Conjectural Variations Equilibria

Part II: Dynamic Equilibria

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Contents

Dynamic conjectures, bounded rationality and learning

- The principle.
- A learning model.
- Friedman Mezzetti model.

Consistent conjectures in a dynamic setting

- The principle.
- Consistent conjectures in differential games.
- Theoretical framework in discrete time, infinite horizon.

Dynamic conjectures, bounded rationality and learning

The idea

Ingredients

- Dynamic conjectures
- Limited rationality
- Updating of conjectures

Conjecture adjustment process

$$\dot{r}_{ij}(t) = \mu_i(r'_{ij}(t) - r_{ij}(t)), \quad r_{ij}(t+1) = (1-\mu_i)r_{ij}(t) + \mu_i r'_{ij}(t)$$

μ_i $\longrightarrow$ speed of adjustment.
$r_{ij}(t)$ $\longrightarrow$ conjecture of i about j.
$r'_{ij}(t)$ $\longrightarrow$ conjecture to be used, based on observations.

The learning model

- n players, e_i strategy of i, e profile of strategies,
- e^b a given benchmark strategy,
- V^i instantaneous payoff of player i.

Player i makes a conjecture about j of the form

$$e_j = e_j^b + r_{ij}(e_i - e_i^b), \quad r_{ij} \in \mathbb{R}$$

and solves

$$\max_{e_i} V^i(e_i, (e_j^b + r_{ij}(e_i - e_i^b))_{i \neq j}) \ .$$

There exists a unique solution $e_i = \phi_i(e^b; r_i)$, ($r_i = (r_{ij})_{i \neq j}$).

Learning model (continued)

i observes that j has played e_j and concludes that her conjecture should have been r'_{ij} /

$$e_j = e_j^b + r'_{ij}\,(e_i - e_i^b), \qquad \Longrightarrow \qquad r'_{ij} = \frac{e_j - e_j^b}{e_i - e_i^b}$$

Adjustment process of conjectures

$$r_{ij}(t+1) = (1-\mu_i)r_{ij}(t) + \mu_i \frac{e_j(t) - e_j^b}{e_i(t) - e_i^b}$$

with $e_i(t) = \phi_i(e^b, r_i(t))$.

Properties of fixed points

Proposition 1: If $r_{ij}(t) \to r_{ij}$ as $t \to \infty$, then

$$r_{i_1 i_2} r_{i_2 i_3} \ldots r_{i_p i_1} = 1 \quad \forall i_1 \ldots i_p$$

in particular

$$r_{ji} = (r_{ij})^{-1}$$

The vector $(r_{i1} \ldots r_{ii-1}, 1, r_{ii+1} \ldots r_{in})$ is the direction of the line (passing through e^b) of the space of strategy profiles, on which player i chooses her own strategy.

$e_i = \phi_i(e^b, r_i)$ is the strategy played by i in the limit.

Properties of fixed points (continued)

Proposition 2: Pareto optimality
If e is a limit point obtained by the convergence of the adjustment recurrence then e is a candidate Pareto-optimal solution.
candidate i.e. it verifies necessary optimal conditions.

Proposition 3: In the case of identical players:
$\phi_i(e^b, r) = \phi(e^b, r)$, $e_i^b = e^b \; \forall i$; the recurrence converges to 1 for any $0 < \mu < 1$ and any (common) initial condition.

Example

Cournot's oligopoly:

$$V^i(e_i, e_{-i}) = (\alpha - \beta \sum_j e_j)e_i - (be_i + c) = \beta e_i(\Gamma - \sum_j e_j) - c.$$

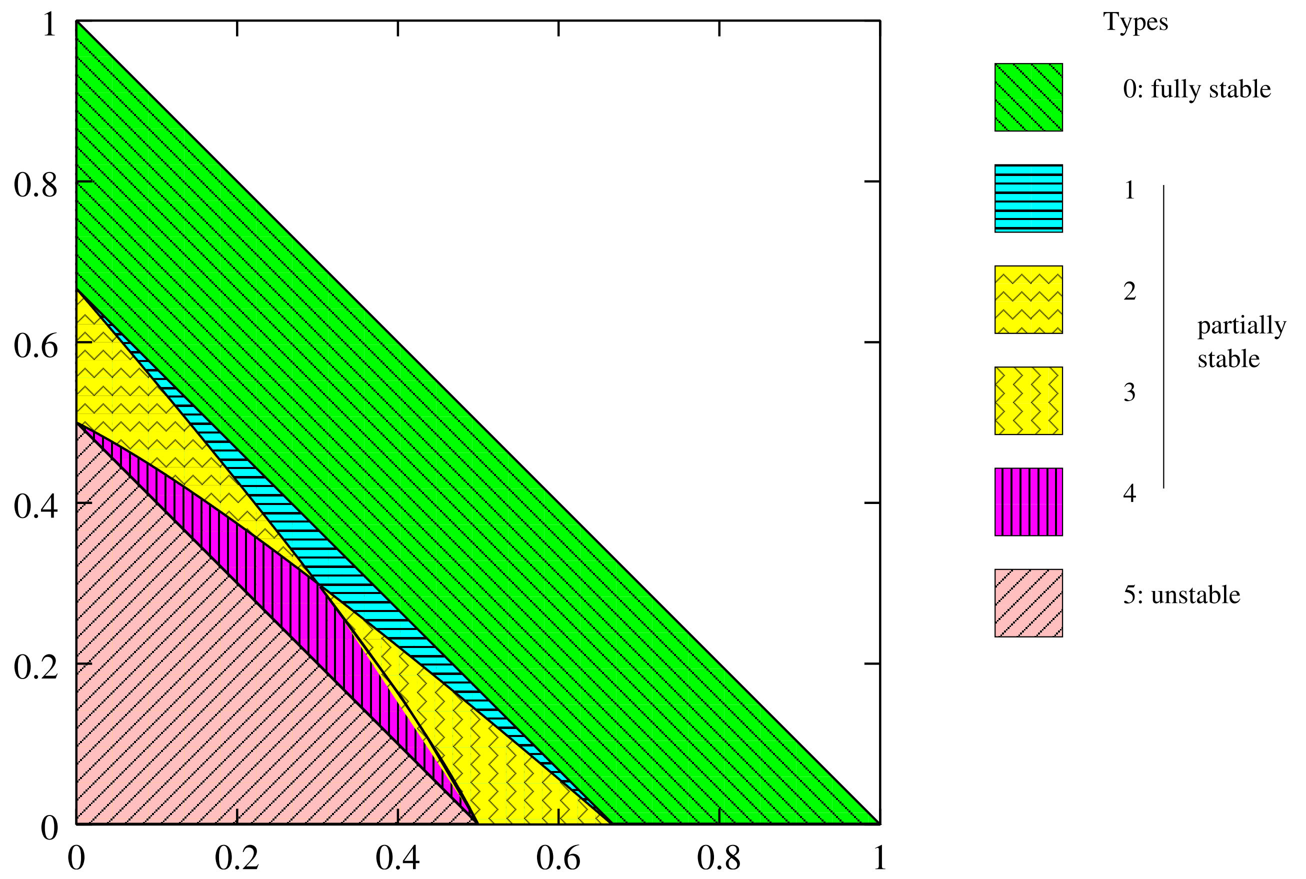
Where $\Gamma = \frac{\alpha - b}{\beta} > 0$.

Theorem: the unique fixed point of the adjustment process are $r_{ij} = e_j^b / e_i^b$ and the corresponding strategies are Pareto optima.

The learning procedure selects among the Pareto outcomes the only one with quantities proportional to that of the reference point.

Zones of stability

Zones of stability in the Cournot case ($\Gamma = 1$).



The Friedman-Mezzetti model

Friedman-Mezzetti (2002) study a discounted repeated game, discrete time, infinite horizon, where agents form fixed conjectures about the others agents but they update the reference point.

$$e_j(t+1) = e_j(t) + r_{ij}(e_i(t) - e_i(t-1))$$

Optimization

$$e_i(t) = \phi_t^i(e(t-1), e_i(t-2))$$

Optimal policy $\rightarrow \phi_1^i$ at time $t = 1$, she observes $e(1)$ and applies ϕ_1^i.

$$e_i(t) = \phi_1^i(e(t-1), e_i(t-2)),\ i = 1...n$$

Result

Theorem: let $e_i^s(r,\theta)$ be a fixed point of the dynamical system

$$e_i(t) = \phi_1^i(e(t-1), e_i(t-2)),\ i = 1...n$$

for a fixed vector of conjectures r. Let $e_i^c(r)$ be a conjectural variations equilibrium with constant vector of conjectures r, for the associated static game.

If there exists $e_i^s(r,\theta)$, then there exists a $e_i^c(\theta r)$, and conversely. If both are unique then

$$e_i^s(r,\theta) = e_i^c(\theta r)$$

Adapting reference point in our learning model

$$e_j(t+1) = e_j(t) + r_{ij}(e_i(t) - e_i(t-1))$$

optimization

$$e_i(t+1) = \phi_i(e(t), r_i)$$

if the recurrence converges to $\bar{e}$

$$\bar{e_i} = \phi_i(\bar{e}, r_i)$$

Adapting the reference point in Cournot's duopoly

$$V^i(e_i, e_{-i}) = \beta e_i(\Gamma - \sum_j e_j) - c.$$

$$e_i = \frac{(1 + r_{ij})\Gamma}{(2 + r_{12})(2 + r_{21}) - 1}$$

$$(e_1, e_2) \text{ Pareto } \iff r_{12}r_{21} = 1$$

EXTENSION: Adapting conjectures and reference points.

Consistent conjectures in a dynamic setting

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Consistent conjectures in a dynamic setting

Ingredients

- Dynamic game. Repeated game
- Conjectures on how the other players react
- Consistency: conjectures of each player $\equiv$ best response reactions of the others players

Principle

- n players, time horizon T
- $x(t) = (x_1(t), ...x_n(t)) \in \mathbb{R}^m$ state variable
- $e_i(t)$ control variable of i in $[t, t+1]$, $e(t)$

Dynamics

$$x(t+1) = f(x(t), e(t)), \quad x(0) = x_0$$

(repeated game $\rightarrow x(t+1) = e(t)$)

Payoff

$$V^i(x_0, e(0), ...e(T-1)) = \sum_{t=1}^{T} \theta^{t-1} \Pi^i(x(t), e(t))$$

Principle (continued)

Conjecture of i

$$e_j^c(t) = \phi_t^{ij}(x(t)) \quad \longrightarrow \quad x(t+1) = \tilde{f}_i(x(t), e_i(t)) \; .$$

optimal control problem

optimal policy $e_i^{i*}(t)$ that we suppose unique. Player i can compute $e_j^{i*}(t)$ and $x^{i*}(t)$ via ϕ_t^{ij}.

Call $x^a(t)$ the actual trajectory (replacing e_i^{i*} in the dynamics).

Different definitions of consistency

Definition 1: $\phi_t^1, ...\phi_t^n$ is a state-consistent conjectural equilibrium $\Longleftrightarrow$

$$x^{i*}(t) = x^a(t), \quad \forall i,\ t,\ x(0) = x_0$$

Definition 2: $\phi_t^1, ...\phi_t^n$ is a (weak) control-consistent conjectural equilibrium $\Longleftrightarrow$

$$e^{i*}(t) = e^{j*}(t), \quad \forall i \neq j,\ t,\ x(0) = x_0 \ \ (a\ x(0)\ given)$$

control-consistent c.e. $\Longrightarrow$ state-consistent c.e.

Different definitions of consistency (continued)

Optimization problem: $\rightarrow e_i^{i*}(t) = \psi_t^i(x(t))$

Definition 3: $\phi_t^1, ...\phi_t^n$ is a feedback-consistent conjectural equilibrium $\iff$

$$\psi_t^i = \phi_t^{ji}, \quad \forall i \neq j,\ t,\ x(0) = x_0$$

as a consequence

$$\phi_t^{ji} = \phi_t^{ki}, \quad \forall i \neq j \neq k,\ t.$$

Consistency in differential games

Fershman and Kamien (1985) define consistent conjectures in differential games.

- Open-loop Nash equilibria are weak control-consistent conjectural equilibria
- Control-consistent conjectural equilibria and feedback Nash equilibria coincide

The model of Friedman (1968)

Discrete time, infinite horizon, repeated game.

$$V^i(x_0, e(0), ...) = \sum_{t=1}^{\infty} \theta^{t-1} \Pi^i(x(t))$$
$$x(t+1) = e(t)$$
$$x_j(t+1) = \phi^i(x(t))$$

Solution: Solve the control problem with finite horizon T and let T goes to infinity.

Repeated static Nash equilibria is a feedback-consistent conjectural equilibria

$\exists$ other feedback-consistent conjectural equilibria?

The linear-quadratic case: setting of the problem

Instantaneous payoff

$$\Pi^i(x) = \frac{1}{2} x^t K^i x + L^i x + M^i$$

Discounted payoff

$$V^i(x_0, e(0), \ldots) = \sum_{t=1}^{\infty} \theta^{t-1} \Pi^i(x(t))$$

Dynamics

$x_i(t+1) = e_i(t)$

$x_j(t+1) = e_j^c(t) = \sum_{k=1}^{n} f_{jk}^i(\tau) x_k(t) + g_j^i(\tau)$

$\tau = T - (t+1)$ is the number of time units left before the end of the game.

optimization problem

The L-Q case: setting of the problem (continued)

The optimization problem for player i, for a finite time T is:

$$W_T(x_0) = \max_{e(0),...e(T-1)} V^i(x_0, x(1), ...x(T))$$

such that

$x(t+1) = e(t)b_i + F^i(\tau)x(t) + g^i(\tau), \quad x(0) = x_0$.

$b_i = (0, ..., 1, ...0)^t$ with '1' in position i.

The function W_T is the value function of the control problem.

We can obtain

- recurrence formulas for the optimal reaction function
- necessary and sufficient conditions of convergence when $T \to \infty$.

Results

- The repeated static Nash equilibrium is the unique feedback consistent conjectural equilibrium in quadratic symmetric Cournot and Bertrand oligopoly.

- Consider a distance game, that is a game where player 1 wishes to minimize her distance to point (1,0) whereas player 2 wishes to minimize her distance to (0,1). We can prove that there exists an infinity of feedback consistent conjectural equilibria.

Other examples with finite number of feedback consistent conjectural equilibria?

Conclusions

Existing results call for further studies on:

- More examples of feedback-consistent equilibria.
- Learning with adaptation of conjectures and the reference point.
- Evolutionary games.
  Dixon and Somma (2001) have proved in Cournot's duopoly that the unique evolutionary stable strategy is the *consistent* CVE of the static game.

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