IFORS Edinburgh

9 July 2002

Polynomial algorithms for Taylor Expansions of $(max, +)$ systems

Michael Heusch

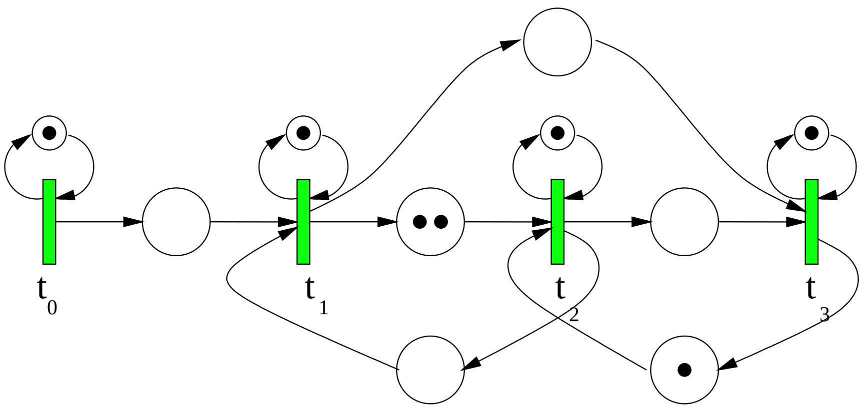
Alain Jean-Marie

LIRMM

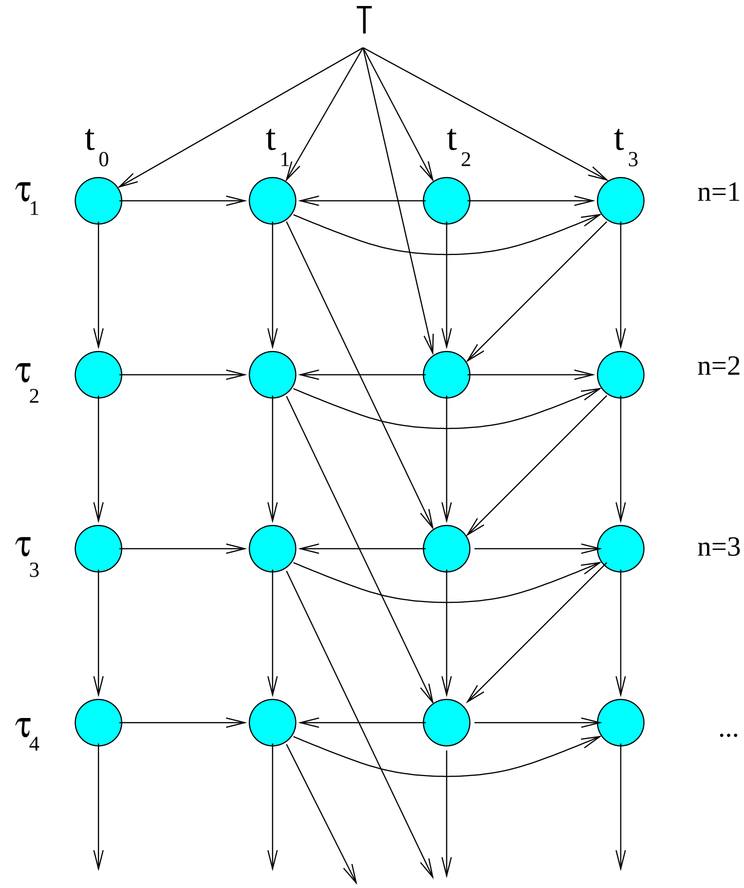
Université Montpellier II & CNRS

161, rue Ada, 34392 Montpellier, FRANCE

Context: stochastic (max, +) systems



Petri Net representation



Task Graph representation

Context

With state vector: $X_n = (X_n^1, \ldots, X_n^\alpha)$,

- X_n^i: the time at which activity i completes for the n-th time

and input time sequence

- T_n: the time at which the n-th start signal (token/customer/...) arrives to the system

Then the state vector evolves as:

- $X_{n+1} = A_n \otimes X_n \oplus B_{n+1} \otimes T_{n+1}$

We are interested in the *waiting time* for the i-th activity:

$$\boxed{W_n^i \quad = \quad X_n^i - T_n}$$

Outline of the talk

1. Presentation of Baccelli & Schmidt's approach

→ Polynomials p_k

→ Recurrences.

2. Presentation of the direct computation for Poisson/Deterministic systems.

3. Derivation of formulas

→ Formal evaluation

→ Numerical

→ New characterization and recurrences for the polynomials p_k

4. Applications

5. Discussion of numerical issues

Taylor expansions for statistics of W_n

F. Baccelli & V. Schmidt: Taylor series expansions for Poisson driven (max,+) linear systems. *Annals of Applied Probability*, 6, (1996), 138–186.

Conditionned on the duration of the length of the activities, there exist for all i a sequence (increasing) of times: $b_0, b_1, b_2, \ldots$ such that:

$$\begin{aligned} \mathbb{E}W_n &= \sum_{k=0}^{n} \lambda^k p_{k+1}(b_0, \ldots, b_k) \\ &\quad + \sum_{k=n+1}^{\infty} \lambda^k p_{k+1}(b_0, \ldots, b_{n-1}, b_n, \ldots, b_n) \\ \mathbb{E}W &= \sum_{k=0}^{n} \lambda^k p_{k+1}(b_0, \ldots, b_k) + \mathcal{O}(\lambda^{n+1}) \ . \end{aligned}$$

The functions p_k are *multivariate polynomials*, defined by:

$$p_k(x_0, \ldots, x_{k-1}) = \sum_{(i_0,\ldots,i_{k-1}) \in S_k} (-1)^{q_k(i_0,\ldots,i_{k-1})} \prod_{j=0}^{k-1} \frac{x_j^{i_j}}{i_j!}$$

and obeying the recurrence:

$$p_{k+1}(x_0, x_1, \ldots, x_k)$$

$$= \frac{1}{k+1} \sum_{j=0}^{k+1} (x_j - x_{(j-1 \ mod \ k+1)}) p_k(x_{(j \ mod \ k+1)}, \ldots, x_{(j+k-1 \ mod \ k+1)})$$

Expression of the first few p_k in expanded form:

$$\begin{aligned}
p_2(b_0, b_1) &= \frac{1}{2}\Big(b_0^2 + b_1^2 - 2b_0b_1\Big) \\
p_3(b_0, b_1, b_2) &= \frac{1}{3!}\Big(b_0^3 + b_1^3 + b_2^3 \\
&\quad - 3(b_0^2b_1 + b_1^2b_2 + b_2^2b_0) \\
&\quad + 6\ b_0b_1b_2\Big) \\
p_4(b_0, b_1, b_2, b_3) &= \frac{1}{4!}\Big(b_3^4 + b_2^4 + b_1^4 + b_0^4 \\
&\quad - 4(b_0b_3^3 + b_1b_0^3 + b_2b_1^3 + b_3b_2^3) \\
&\quad - 6(b_3^2b_1^2 + b_2^2b_0^2) \\
&\quad + 12(b_1b_2b_0^2 + b_2b_3b_1^2 + b_3b_0b_2^2 + b_0b_1b_3^2) \\
&\quad - 24\ b_1b_2b_3b_0\Big)
\end{aligned}$$

$$\begin{aligned}
p_5(b_0, b_1, b_2, b_3, b_4) \quad = \quad & \frac{1}{5!}\Big(b_0^5 + b_1^5 + b_2^5 + b_3^5 + b_4^5 \\
& - 5(b_0^4 b_1 + b_1^4 b_2 + b_2^4 b_3 + b_3^4 b_4 + b_4^4 b_0) \\
& - 10(b_0^3 b_2^2 + b_1^3 b_3^2 + b_2^3 b_4^2 + b_3^3 b_0^2 + b_4^3 b_1^2) \\
& + 20(b_0^3 b_1 b_2 + b_1^3 b_2 b_3 + b_2^3 b_3 b_4 + b_3^3 b_4 b_0 + b_4^3 b_0 b_1) \\
& + 30(b_0^2 b_2^2 b_3 + b_1^2 b_3^2 b_4 + b_2^2 b_4^2 b_0 + b_3^2 b_0^2 b_1 + b_4^2 b_1^2 b_2) \\
& - 60(b_0^2 b_1 b_2 b_3 + b_1^2 b_2 b_3 b_4 + b_2^2 b_3 b_4 b_0 + b_3^2 b_4 b_0 b_1 + b_4^2 b_0 b_1 b_2) \\
& + 120\; b_0 b_1 b_2 b_3 b_4 \;\Big)
\end{aligned}$$

Taylor expansions

The same polynomials appear in expansions for other statistics. For instance:

$$\begin{aligned} \mathbb{E}[e^{-s\ W}] &= \sum_{k=0}^{n} \lambda^k q_{k+1}(b_0,\ldots,b_k) + \mathcal{O}(\lambda^{n+1}) \ , \\ q_{k+1}(b_0,\ldots,b_k) &= \frac{1}{s}\left(q_k(b_0,\ldots,b_{k-1}) - q_k(b_1,\ldots,b_k)\right) \\ &\quad -e^{-s\ x_0}\left(p_k(b_0,\ldots,b_{k-1}) - p_k(b_1,\ldots,b_k)\right) \end{aligned}$$

S. Hasenfuss, F. Baccelli and V. Schmidt. Transient and stationary waiting times in (max;+)–linear systems with Poisson input. *QUESTA*, 26, (1997), 301–342.

H. Ayhan, D.-W. Seo. Tail probability of transient and stationary waiting times in $(max,+)$-linear systems. *Trans. on Automatic Control*, 47-1, (2002), 151-157.

H. Ayhan and D.-W. Seo, Laplace Transform and Moments of Waiting Times in (Max,+) Linear Systems with Poisson Input, *QUESTA*, 37, (2001), 405-438.

The direct approach

A. Jean-Marie. The waiting time distribution in poisson-driven deterministic systems. Technical report, Report 3083, INRIA, 1997.

Using a direct recursive computation, one gets

$$
\begin{aligned}
\mathbb{E}W_n &= b_n + \frac{1}{\lambda}\sum_{v=0}^{n} \chi_{v,v}^{(n)} - \frac{n+1}{\lambda}\ , \ \textit{with} \\
\chi_{v,v}^{(n)} &= \sum_{\ell=0}^{v-1} \frac{\lambda^\ell}{\ell!}(b_n - b_{n-v})^\ell e^{-\lambda(b_n - b_{n-v})} \\
&\quad - \sum_{\ell=1}^{v-1} \frac{\lambda^\ell}{\ell!}(b_{n-v+\ell} - b_{n-v})^\ell e^{-\lambda(b_{n-v+\ell} - b_{n-v})} \chi_{v-\ell,v-\ell}^{(n)}\ , \\
\chi_{0,0}^{(n)} &= 1\ .
\end{aligned}
$$

Direct calculations

The numbers $\chi_{u,v}$ are probabilities:

$$\begin{aligned}
\chi_{u,v}^{(n)} &= \mathbb{P}(b_{n-u} \leq b_{n-v}; b_{n-u+1} \leq b_{n-v} + \tau_1; \\
&\quad b_{n-u+2} \leq b_{n-v} + \tau_1 + \tau_2; \ldots ; \\
&\quad b_n \leq b_{n-v} + \tau_1 + \ldots + \tau_n) \ .
\end{aligned}$$

A new construction of the polynomials p_k

Since:

$$\begin{aligned}\mathbb{E}W_n &= b_n + \frac{1}{\lambda}\sum_{v=0}^{n}\chi_{v,v}^{(n)} - \frac{n+1}{\lambda} \\ &= \sum_{k=0}^{n}\lambda^k p_{k+1}(b_0,\ldots,b_k) + \sum_{k=n+1}^{\infty}\lambda^k p_{k+1}(b_0,\ldots,b_{n-1},b_n,\ldots,b_n)\end{aligned}$$

then

$$p_{n+1}(b_0,\ldots,b_n) = \sum_{v=0}^{n}[\lambda^{n+1}]\ \chi_{v,v}^{(n)}\ .$$

Direct calculations

Now write:

$$\chi_{v,v}^{(n)} = e^{-\lambda(b_n - b_{n-v})} \sum_{j=0}^{v-1} a_{v,j} \, \lambda^j \; ,$$

then the coefficients are computed recursively as

$$a_{v,k} = \frac{(b_n - b_{n-v})^k}{k!} - \sum_{i=v-k}^{v-1} a_{i,i-(v-k)} \frac{(b_{n-i} - b_{n-v})^{v-i}}{(v-i)!}$$

Finally,

$$p_{n+1}(b_0, b_1, \ldots, b_n) = \sum_{v=1}^{n} \sum_{k=0}^{v-1} (-1)^k \, \frac{(b_n - b_{n-v})^{n+1-k}}{(n+1-k)!} \, a_{v,k}^n \; .$$

Complexity of the evaluation

What complexity?

- *arithmetic* complexity: numerical, fixed precision complexity, counting elementary operations (+, ×, powers...)
- *formal* complexity, size of the symbolic expressions obtained.

The polynomial $p_n(b_0, \ldots, b_n)$, once developped, has $2^n - 1$ monomials.

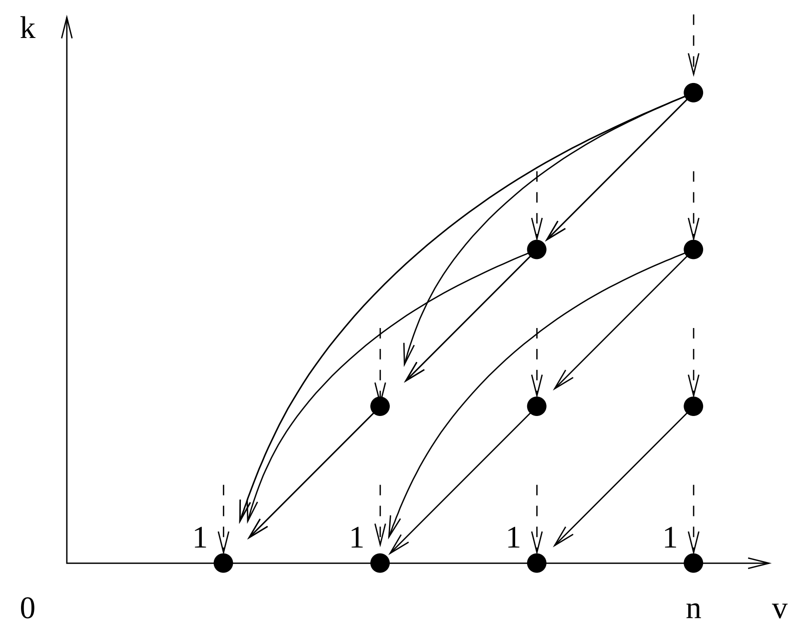
The numerical evaluation through the recurrence costs $\lfloor (n)!(2e-3) - 1 \rfloor$ multiplications (S. Hasenfuss. *Performance Analysis of (max,+)-Linear Systems via Taylor Series Expansions.* PhD thesis, Univ. of Ulm, 1997.)

There could be a faster way: like

$$\prod_{i=1}^{n}(a_i + b_i) \; = \; \sum_{2^n \text{terms}} \prod \cdots$$

Direct calculations

The recurrence structure for the numbers $a_{n,k}$ is:



They can be evaluated numerically in $\frac{1}{3}\, n^3 + o(n^2)$ operations.

However, formally, the expression remains of the order of 2^n.

Relationship between the $a_{v,k}$ and the p_k

Another expression for the $a_{v,k}$. From Hasenfuss:

$$\begin{aligned} W_n &= b_n + \frac{1}{\lambda}\sum_{m=0}^{n-1}\left[e^{-\lambda(b_n-b_m)}-1\right] \\ &\quad + \sum_{j=0}^{n-2}\lambda^j \sum_{m=0}^{n-j-2}\Big[e^{-\lambda(b_n-b_m)}\{p_{j+1}(b_n,b_{m+1},\dots,b_{m+j}) \\ &\qquad\qquad - p_{j+1}(b_{m+1},\dots,b_{m+j+1})\}\Big] \\ &= b_n - \frac{n+1}{\lambda} + \frac{1}{\lambda}\sum_{v=1}^{n} e^{-\lambda(b_n-b_{n-v})}\Big\{\sum_{k=0}^{v-1} a_{v,k}\lambda^k\Big\}\,. \end{aligned}$$

$$\Rightarrow \boxed{a_{v,k} = p_k(b_n,b_{n-v+1},\dots,b_{n-v+k-1}) - p_k(b_{n-v+1},\dots,b_{n-v+k})\,.}$$

Formal results: a partially factored expression:

$$\begin{aligned}
p_2(b_0, b_1) &= \frac{1}{2!}(b_0 - b_1)^2 \\
p_3(b_0, b_1, b_2) &= \frac{1}{3!}(b_1 - b_2)^3 + \frac{1}{3!}(-b_2 + b_0)^3 \\
&\quad + \frac{1}{2!}(b_2 - b_1)(-b_2 + b_0)^2 \\
p_4(b_0, b_1, b_2, b_3) &= \frac{1}{4!}(-b_3 + b_2)^4 + \frac{1}{4!}(-b_3 + b_1)^4 + \frac{1}{4!}(-b_3 + b_0)^4 \\
&\quad + \frac{1}{3!}(b_3 - b_2)(-b_3 + b_1)^3 + \frac{1}{3!}(b_3 - b_1)(-b_3 + b_0)^3 \\
&\quad + \frac{1}{2!}\left(\frac{1}{2!}(b_3 - b_0)^2 - (b_1 - b_0)(b_3 - b_2) - \frac{1}{2!}(b_2 - b_0)^2\right)(-b_3 + b_0)^2
\end{aligned}$$

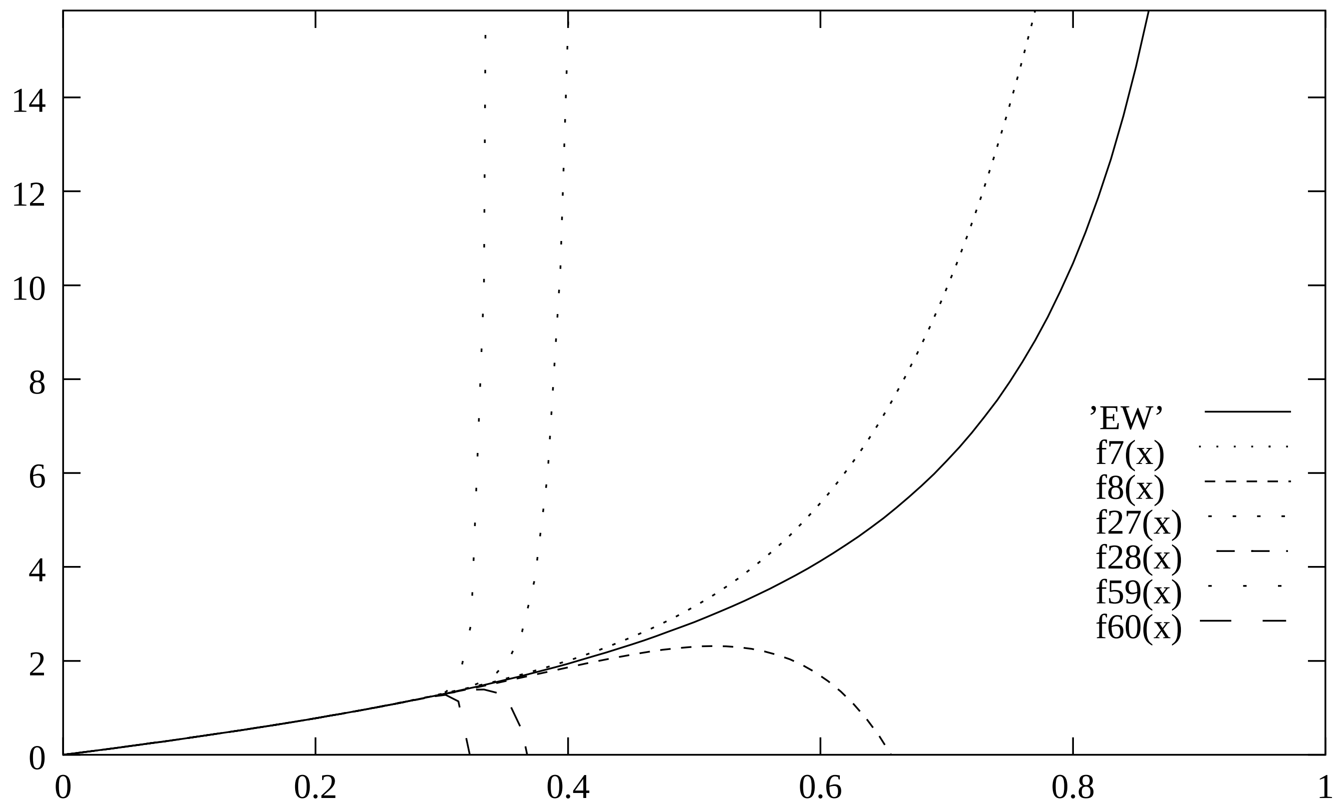
Direct calculations

$$
\begin{aligned}
& p_5(b_0, b_1, b_2, b_3, b_4) \\
= \quad & \frac{1}{5!}(-b_4+b_3)^5 + \frac{1}{5!}(-b_4+b_2)^5 + \frac{1}{5!}(-b_4+b_1)^5 + \frac{1}{5!}(-b_4+b_0)^5 \\
& + \frac{1}{4!}(b_4-b_3)(-b_4+b_2)^4 + \frac{1}{4!}(b_4-b_2)(-b_4+b_1)^4 + \frac{1}{4!}(b_4-b_1)(-b_4+b_0)^4 \\
& + \frac{1}{3!}\Big(\frac{1}{2!}(b_4-b_1)^2 - (b_2-b_1)(b_4-b_3) - \frac{1}{2!}(b_3-b_1)^2\Big)(-b_4+b_1)^3 \\
& + \frac{1}{3!}\Big(\frac{1}{2!}(b_4-b_0)^2 - \frac{1}{2!}(b_2-b_0)^2 - (b_1-b_0)(b_4-b_2)\Big)(-b_4+b_0)^3 \\
& + \frac{1}{2!}\Big\{ -(b_1-b_0)\Big(\frac{1}{2!}(b_4-b_1)^2 - (b_2-b_1)(b_4-b_3) - \frac{1}{2!}(b_3-b_1)^2\Big) \\
& \qquad + \frac{1}{3!}(b_4-b_0)^3 - \frac{1}{2!}(b_2-b_0)^2(b_4-b_3) - \frac{1}{3!}(b_3-b_0)^3\Big\}(-b_4+b_0)^2
\end{aligned}
$$

The queue $M/D_2/1$

A queue with alternate services: $\sigma_{2n} = 2$, $\sigma_{2n+1} = 8$.

S. Hasenfuss has computed the expansion up to $n = 28$.



Singularity at $\rho \simeq 0.278$.

An exploration of numerical instabilities

Test on the identity: $p_k(1, 2, \ldots, k-1) = \frac{1}{2}$.

With standard double precision (32 bits)

```
 p(2)    =    0.500000000000000000000000000000
 p(3)    =    0.500000000000000000000000000000
 p(4)    =    0.500000000000000000000000000000
 p(5)    =    0.499999999999999999945789891376
 p(6)    =    0.499999999999999999783159565503
 p(7)    =    0.500000000000000000216840434497
 p(8)    =    0.499999999999999999566319131006
 p(9)    =    0.499999999999999998265276524023
p(10)    =    0.5000000000000000013877787807814
p(11)    =    0.4999999999999999954897189624603
p(12)    =    0.5000000000000000180411241501588
```

Numbers are not useable for $n > 27$...

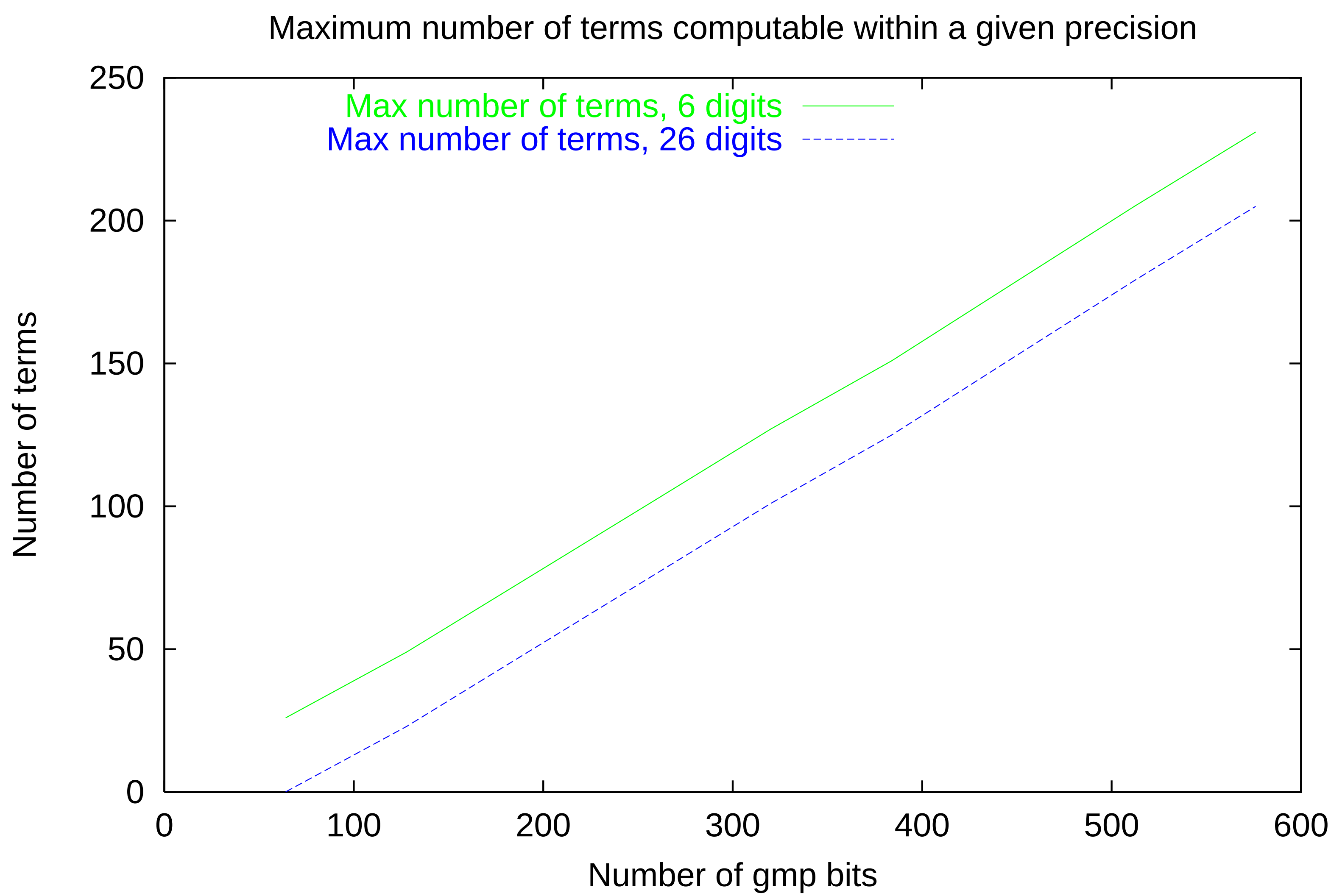
Numerical Instabilities

Experiments with `gmp` (Gnu MultiPrecision).

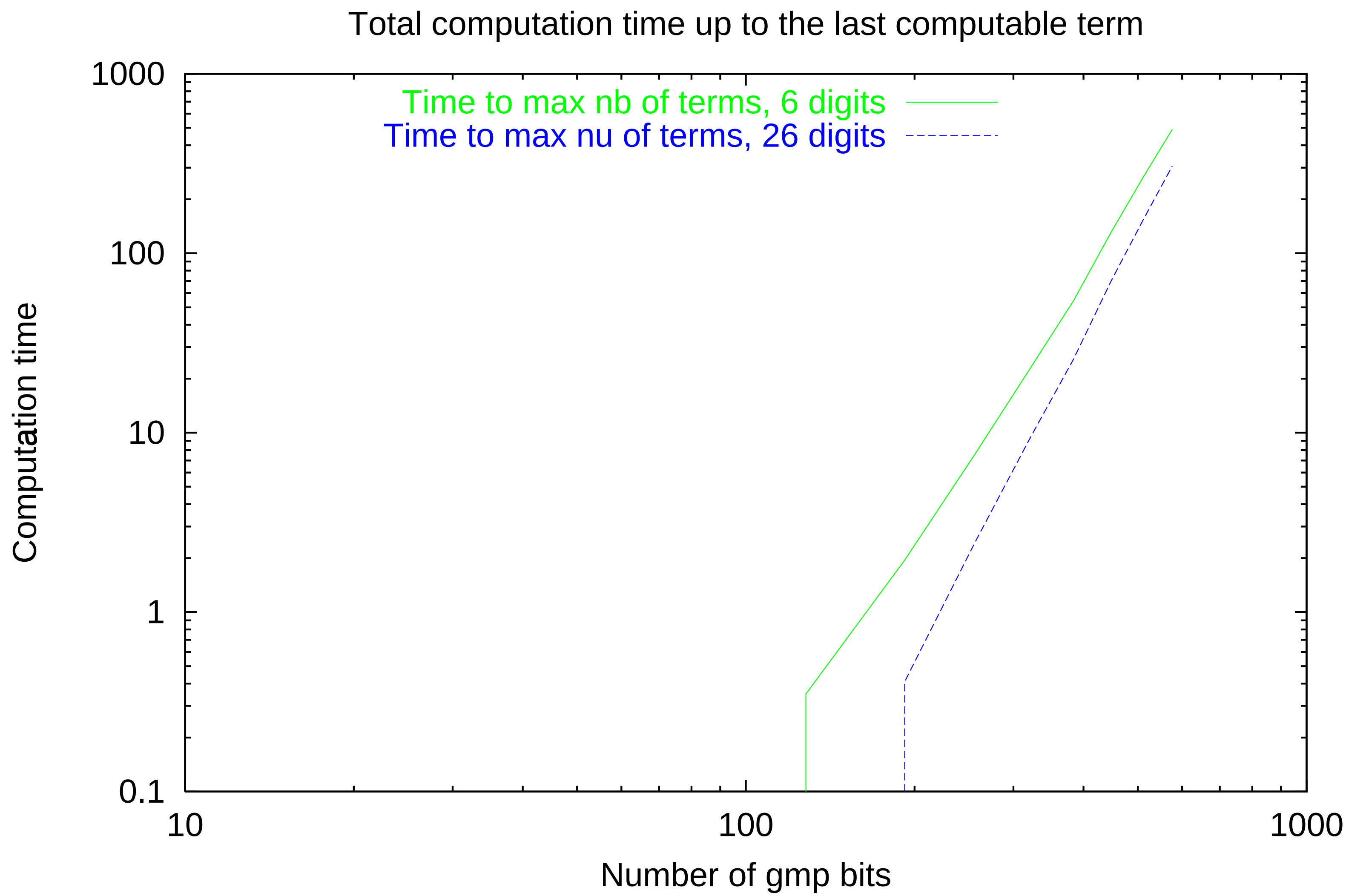
Curves relating the number of bits of `gmp` with

- the maximum number of terms $p_n(\dots)$ computed within a precision of 6 and 26 digits,
- the computation time.

Numerical Instabilities



Numerical Instabilities



Numerical Instabilities

Another direction for increasing precision: using more complex but robust recurrences.

$$\chi_{u,v}^{(n)} = \sum_{s=0}^{u-v} \frac{\lambda^s}{s!}(b_{n-v+1} - b_{n-v})^s e^{-\lambda(b_{n-v+1}-b_{n-v})} \chi_{u-s,v-1}^{(n)} \quad 1 \le v \le u,$$

$$= e^{-\lambda(b_n - b_{n-v})} \; \kappa_{u,v}^{(n)}$$

$$\kappa_{u,v}^{(n)} = \sum_{s=0}^{u-v} \frac{\lambda^s}{s!}(b_{n-v+1} - b_{n-v})^s \kappa_{u-s,v-1}^{(n)} \quad 1 \le v \le u,$$

$\kappa_{u,v}^{(n)}$ is a polynomial of λ of degree $u-1$, with positive coefficients:

$$\kappa_{u,v}^{(n)} = \sum_{j=0}^{u-1} a_{(u,v),\ j}^{(n)} \; \lambda^j$$

$\Rightarrow$ a longer, but still polynomial algorithm.

Conclusion

We have provided an algorithm for computing the Taylor expansions for functions of W_n in polynomial time.

New formulas and relationships may help undestand them better, or find other types of expansions.

Numerical instabilities to be mastered (arbitrary precision, clustering of terms, recurrence with positive coefficients...)