic conservation at each point in space to each traffic class as expressed in the following:

$$\nabla \cdot \mathbf{T}^j(\mathbf{x}) = \rho^j(\mathbf{x}), \qquad \forall \mathbf{x} \in \Omega. \tag{4}$$

The function $\mathbf{T}^j$ is the flow distribution of class j and ρ^j corresponds to the distribution of the external sources and/or sinks.

Let $\mathbf{T}(\mathbf{x})$ be the total flow vector at point $\mathbf{x} \in \Omega$. A generic multi-class optimization problem would then be: minimize Z over the flow distributions $\{T_i^j\}$

$$Z = \int_\Omega g(\mathbf{x}, \mathbf{T}(\mathbf{x})) \mathrm{d}x_1 \mathrm{d}x_2 \quad \text{subject to} \quad \nabla \cdot \mathbf{T}^j(\mathbf{x}) = \rho^j(\mathbf{x}), \; j = 1, ..., \nu \qquad \forall \mathbf{x} \in \Omega. \tag{5}$$

4 Directional Antennas and Global Optimization

Unlike the previous work that we described on massively dense ad-hoc networks, we introduce a model that uses directional antennas. The approach that we follow is inspired by the work of Dafermos (see [6]) on road traffic. An alternative approach based on road traffic tools can be found in [1, 23].

For energy efficiency, it is assumed that each terminal is equipped with one or with two directional antennas, allowing transmission at each hop to be directed either from North to South or from West to East. The model we use extends that of [6] to the multi-class framework. We thus consider ν classes of flows $T_1^j \geq 0$, $T_2^j \geq 0$, $j = 1, ..., \nu$. To be compatible with Dafermos [6], we use her definitions of orientation according to which the directions North to South and West to East are taken positive. In the dense limit, a curved path can be viewed as a limit of a path with many such hops as the hop distance tends to zero.

Some assumptions on the cost:

- **Individual cost:** We allow the cost for a horizontal (West-East) transmission from a point $\mathbf{x}$ to be different than the cost for a vertical transmission (North-South). It is assumed that a packet located at the point $\mathbf{x}$ and traveling in the direction of the axis x_i incurs a **transportation cost** g_i and such transportation cost depends upon the position $\mathbf{x}$ and the traffic flow $\mathbf{T}(\mathbf{x})$. We thus allow for a vector valued cost $\mathbf{g} := \mathbf{g}(\mathbf{x}, \mathbf{T}(\mathbf{x}))$.

- The local cost corresponding to the global optimization problem is given by $g(\mathbf{x}, \mathbf{T}(\mathbf{x})) = \mathbf{g}(\mathbf{x}, \mathbf{T}(\mathbf{x})) \cdot \mathbf{T}(\mathbf{x})$ if it is perceived as the sum of costs of individuals.
- The global cost will be the integral of the local cost density.
- The local cost $g(\mathbf{x}, \mathbf{T}(\mathbf{x}))$ is assumed to be non-negative, convex increasing in each of the components of $\mathbf{T}$ (T_1 and T_2 in our 2-dimensional case).

The **boundary conditions** will be determined by the options that travelers have in selecting their origins and/or destinations. Examples of the boundary conditions are:

- *Assignment problem*: users of the network have predetermined origins and destinations and are free to choose their travel paths.
- *Combined distribution and assignment problem*: users of the network have predetermined origins and are free to choose their destinations (within a certain destination region) as well as their paths.
- *Combined generation, distributions and assignment problem*: users are free to choose their origins, their destinations, as well as their travel paths.

The problem formulation is again to minimize Z as defined in (5). The natural choice of functional spaces to make that problem precise, and take advantage of the large body of theory developped with Sobolev spaces in the PDE community, is to seek T_i^j in $L^2(\Omega)$, so that ρ may be in $H^{-1}(\Omega)$, allowing for some localized mass of traffic source or sink.

Kuhn-Tucker conditions. Define the Lagrangian as

$$L^\zeta(\mathbf{x}, \mathbf{T}) := \int_\Omega \ell^\zeta(\mathbf{x}, \mathbf{T})\, \mathrm{d}\mathbf{x} \quad \text{where } \ell^\zeta(\mathbf{x}, \mathbf{T}) = g(\mathbf{x}, \mathbf{T}(\mathbf{x})) - \sum_{j=1}^{\nu} \zeta^j(\mathbf{x}) \Big[\nabla\cdot \mathbf{T}^j(\mathbf{x}) - \rho^j(\mathbf{x})\Big]$$

where the $\zeta^j(\mathbf{x}) \in H^1(\Omega)$ are Lagrange multipliers. The criterion is convex, and the constraint (4) affine. Therefore the Kuhn-Tucker theorem holds, stating that the Lagrangian is minimum at the optimum. A variation $\delta\mathbf{T}(\cdot)$ will be admissible if $\mathbf{T}(\mathbf{x}) + \delta\mathbf{T}(\mathbf{x}) \geq 0$ for all $\mathbf{x}$, hence in particular, $\forall \mathbf{x} : T_i^j(\mathbf{x}) = 0$, $\delta T_i^j(\mathbf{x}) \geq 0$.

Let $\mathrm{D}L^\zeta$ denote the Gteaux derivative of L^ζ w.r. to $T(\cdot)$. Euler's inequality reads

$$\forall \delta\mathbf{T} \text{ admissible}\,, \mathrm{D}L^\zeta \cdot \delta\mathbf{T} \geq 0\,,$$

therefore here

$$\int_\Omega \sum_j \langle \nabla_{\mathbf{T}^j} g(\mathbf{x}, \mathbf{T}(\mathbf{x})), \delta\mathbf{T}^j(\mathbf{x})\rangle\, \mathrm{d}\mathbf{x} - \int_\Omega \sum_j \zeta^j(\mathbf{x}) \nabla\cdot\delta\mathbf{T}^j(\mathbf{x})\, \mathrm{d}\mathbf{x} \geq 0.$$

Integrating by parts with Green's formula, this is equivalent to

$$\int_\Omega \sum_j \left[\langle \nabla_{\mathbf{T}^j} g, \delta\mathbf{T}^j\rangle + \langle \nabla_{\mathbf{x}} \zeta^j, \delta\mathbf{T}^j\rangle\right] \mathrm{d}\mathbf{x} - \int_{\partial\Omega} \sum_j \zeta^j \langle \delta\mathbf{T}^j, \mathbf{n}\rangle\, \mathrm{d}\ell \geq 0\,.$$

We may choose all the $\delta\mathbf{T}^k = 0$ except $\delta\mathbf{T}^j$, and choose that one in $(H_0^1(\Omega))^2$, *i.e.* such that the boundary integral be zero. This is always feasible and admissible. Then the last term above vanishes, and it is a classical fact that the inequality implies for $i = 1, 2$:

$$\frac{\partial g(\mathbf{x},\mathbf{T})}{\partial T_i^j} + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} = 0 \quad \text{if } T_i^j(\mathbf{x}) > 0 \tag{6a}$$

$$\frac{\partial g(\mathbf{x},\mathbf{T})}{\partial T_i^j} + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} \geq 0 \quad \text{if } T_i^j(\mathbf{x}) = 0. \tag{6b}$$

Placing this back in Euler's inequality, and using a $\delta\mathbf{T}^j$ non zero on the boundary, it follows that necessarily [3] $\zeta^j(\mathbf{x}) = 0$ at any $\mathbf{x}$ of the boundary $\partial\Omega$ where $T(\mathbf{x}) > 0$. This provides the boundary condition to recover ζ^j from the condition (4).

Remark: The Kuhn-Tucker type characterization (6a)-(6b) is already stated in [6] for the single class case. However, as Dafermos states explicitly, its rigorous derivation is not available there.

Consider the following special cases that we shall need later. We assume a single traffic class, but this could easily be extended to several. Let

$$g(\mathbf{x},\mathbf{T}(\mathbf{x})) = \sum_{i=1,2} g_i(\mathbf{x},\mathbf{T}(\mathbf{x}))T_i(\mathbf{x}).$$

1. Monomial cost per packet:

   $$g_i(\mathbf{x},\mathbf{T}(\mathbf{x})) = k_i(\mathbf{x})\Big(T_i(\mathbf{x})\Big)^{\beta} \tag{7}$$

   for some $\beta > 1$. Then (6a)-(6b) simplify to

   $$(\beta+1)k_i(\mathbf{x})\,(T_i(\mathbf{x}))^{\beta} + \frac{\partial \zeta(\mathbf{x})}{\partial x_i} = 0 \quad \text{if } T_i(\mathbf{x}) > 0 \tag{8a}$$

   $$(\beta+1)k_i(\mathbf{x})\,(T_i(\mathbf{x}))^{\beta} + \frac{\partial \zeta(\mathbf{x})}{\partial x_i} \geq 0 \quad \text{if } T_i(\mathbf{x}) = 0. \tag{8b}$$

   In that case, recovery of ζ to complete the process is difficult, at best. Things are simpler in the next case.

2. Affine cost per packet:

   $$g_i(\mathbf{x},\mathbf{T}(\mathbf{x})) = \frac{1}{2}k_i(\mathbf{x})T_i(\mathbf{x}) + h_i(\mathbf{x}). \tag{9}$$

   Then (6a)-(6b) simplify to

   $$k_i(\mathbf{x})T_i(\mathbf{x}) + h_i(\mathbf{x}) + \frac{\partial \zeta(\mathbf{x})}{\partial x_i} = 0 \quad \text{if } T_i(\mathbf{x}) > 0$$

   $$k_i(\mathbf{x})T_i(\mathbf{x}) + h_i(\mathbf{x}) + \frac{\partial \zeta(\mathbf{x})}{\partial x_i} \geq 0 \quad \text{if } T_i(\mathbf{x}) = 0.$$

[3] This is a complementary slackness condition on the boundary.

Assume that the $k_i(\cdot)$ are everywhere positive and bounded away from 0. For simplicity, let $a_i = 1/k_i$, and b be the vector with coordinates $b_i = h_i/k_i$, all assumed to be square integrable. Assume that there exists a solution where $T(\mathbf{x}) > 0$ for all $\mathbf{x}$. Then

$$T_i(\mathbf{x}) = -\left(a_i(\mathbf{x})\frac{\partial\zeta(\mathbf{x})}{\partial x_i} + b_i(\mathbf{x})\right).$$

As a consequence, from (4) and the above remark, we get that $\zeta(\cdot)$ is to be found as the solution in $H_0^1(\Omega)$ of the elliptic equation (an equality in $H^{-1}(\Omega)$)

$$\sum_i \frac{\partial}{\partial x_i}\left(a_i(\mathbf{x})\frac{\partial\zeta}{\partial x_i}\right) + \nabla\cdot b(\mathbf{x}) + \rho(\mathbf{x}) = 0\,.$$

This is a well behaved Dirichlet problem, known to have a unique solution in $H_0^1(\Omega)$, furthermore easy to compute numerically.

5 User optimization and congestion independent costs

We expand on the shortest path approach for optimization that has already appeared using geometrical optics tools [17]. We present general optimization frameworks for handling shortest path problems and more generally, minimum cost paths.

We consider the model of Section 4. We assume that the local cost depends on the direction of the flow but not on its size. The cost is $c_1(\mathbf{x})$ for a flow that is locally horizontal and is $c_2(\mathbf{x})$ for a flow that is locally vertical. We assume in this section that c_1 and c_2 do not depend on $\mathbf{T}$. The cost incurred by a packet transmitted along a path p is given by the line integral

$$\mathbf{c}_p = \int_p \mathbf{c}\cdot\mathbf{dx}. \tag{11}$$

Let $V^j(\mathbf{x})$ be the minimum cost to go from a point $\mathbf{x}$ to a set B^j, $j = 1, ..., \nu$. Then

$$V^j(\mathbf{x}) = \min\left(c_1(\mathbf{x})\mathrm{d}x_1 + V^j(x_1+\mathrm{d}x_1, x_2), c_2(\mathbf{x})\mathrm{d}x_2 + V^j(x_1, x_2+\mathrm{d}x_2)\right) \tag{12}$$

This can be written as

$$0 = \min\left(c_1(\mathbf{x}) + \frac{\partial V^j(\mathbf{x})}{\partial x_1}, c_2(\mathbf{x}) + \frac{\partial V^j(\mathbf{x})}{\partial x_2}\right), \qquad \forall \mathbf{x}\in B^j\,,\ V^j(\mathbf{x}) = 0\,. \tag{13}$$

If V^j is differentiable then, under suitable conditions, it is the unique solution of (13). In the case that V^j is not everywhere differentiable then, under suitable conditions, it is the unique viscosity solution of (13) (see [3, 8]).

There are many numerical approaches for solving the HJB equation. One can discretize the HJB equation and obtain a discrete dynamic programming for which efficient solution methods exist. If one repeats this for various discretization steps, then we know that the solution of the discrete problem converges to the viscosity solution of the original problem (under suitable conditions) as the step size converges to zero [3].

6 Geometry of minimum cost paths

We begin by introducing the standard attribute (plus or minus) to a path according to the direction of the movement along it. The definition is different than in [6] (which we used in Section 4).

Definition 6.1 *[18]. (i) Let C be some simple closed curve surrounding some region R. Then C^+ corresponds to a counterclockwise movement; more precisely, it corresponds to moving so that the region R is to our left. The opposite orientation along C is denoted by C^-.*
(ii) The orientation of path segments which are not closed are defined differently. A "plus" indicates an orientation of left to right or bottom to top, and the "minus" indicates curves oriented from right to left or from top to bottom.

We consider now our directional antenna model in a given rectangular area R on a region Ω, defined by the simple closed curve $\partial R^+ = \Gamma_1^+ \cup \Gamma_2^+ \cup \Gamma_3^- \cup \Gamma_4^-$ (see Fig. 1).

We obtain below **optimal paths** defined as paths that achieve the minimum cost in (11). We shall study two problems:

- **Point to point optimal path:** we seek the minimum cost path between two points.
- **Point to boundary optimal path:** we seek the minimum cost path on a given region that starts at a given point and is allowed to end at any point on the boundaries.

Define the function

$$U(\mathbf{x}) = \frac{\partial c_2}{\partial x_1}(\mathbf{x}) - \frac{\partial c_1}{\partial x_2}(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega.$$

It will turn out that the structure of the minimum cost path depends on the costs through the sign of the function U. Now, if the function $\mathbf{c} \in \mathcal{C}^1(\Omega)$ then U is a continuous function on Ω. This motivates us to study cases in which U has the same sign everywhere (see Fig. 2), or in which there are two regions in R, one with $U > 0$ and one with $U < 0$, separated by a curve on which $U = 0$ (e.g. Fig. 3).

$$\begin{aligned}
\Gamma_1^+ &= \{0 \le x_1 \le a, \quad x_2 = 0\} \\
\Gamma_2^+ &= \{x_1 = a, \quad 0 \le x_2 \le b\} \\
\Gamma_3^- &= \{0 \le x_1 \le a, \quad x_2 = b\} \\
\Gamma_4^- &= \{x_1 = 0, \quad 0 \le x_2 \le b\}.
\end{aligned}$$

Figure 1: The boundaries of the region R.

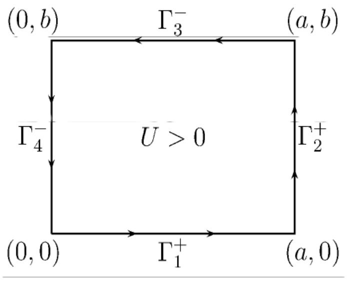

Figure 2: The region R. The case where $U > 0$.

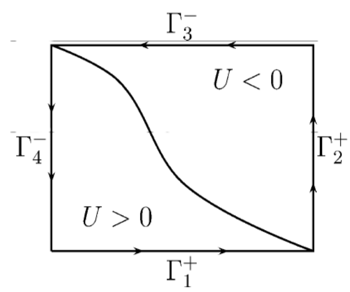

Figure 3: The case of two regions separated by a curve. Case 1.

We shall assume throughout that the function $\mathbf{c} \in \mathcal{C}^1(\Omega)$, and that, if non empty, the set $M = \{\mathbf{x} \mid U(\mathbf{x}) = 0\}$ is a smooth line. (This is true, e.g., if $\mathbf{c} \in \mathcal{C}^2$ and $\nabla U \neq 0$ on M.)

6.1 The function U has the same sign over the whole region R

Theorem 6.1 *(Point to point optimal path) Suppose that a point* $\mathbf{x}^O = (x_1^O, x_2^O)$ *in* $\mathring{R}$ *(the interior of* R*), wants to send a packet to a point* $\mathbf{x}^D = (x_1^D, x_2^D)$ *in* $\mathring{R}$*.*

i. If $U > 0$ *in the region* $R_{OD} = \{(x_1, x_2) \text{ such that } x_1^O \le x_1 \le x_1^D, x_2^O \le x_2 \le x_2^D\}$*, except perhaps from a set of Lebesgue measure zero, then there is an optimal path given by (see Fig. 4):*

$$\gamma^{opt} = \gamma_H \cup \gamma_V \text{ where}$$

$$\gamma_H = \{(x_1, x_2) \text{ such that } x_1^O \le x_1 \le x_1^D, x_2 = x_2^O\}$$
$$\gamma_V = \{(x_1, x_2) \text{ such that } x_1 = x_1^D, x_2^O \le x_2 \le x_2^D\}.$$

ii. If $U < 0$ *in that region except perhaps from a set of Lebesgue measure zero, then there is an optimal path given by (see Fig. 5):*

$$\gamma^{opt} = \gamma_V \cup \gamma_H \text{ where}$$

$$\gamma_V = \{(x_1, x_2) \text{ such that } x_1 = x_1^O, x_2^O \le x_2 \le x_2^D\}$$
$$\gamma_H = \{(x_1, x_2) \text{ such that } x_1^O \le x_1 \le x_1^D, x_2 = x_2^D\}.$$

iii. In both cases, γ^{opt} *is unique up to a zero Lebesgue measure. (i.e. the Lebesgue measure of the area between* γ^{opt} *and any other optimal path is zero).*

Proof.- Consider an arbitrary path[4] γ_C joining $\mathbf{x}^O$ to $\mathbf{x}^D$, and assume that the Lebesgue measure of the area between γ^{opt} and γ_C is nonzero. We call such path, the comparison path (see Fig. 4 for the case $U > 0$ and Fig. 5 for $U < 0$).

(i) Showing that the cost over path γ^{opt} is optimal is equivalent to showing that the integral of the cost over the closed path ξ^- is negative, where ξ^- is given by following γ^{opt} from the source $\mathbf{x}^O$ to the destination $\mathbf{x}^D$ and then returning from $\mathbf{x}^D$ to $\mathbf{x}^O$ by moving along the path γ_C in the reverse direction. This closed path is written as $\xi^- = \gamma_H^+ \cup \gamma_V^- \cup \gamma_C^+$ and Ω_1 denotes the bounded area described by ξ^-. Using Green Theorem (see Appendix) we obtain

$$\oint_{\xi^-} \mathbf{c} \cdot \mathbf{dx} = -\int_{\Omega_1} U(\mathbf{x}) dS$$

which is strictly negative since $U > 0$ a.e. on R. Decomposing the left integral, this concludes the proof of (i), and establishes at the same time the corresponding statement on uniqueness in (iii).
(ii) is obtained similarly. ■

[4]Respecting that each subpath can be decomposed in sums of paths either from North to South or from West to East (or is a limit of such paths). From now on, we will call a path valid if it satisfies that condition.

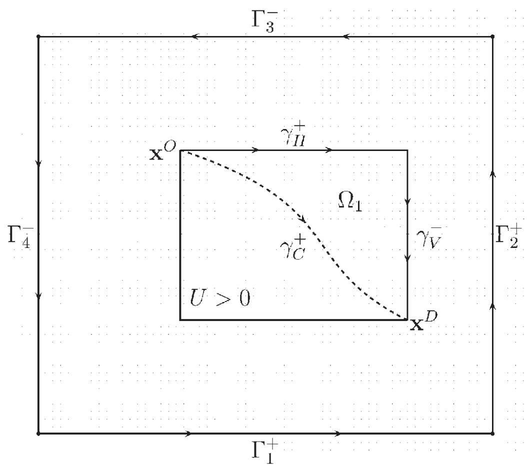

Figure 4: Optimal path for $U > 0$.

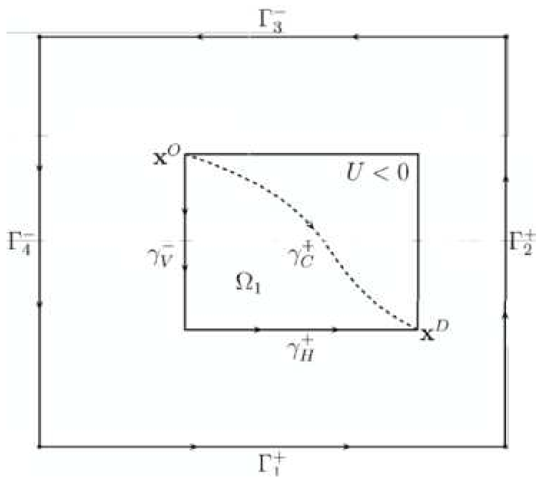

Figure 5: Optimal path for $U < 0$.

Theorem 6.2 *(Point to boundary optimal path)*
Consider the problem of finding an optimal path from a point $\bar{\mathbf{x}} \in \mathring{R}$ to the boundary $\Gamma_1 \cup \Gamma_2$.

i. Assume that $U(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathring{R}$ except perhaps for a set of Lebesgue measure zero. Assume that the cost on Γ_1 is non-negative and that the cost on Γ_2 is non-positive. Then the optimal path is the straight vertical line.

ii. Assume that $U(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathring{R}$ except perhaps for a set of Lebesgue measure zero. Assume that the cost on Γ_1 is non-positive and that the cost on Γ_2 is non-negative. Then the optimal path is the straight horizontal line.

Proof.-

(i) Denote by γ_V the straight vertical path joining $\bar{\mathbf{x}}$ to Γ_1. Consider another arbitrary valid path γ_C joining $\bar{\mathbf{x}}$ to any point $\mathbf{x}^*$ on $\Gamma_1 \cup \Gamma_2$, and assume that the Lebesgue measure of the area between γ^{opt} and γ_C is nonzero. We call such path, the comparison path.

Assume first that $\mathbf{x}^*$ is on Γ_2. Denote $\mathbf{x}^D := \Gamma_1 \cap \Gamma_2$. Then by Theorem 6.1 (ii), the cost to go from $\bar{\mathbf{x}}$ to $\mathbf{x}^D$ is smaller when using γ_V^- and then continuing eastwords (along Γ_1^+) than when using γ_C^+ and then southwords (along Γ_2^*). Due to our assumptions on the costs over the boundaries, this implies that the cost along γ_V is smaller than along γ_C.

Next consider the case where $\mathbf{x}^*$ is on Γ_1. Denote by η the section of the boundary Γ_1 that joins $\gamma_V \cap \Gamma_1$ with $\mathbf{x}^*$ (see Figure 6). Then again, by Theorem 6.1 (ii), the cost to go from $\bar{\mathbf{x}}$ to $\mathbf{x}^*$ is smaller when using γ_V^- and then continuing eastwords (along Γ_1^+) than when using γ_C^+. Due to our assumptions that the cost on Γ_1 is non-negative, this implies that the cost along γ_V is smaller than along γ_C.

(ii) is obtained similarly. ■

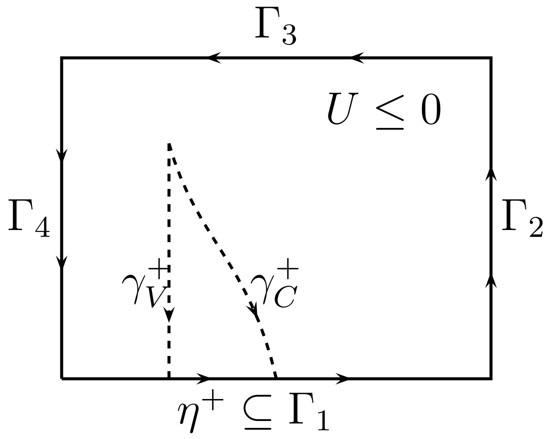

Figure 6: Theorem 6.2 (i)

6.2 The function U changes sign within the region R

Consider the region on the space $M := \{\mathbf{x} \in \Omega \text{ such that } U(\mathbf{x}) = 0\}$. Let us consider the case when M is only a valid path in the rectangular area, such that it starts at the intersection $\Gamma_3 \cap \Gamma_4$, and finishes at the intersection of the sinks $\Gamma_1 \cap \Gamma_2$. Then the space is divided in two areas, and as the function U is continuous we have the following cases:

1. $U(\mathbf{x})$ is negative in the upper area and positive in the lower area (see Fig. 3).
2. $U(\mathbf{x})$ is positive in the upper area and negative in the lower area (see Fig. 7).

Two other cases where the sign of U is the same over Ω are contained in what we solved in the previous section (allowing U to be zero on M which has Lebesgue measure zero)

Case 1: The function $U(\mathbf{x})$ is negative in the upper area and positive in the lower area.

We shall show that this case, M is an atractor.

Proposition 6.1 *Assume that the source* $\mathbf{x}$ *and destination* $\mathbf{y}$ *are both on* M*. Then the path* p_M *that follows* M *is optimal.*

Proof.- Consider an alternative path γ_C that coincides with M only in the source and destination points. First assume γ_C is entirely in the upper (i.e. northen) part and call Ω_1 the surrounded area. Define ξ^+ to be the closed path that follows p_M from $\mathbf{x}$ to $\mathbf{y}$ and then returns along γ_C.

The integral $\int_{\Omega_1} U(\mathbf{x})dS$ is negative by assumption. By Green Theorem it equals $\oint_{\xi^+} \mathbf{c} \cdot \mathbf{dx}$. This implies that the cost along p_M is strictly smaller than along γ_C.

A similar argument holds for the case that γ_C is below p_M.

A path between $\mathbf{x}$ and $\mathbf{y}$ may have several intersections with M. Between each pair of consecutive intersections of M, the subpath has a cost larger than that obtained by following

M between these points (this follows from the previous steps of the proof). We conclude that p_M is indeed optimal. ■

Proposition 6.2 *Let a point* $\bar{\mathbf{x}}^O$ *send packets to a point* $\mathbf{x}^D$*.*

i. Assume both points in the upper region. Denote by γ_1 *the two segments path given in Theorem 6.1 (ii). Then the curve* $\widehat{\gamma}$ *obtained as the maximum between* M *and* γ_1 *is optimal.*[5]

ii. Let both points be in the lower region. Denote by γ_2 *the two segments path given in Theorem 6.1 (i). Then the curve* $\overline{\gamma}$ *obtained as the minimum between* M *and* γ_2 *is optimal.*

Proof.- (i) A straightforward adaptation of the proof of the previous proposition implies that the path in the statement of the proposition is optimal among all those restricted to the upper region. Consider now a path γ_C that is not restricted to the upper region. Then $M \cap \gamma_C$ contains two distinct points such that γ_C is strictly lower than M between these points. Applying Proposition 6.1 we then see that the cost of γ_C can be strictly improved by following M between these points instead of following γ_C there. This concludes (i). (ii) is proved similarly. ■

Proposition 6.3 *Let a point* $\bar{\mathbf{x}}^O$ *send packets to a point* $\mathbf{x}^D$*.*

i. Assume the origin is in the upper region and the destination in the lower one. Then the optimal path has three segments;

1. *It goes straight vertically from* $\bar{\mathbf{x}}^O$ *to* M*,*
2. *Continues as long as possible along* M*, i.e. until it reaches the* x *coordinate of the destination,*
3. *At that point it goes straight vertically from* M *to* $\mathbf{x}^D$*.*

ii. Assume the origin is in the lower region and the destination in the upper one. Then the optimal path has three segments;

1. *It goes straight horizontally from* $\bar{\mathbf{x}}^O$ *to* M*,*
2. *Continues as long as possible along* M*, i.e. until it reaches the* y *coordinate of the destination,*
3. *At that point it goes straight horizontally from* M *to* $\mathbf{x}^D$*.*

[5] By the maximum we mean the following. If γ_1 does not intersect M then $\widehat{\gamma} = \gamma_1$. If it intersects M then $\widehat{\gamma}$ agrees with γ_1 over the path segments where γ_1 is in the upper region and otherwize agrees with M. The minimum is defined similarly

Proof.- The proofs of (i) and of (ii) are the same. Consider an alternative route γ_C. Let $\tilde{\mathbf{x}}$ be some point in $\gamma_C \cap M$. The proof now follows by applying the previous proposition to obtain first the optimal path between the origin and $\tilde{\mathbf{x}}$ and second, the optimal path between $\tilde{\mathbf{x}}$ and the destination. ■

Case 2: The function U is positive in the upper area and negative in the lower area.

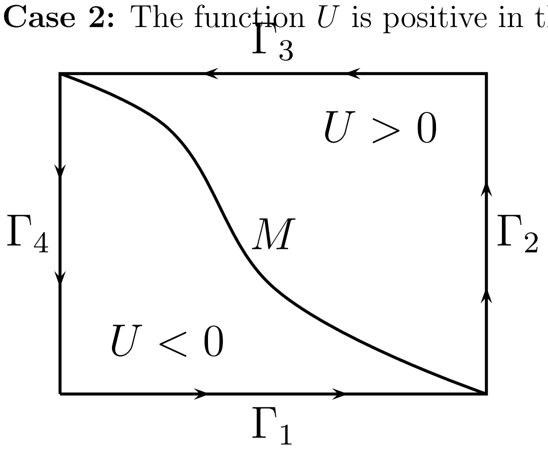

Figure 7: Two regions separated by the curve M. Case 2.

This case turns out to be more complex than the previous one. The curve M has some obvious repelling properties which we state next, but they are not as general as the attractor properties that we had in the previous case.

Proposition 6.4 *Assume that both source and destination are in the same region. Then the paths that are optimal in Theorem 6.1 are optimal here as well if we restrict to paths that remain in the same region.*

Proof.- Given that the source and destination are in a region we may change the cost over the other region so that it has the same sign over all the region R. This does not influence the cost of path restricted to the region of the source-destination pair. With this transformation we are in the scenario of Theorem 6.1 which we can then apply. ■

Discussion.- Note that the (sub)optimal policies obtained in Proposition 6.4 indeed look like being repelled from M; their two segments trajectory guarantees to go from the source to the destination as far as possible from M.

We note that unlike the attracting structure that we obtained in Case 1, one cannot extend the repelling structure to the case where the paths are allowed to traverse from one region to another.

7 User optimization and congestion dependent cost

We go beyond the approach of geometrical optics by allowing the cost to depend on congestion. Shortest path costs can be a system objective as we shall motivate below. But it can also be the result of decentralized decision making by many "infinitesimally small" players where a player may represent a single packet (or a single session) in a context where there is a huge population of packets (or of sessions). The result of such a decentralized decision making can be expected to satisfy the following properties which define the so called, user (or Wardrop) equilibrium:

"*Under equilibrium conditions traffic arranges itself in congested networks such that all used routes between OD pair (origin-destination pair), have equal and minimum costs while all unused routes have greater or equal costs*" [30].

Related work.- Both the framework of global optimization as well as the one of minimum cost path had been studied extensively in the context of road traffic engineering. The use of a continuum network approach was already introduced on 1952 by Wardrop [30] and by Beckmann [4]. For more recent papers in this area, see e.g. [6, 7, 14, 16, 32] and references therein. We formulate it below and obtain some of its properties.

Motivation.- One popular objective in some routing protocols in ad-hoc networks is to assign routes for packets in a way that each packet follows a minimal cost path (given the others' paths choices) [11]. This has the advantage of equalizing source-destination delays of packets that belong to the same class, which allows one to minimize the amount of packets that come out of sequence. (This is desirable since in data transfers, out of order packets are misinterpreted to be lost which results not only in retransmissions but also in drop of systems throughput.)

Traffic assignment that satisfies the above definition is known in the context of road traffic as Wardrop equilibrium [30].

Congestion dependent cost

We now add to c_1 the dependence on T_1 and to c_2 the dependence on T_2, as in Section 4. Let $V^j(\mathbf{x})$ be the minimum cost to go from a point $\mathbf{x}$ to B^j at equilibrium. Equation (12) still holds but this time with c_i that depends on $T_i^j \quad i = 1,2$, and on the total flows $T_i \quad i = 1,2$. Thus (13) becomes, $\forall j \in \{1,\dots,\nu\}$,

$$0 = \min_{i=1,2} \left(c_i(\mathbf{x}, T_i) + \frac{\partial V^j(\mathbf{x})}{\partial x_i} \right), \quad \forall \mathbf{x} \in B^j \,, V^j(\mathbf{x}) = 0 \,. \tag{14}$$

We note that if $T_i^j(\mathbf{x}) > 0$ then by the definition of the equilibrium, i attains the minimum at (14). Hence (14) implies the following relations for each traffic class j, and for $i = 1,2$:

$$c_i(\mathbf{x}, T_i) + \frac{\partial V^j}{\partial x_i} = 0 \quad \text{if} \quad T_i^j > 0, \tag{15a}$$

$$c_i(\mathbf{x}, T_i) + \frac{\partial V^j}{\partial x_i} \geq 0 \quad \text{if} \quad T_i^j = 0. \tag{15b}$$

This is a set of coupled PDE's, actually difficult to analyse further.

Beckmann transformation

As Beckmann et al. did in [5] for discrete networks, we transform the minimum cost problem into an equivalent global minimization one. We shall restrict here to the single class case. To that end, we note that equations (15a)-(15b) have exactly the same form as the Kuhn-Tucker conditions (6a)-(6b), except that $c_i(\mathbf{x}, T_i)$ in the former are replaced by $\partial g(\mathbf{x}, \mathbf{T})/\partial T_i(\mathbf{x})$ in the latter. We therefore introduce a *potential function* ψ defined by

$$\psi(\mathbf{x}, \mathbf{T}) = \sum_{i=1,2} \int_0^{T_i} c_i(\mathbf{x}, s) \mathrm{d}s$$

so that for both $i = 1, 2$:

$$c_i(\mathbf{x}, T_i) = \frac{\partial \psi(\mathbf{x}, \mathbf{T})}{\partial T_i} .$$

Then the user equilibrium flow is the one obtained from the global optimization problem where we use $\psi(\mathbf{x}, \mathbf{T})$ as local cost. Hence, the Wardrop equilibrium is obtained as the solution of

$$\min_{T(\cdot)} \int_\Omega \psi(\mathbf{x}, \mathbf{T})\, d\mathbf{x} \qquad \text{subject to } \nabla \cdot \mathbf{T}(\mathbf{x}) = \rho(\mathbf{x}), \qquad \forall \mathbf{x} \in \Omega.$$

In the special case where costs are given as a power of the flow as defined in eq. (7), we observe that equations (15a)-(15b) coincide with equations (8a)-(8b) (up-to a multiplicative constant of the cost). We conclude that for such costs, the user equilibrium and the global optimization solution coincide.

8 Numerical Example

The following example is an adaptation of the road traffic problem solved by Dafermos in [6] to our ad-hoc setting. We therefore use the notation of [6] for the orientation, as we did in Section 4. Thus the direction from North to South will be our positive x_1 axis, and from West to East will be the positive x_2 axis. The framework we study is the user optimization with congestion cost. For each point on the West and/or North boundary we consider the point to boundary problem. We thus seek a Wardrop equilibrium where each user can choose its destination among a given set. A flow configuration is a Wardrop equilibrium if under this configuration, each origin chooses a destination and a path to that destination that minimize that users cost among all its possible choices.

Consider the rectangular area R on the bounded domain Ω defined by the simple closed curve $\partial R^+ = \Gamma_1^+ \cup \Gamma_2^+ \cup \Gamma_3^- \cup \Gamma_4^-$ where

$$\begin{aligned} \Gamma_1 &= \{0 \le x_1 \le a, \quad x_2 = 0\}, \quad \Gamma_2 = \{x_1 = a, \quad 0 \le x_2 \le b\}, \\ \Gamma_3 &= \{0 \le x_1 \le a, \quad x_2 = b\}, \quad \Gamma_4 = \{x_1 = 0, \quad 0 \le x_2 \le b\}. \end{aligned}$$

Assume throughout that $\rho = 0$ for all $\mathbf{x} \in \mathring{\Omega}$, and that the costs of the routes are linear, i.e.

$$c_1 = k_1 T_1 + h_1 \quad \text{and} \quad c_2 = k_2 T_2 + h_2, \tag{16}$$

with $k_1 > 0$, $k_2 > 0$, h_1, and h_2 constant over Ω.

We are precisely in the framework of section 7 and 4 with affine costs per packet. As a matter of fact, the potential function associated with these costs is

$$\psi(\mathbf{T}) = \sum_{i=1}^{2} \int_0^{T_i} (k_i s + h_i)\, ds = \sum_{i=1}^{2} (\frac{1}{2} k_i T_i + h_i) T_i \,.$$

Now, we want to handle a condensation of sources or sinks along the boundary. While this is feasible with the framework of section 4, it is rather technical. We rather use a more direct path below.

Notice that we have in $\mathring{\Omega}$, we have

$$\frac{\partial T_1}{\partial x_1} + \frac{\partial T_2}{\partial x_2} = 0.$$

Take any closed path γ surrounding a region ω. Then by Green formula,

$$\oint_\gamma T_1 d\xi_2 - T_2 d\xi_1 = \int_\omega \frac{\partial T_1}{\partial x_1} + \frac{\partial T_2}{\partial x_2} = 0$$

Therefore we can define

$$\phi(\mathbf{x}) := \int_{\mathbf{x^o}}^{\mathbf{x}} T_1 d\xi_2 - T_2 d\xi_1$$

the integral will not depend on the path between $\mathbf{x^o}$ and $\mathbf{x}$ and ϕ is thus well defined, and we have

$$\frac{\partial \phi(\mathbf{x})}{\partial x_2} = T_1(\mathbf{x}) \qquad \frac{\partial \phi(\mathbf{x})}{\partial x_1} = -T_2(\mathbf{x})\,. \tag{17}$$

We now make the assumption that there is sufficient demand and that the congestion cost is not too high so that at equilibrium the traffic T_1 and T_2 are strictly positive over all Ω [6]. It turns out that all paths to the destination are used. Thus, from Wardrop's principle, the cost $\int \mathbf{c}\, dx$ is equalized between any two paths. And therefore,

$$\frac{\partial c_1}{\partial x_2} = \frac{\partial c_2}{\partial x_1}.$$

Using the equations in (16) then

$$k_1 \frac{\partial T_1}{\partial x_2} = k_2 \frac{\partial T_2}{\partial x_1},$$

and from equations in (17) we have

$$k_1 \frac{\partial^2 \phi}{\partial x_2^2} + k_2 \frac{\partial^2 \phi}{\partial x_1^2} = 0.$$

Let $k_i = K_i^2$. Divide the above equation by $k_1 k_2$. One obtains

$$\frac{1}{K_1^2} \frac{\partial^2 \phi}{\partial x_1^2} + \frac{1}{K_2^2} \frac{\partial^2 \phi}{\partial x_2^2} = 0.$$

Following the classical way of analyzing the Laplace equation, (see[31]) we attempt a separation of variables according to

$$\phi(x_1, x_2) = F_1(K_1 x_1) F_2(K_2 x_2)\,.$$

We then get that

$$\frac{F_1''(K_1 x_1)}{F_1(K_1 x_1)} = -\frac{F_2''(K_2 x_2)}{F_2(K_2 x_2)} = s^2\,.$$

In that formula, since the first term in independent on x_2 and the second on x_1, then both must be constant. We call s^2 that constant, but we do not know its sign. Therefore, s may be imaginary or real. All solutions of this system for a given s are of the form

$$F_1(x) = A\cos(isx) + B\sin(isx)\,, \qquad F_2 = C\cos(sx) + D\sin(sx)\,.$$

As a matter of fact, ϕ may be the sum of an arbitrary number of such multiplicative decompositions with different s. We therefore arrive at general formula such as

$$\phi(x_1, x_2) = \int [A(s)\cos(isK_1x_1) + B(s)\sin(isK_1x_1)][C(s)\cos(sK_2x_2) + D(s)\sin(sK_2x_2)]\,ds.$$

From this formula, we can write T_1 and T_2 as integrals also. The flow T at the boundaries should be orthogonal to the boundary, and have the local source density for inward modulus (it is outward at a sink). There remains to expand these boundary conditions in Fourier integrals to identify the functions A, B, C, and D. (Surely not a simple matter!) (It is advisable to represent the integrals of the boundary densities as Fourier integrals, since then the boundary conditions themselves will be of the form $s\int R(s)\,ds$, closely matching the formulas we obtain for the T_i's.)

9 Conclusions

Routing in ad-hoc networks have received much attention in the massively dense limit. The main tools to describe the limits had been electrostatics and geometric optics. We exploited another approach for the problem that has its roots in road traffic theory, and presented both quantitative as well as qualitative results for various optimization frameworks.

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10 Appendix: Mathematical Tools

Theorem 10.1 (Green's Theorem) *Let $\Omega \subseteq \mathcal{X}$ be a region of the space, and let Γ be its boundary. Suppose that $P, Q \in \mathcal{C}^1(\Omega)$ (We denote $\mathcal{C}^1(\Omega)$ the set of functions that are differentiable and whose partial derivatives are continuous on Ω.) Then*

$$\oint_{\Gamma^+} P dx + Q dy = \int_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy. \tag{18}$$