Conjectural Variations Equilibria

Part I: Static Equilibria

Alain Jean-Marie   Mabel Tidball

ajm@lirmm.fr   tidball@inra.ensam.fr

INRIA/LIRMM CNRS University of Montpellier 2

INRA/LAMETA Montpellier

Context

What is a Conjectural Variations Equilibrium?
A game-theoretic concept in which players have a conjecture about the behavior of their opponents: they think the others will play in function of their own decision.

What is it useful for?

- As a possible alternative to Nash Equilibria...
  - behavioral model
  - explicative model (suitable for empirical studies)
- especially when information is incomplete
- or to model implicit cooperation.
- A shorthand for dynamic interactions

Contents

Static Conjectural Variations Equilibria (CVE)

- Definition
- Properties
- Examples
- Consistency

CVE as shorthands for dynamic interactions

- Repeated games with trigger strategies
- Dynamic games with open-loop strategies
- Dynamic games with closed-loop strategies

Static conjectural variations equilibria

Preliminaries

Consider a two-player game with

- V^i the payoff of player i,
- e_i the strategy of player i in some set $\mathcal{E}$.

Consider some benchmark strategy profile $e^b = (e_1^b, e_2^b)$.
Assume that player i thinks that if she deviates from e_i^b by some infinitesimal quantity de_i, then player j will “react” by deviating from e_j^b by the quantity:

$$r_j(e_i^b, e_j^b)\ \mathrm{d}e_i\ ,$$

for some function r_j.
r_j is the conjectural variation assumed by player i.

Preliminaries (continued)

Non infinitesimal deviations?
Player i is logically led to think that if she deviates by Δe_i from e_i^b, then player j will play:

$$e_j \;=\; \rho_j^c(e_i^b + \Delta e_i)\ ,$$

where ρ_j^c is solution of:

$$\frac{\partial \rho_j^c(e_i; e_i^b, e_j^b)}{\partial e_i} \;=\; r_j(e_i, \rho_j^c(e_i; e_i^b, e_j^b))\ .$$

with initial condition $\rho_j^c(e_i^b; e_i^b, e_j^b) = e_j^b$.
This function is the conjectured reaction function.

Definition

Definition: Conjectural Variations Equilibrium
A pair of variational conjectures $r_i(e_j, e_i)$ $i = 1, 2$, together with a pair of strategies $(e_1^c, e_2^c) \in \mathcal{E}$ is a General Conjectural Variations Equilibrium (GCVE) if (e_1^c, e_2^c) is solution of the optimization problem:

$$\max\{V^i(e_i, e_j) \mid (e_1, e_2) \in \mathcal{E} \text{ and } e_j \;=\; \rho_j^c(e_i; e_i^c, e_j^c)\} \;,$$

simultaneously for $i = 1, 2$.

First order conditions

Theorem: Assume that V^i is twice differentiable.
If (r_1, r_2) and (e_1^c, e_2^c) is a GCVE, then it satisfies

$$e_i^c = \chi_i(e_j^c) \qquad i \neq j \ ,$$

where the function $\chi_i(e_j)$ is implicitly defined by the solution of the following first order conditions for each player:

$$\boxed{V_i^i(e_i, e_j) \ + \ r_j(e_i, e_j)\, V_j^i(e_i, e_j) = 0 \ .}$$

Observations

Several comments:

- A CVE can be seen as the fixed point of the mapping

$$(e_i, e_j) \qquad \mapsto \qquad (\chi_i(e_j), \chi_j(e_i)) \; .$$

  By analogy with Nash's best response functions, χ_i is called the conjectural best response function.
- When the benchmark strategy is a CVE, no player has an incentive to deviate from it, according to her own beliefs.
- There are appropriate second order conditions.

CVE, Nash Equilibria, Pareto Outcomes

CVEs generalize Nash equilibria

Property: A Nash equilibrium is a CVE for the conjectural variations

$$r_j = 0 .$$

Pareto outcomes may be CVE

Property: in a symmetric game, the strategy e that maximizes $V(e, e)$ is a CVE with conjectures

$$r_j = 1 .$$

Example 1: Cournot's duopoly

Bowley (1924) introduced CVE in Cournot's duopoly. Assume linear cost and inverse demand. The profit function has the form:

$$V^i(e_i, e_j) \;=\; (a - b(e_i + e_j))e_i \;-\; ce_i \;.$$

With constant variations r, the first order conditions are:

$$0 \;=\; V_i^i + rV_j^i \;=\; a - (2e_i + e_j) - c - rbe_i \;.$$

The solution is therefore:

$$e_i^c \;=\; e_j^c \;=\; \frac{a-c}{b(3+r)} \;.$$

Example 2: Public goods

Itaya and Dasgupta (1995) have proposed the following model of a voluntary contribution to a public good.
The construction leads to the (Cobb-Douglas) payoff function:

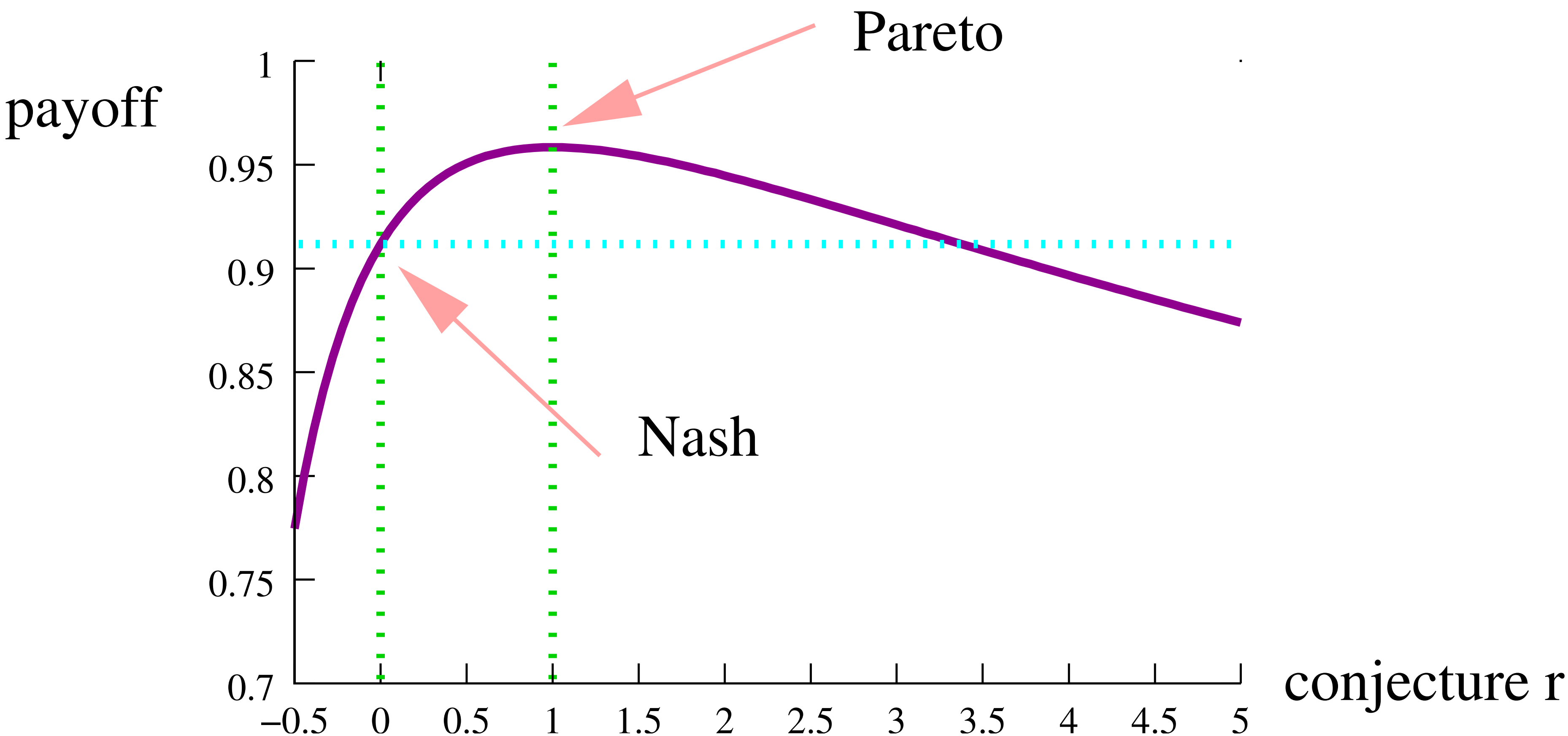
$$V^i(e_i, e_j) = (I - e_i)^\alpha (e_i + e_j)^{1-\alpha}$$

with $\alpha < 1/2$.
There exists a unique CVE with constant conjectural variations r:

$$e_1^c = e_2^c \;=\; I\,\frac{(1-\alpha)(1+r)}{(1-\alpha)(1+r)+2\alpha}\;.$$

Example 2 (continued)

Variation of the payoff in function of the conjecture:



Objections

Several objections have been raised against CVE.

Individual Rationality.
In situations of complete information and common knowledge of rationality, players should consider only strategies remaining after the iterated elimination of dominated strategies. Often, only Nash equilibria remain...

$\longrightarrow$ But if incomplete information?

Objections (continued)

Credible behavior.
Both players are assumed to act à la Stackelberg. However there is not sequentiality. No player observes the opponent's play.

$\longrightarrow$ But if repeated interactions?

Refutability.
By selecting carefully the conjectures, any outcome may be a CVE. The theory accounts for all observable results!

$\longrightarrow$ Definition of consistent conjectures. Conjectures "endogeneized".

Consistency

Consistency is the requirement that conjectural best responses and conjectured reactions coincide.

Definition: Consistent GCVE
A pair of strategies (e_1^c, e_2^c) and the variational conjectures $r_i(e_1, e_2)$, $i = 1, 2$ are a Consistent General Conjectural Variations Equilibrium if:

i) (e_1^c, e_2^c) is a GCVE for conjectures (r_1, r_2);

ii) $\chi_i(e_j)$ being a solution in e_i of

$$\boxed{V_i^i(e_i, e_j) \; + \; r_j(e_i, e_j)\, V_j^i(e_i, e_j) = 0 \; .}$$

Consistency

Consistency is the requirement that conjectural best responses and conjectured reactions coincide.

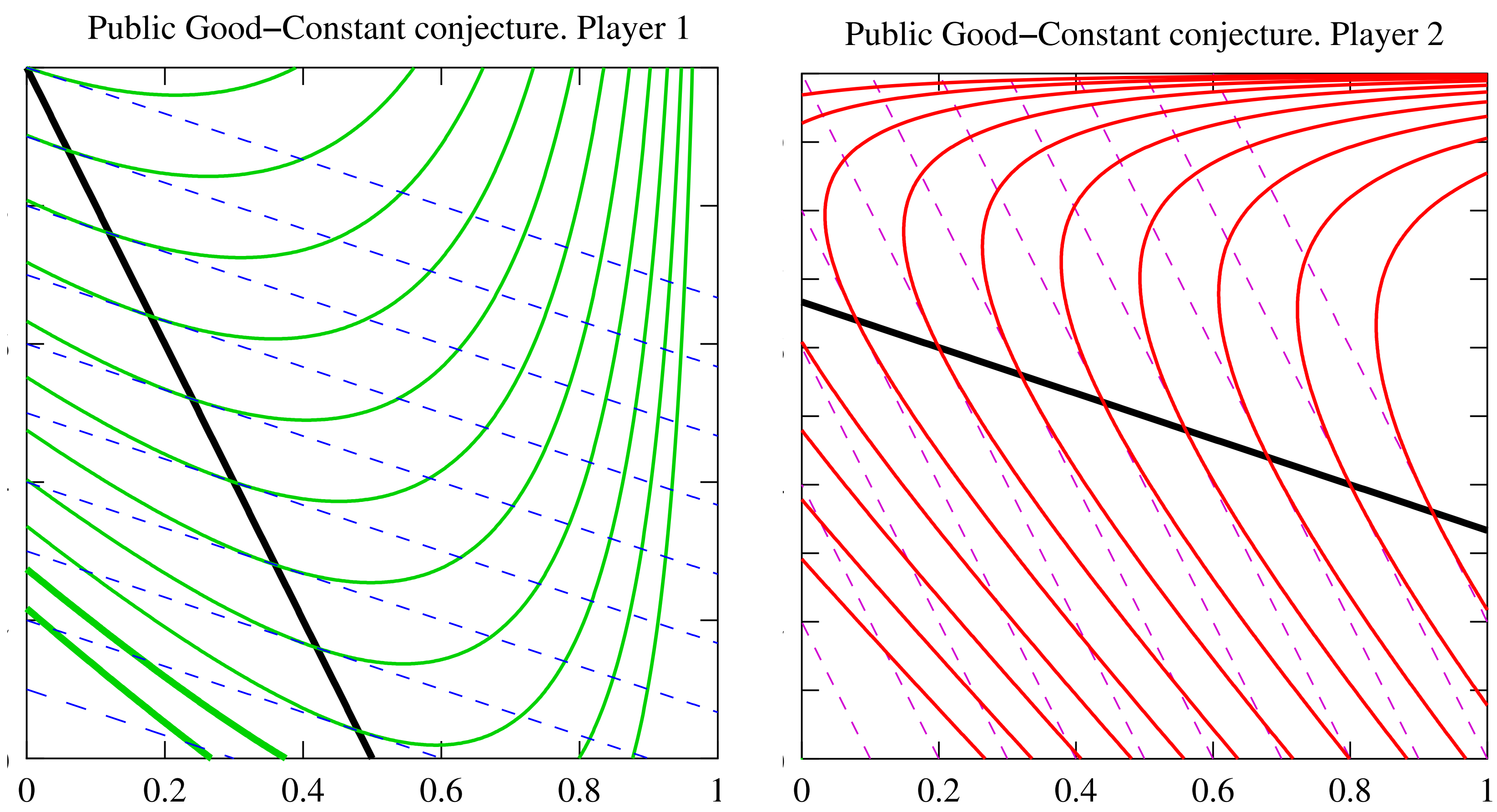
Definition: Consistent GCVE
A pair of strategies (e_1^c, e_2^c) and the variational conjectures $r_i(e_1, e_2)$, $i = 1, 2$ are a Consistent General Conjectural Variations Equilibrium if:

i) (e_1^c, e_2^c) is a GCVE for conjectures (r_1, r_2);

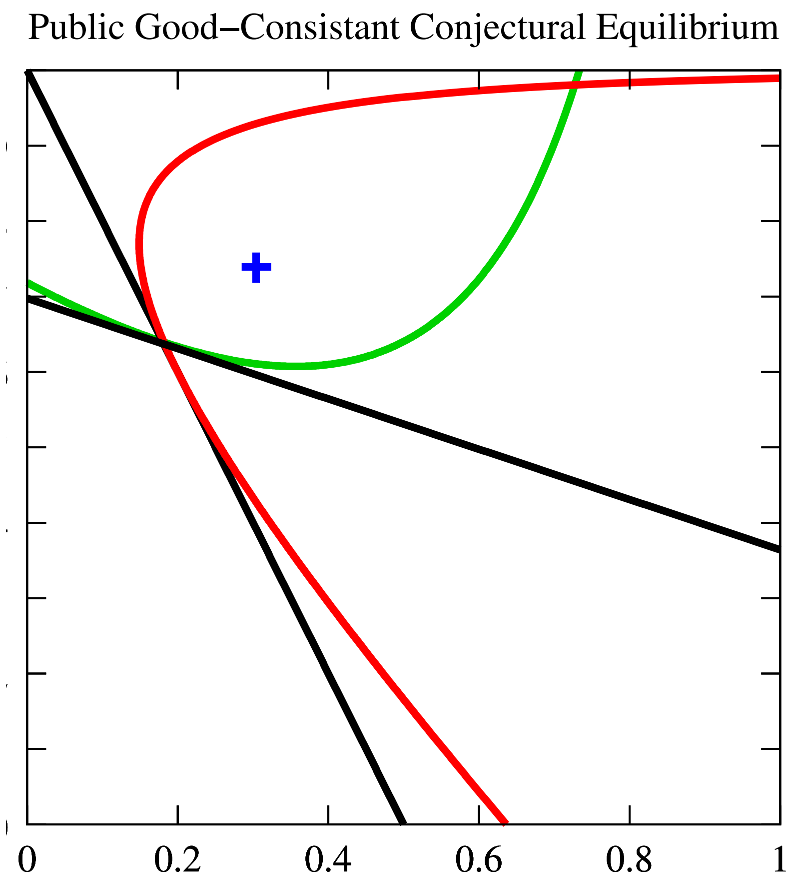
ii) $\chi_i(e_j)$ being the conjectural best response of player i, then in some neighborhood of (e_i^c, e_j^c),

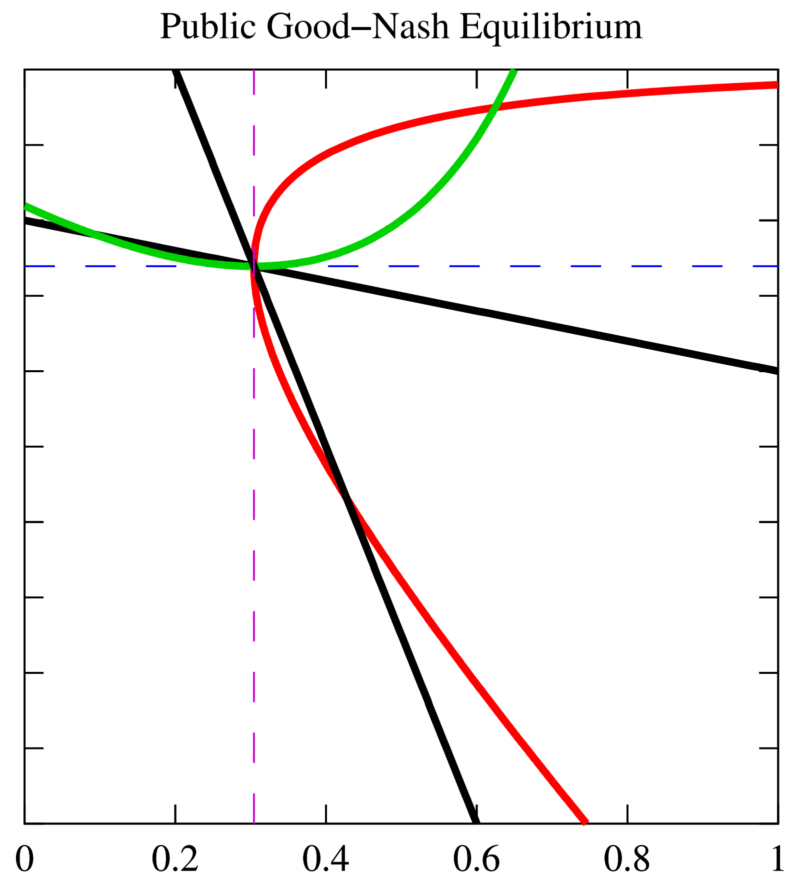
$$\boxed{\chi_i'(e_j) = r_i(\chi_i(e_j), e_j)\ .}$$

Geometry of consistency



Geometry of consistency (continued)





Computing consistent CVE

In general, computing CVE requires solving systems of difference-differential equations: find f_i and f_j,

$$\begin{aligned} 0 &= (1 + (f_i)'(e_j)(f_j)'(e_i))\, V^i_{ij}(e_i, e_j) \;+\; (f_i)'(e_j) V^i_{ii}(e_i, e_j) \\ &\quad + \; (f_j)'(e_i) V^i_{jj}(e_i, e_j) \;+\; (f_i)'(e_j)\, (f_j)''(e_i)\, V^i_j(e_i, e_j)\ , \end{aligned}$$

with $e_i = f_i(e_j)$.

- Few solutions are known, all with constant conjectures.
- In the duopoly, Bresnahan (1981) finds there are no analytic solutions. Olsder (1981) finds a multiplicity of solutions. To be investigated...

Conjectural variations equilibria as shorthands for dynamic interactions

Contents

Static Conjectural Variations Equilibria (CVE)

CVE as shorthands for dynamic interactions

- Principle
- Model of voluntary contribution to a public good
- Duopoly
- Conclusions

Principle

Ingredients

- A game with repeated interactions, with or without dynamics,
- some family of policies in which an optimal is sought
- a resulting stationary state.

Idea: the stationary state may be “summarized” by a static conjectural variations equilibrium. The conjecture captures the inter-temporal forces of the game.

When does it work?

The model of Itaya and Okamura

Itaya and Okamura (2003) consider a repeated game with quadratic payoff function.
The problem of player i is:

$$\max_{e_i(\cdot)} \sum_{t=0}^{\infty} \theta^t \left[I_i - e_i(t) \; + \; G(t)(A - G(t)) \right] \; ,$$

where e_i is i's contribution to a public good: $G(t) = \sum_i e_i(t)$.

Players adopt trigger strategies: given a profile $(e_1^*, \ldots, e_n^*)$

$$e_i(t) \;\; = \;\; \begin{cases} e_i^* & \text{if } e_j(t-1) = e_j^*, \forall j \neq i \\ 0 & \text{if } e_j(t-1) \neq e_j^*, \text{ for some } j \end{cases}$$

Itaya and Okamura (continued)

Among such strategies that are Nash Eq., the best is:

$$e_i^* = \frac{1}{2}(A-1) - E_{-i}^* + \sqrt{\frac{\theta}{1+\theta}E_{-i}^*}.$$

Also, a CVE with constant variation r satisfies:

$$e_i^c = \frac{1}{2}\left(A - \frac{1}{1+r}\right) - E_{-i}^c \ .$$

Identification yields (symmetric case):

$$\frac{r}{1+r} = 2\sqrt{\frac{\theta}{1+\theta}(n-1)\,e^*} \ .$$

Public goods, dynamic game

Feshtman and Nitzan (1991), Itaya and Shimomura (2001) have considered the game:

$$\max_{e_i(\cdot)} \int_0^\infty e^{-\theta t} U^i(e_i(t), G(t))\, \mathrm{d}t \ ,$$

$$\dot{G} \ = \ \sum_i e_i(t) \ - \ \delta G \ .$$

The associated static game is:

$$\max_{e_i} U^i(e_i, G) \qquad \text{with} \quad G \ = \ \sum_i e_i \ .$$

Public goods, dynamic game (continued)

The first order equations write as:

open loop Nash equilibrium
$$U_x^i \;+\; \frac{1}{\theta+\delta}\,U_G^i \;=\; 0$$

feedback Nash equilibrium

with $e_i = \phi_1 + \phi_2 G$
$$U_x^i \;+\; \frac{1}{\delta-(n-1)\phi_2}\,U_G^i \;=\; 0$$

CVE with variation r
$$U_x^i \;+\; r\,U_G^i \;=\; 0$$

Dynamic duopoly

Driskill and McCafferty (1989) and Dockner (1992) consider a dynamic duopoly with adjustment costs:

$$\int_0^{\infty} e^{-\theta\, t}\big[p(e_i(t), e_j(t))e_i(t) - C(e_i(t)) - A\left(x_i\left(t\right)\right)\big]\,\mathrm{d}t\ .$$

where e_i is a stock, Q_i an investment and:

$$\dot{e}_i \;=\; Q_i\ .$$

The associated static game is:

$$\max_{e_i}\ p(e_i, e_j)\ e_i \;-\; C(e_i)\ .$$

Dynamic duopoly (continued)

The first order equations write as:

open loop Nash equilibrium $$p\ [1+\phi]\ =\ C'$$

feedback Nash equilibrium

with $Q_i = \phi_1 + \phi_2 e_i + \phi_3 e_j$ $$p\ \left[1+\left(1+\frac{\phi_3}{\theta-\phi_2}\right)\phi\right]\ =\ C'$$

CVE with variation r $$p\ (E)\,[1+(1+r)\phi]\ =\ C_i'$$

Generalization

Figuières (2000) has studied the general linear-quadratic case:

$$\max_{Q_i(\cdot)} \int_0^\infty e^{-\theta t} \left[P^i(e_i(t), e_j(t)) - C(Q_i(t)) \right] \, \mathrm{d}t$$

with

$$P^i(e_i, e_j) = a_0 + a_1\, e_i + a_2\, e_j + \frac{a_3}{2}\, e_i^2 + a_4\, e_i e_j + \frac{a_5}{2}\, e_j^2 \,,$$

$$C(Q_i) \;=\; \frac{c}{2} Q_i^2 \,+ q\, Q_i \;,$$

$$\dot{e}_i \;=\; Q_i \;-\; \delta e_i \;.$$

Generalization (continued)

The CVE is:

$$e^c = \frac{a_1 + a_2 r}{-(a_3 + a_4) - (a_4 + a_5)\, r} \,.$$

The steady-state feedback and open-loop Nash equilibria are:

$$e^f = \frac{a_1 - (\theta + \delta) q + a_2 \frac{y^*/c}{\theta+\delta-x^*/c}}{(\theta + \delta)\delta c - (a_3 + a_4) - (a_4 + a_5) \frac{y^*/c}{\theta+\delta-x^*/c}} \,,$$

$$e^o = \frac{a_1 - (\theta + \delta) q}{(\theta + \delta)\delta c - (a_3 + a_4)} \,.$$

Conclusion

Equivalence... in some cases.
The connection between dynamic games and static conjectural variations equilibria needs to be investigated further.
Relationship between feedback strategies and consistent conjectural variations?

Pending issues for static conjectural variations:

- existence and uniqueness results of CVE
- existence of consistent CVE
- connexions with other models of “conjectures” (rationalizable...)

Bibliography

C. Figuières, A. Jean-Marie, N. Quérou and M. Tidball, *Theory of Conjectural Variations*, World Scientific Computing, 2004.
C. Figuières, A. Jean-Marie and M. Tidball, *On the effect of conjectures in a strategic setting*, *Ricerche Economiche*, 2004.

- Bowley, A.L. (1924), *The Mathematical Groundwork of Economics*, Oxford University Press, Oxford.
- Bresnahan, T.F. (1981), "Duopoly Models with Consistent Conjectures", *American Economic Review*, Vol. 71 (5), pp. 934–945.
- Dockner, E.J. (1992), "A Dynamic Theory of Conjectural Variations", *The Journal of Industrial Economics*, Vol. XL (4), pp. 377–395.
- Driskill, R.A. and McCafferty, S. (1989), "Dynamic Duopoly with Adjustment Costs: a Differential Game Approach", *Journal of Economic Theory*, Vol. 49, pp. 324–338.
- Fershtman, C. and Nitzan, S. (1991), "Dynamic Voluntary Provision of Public Goods", *European Economic Review*, Vol. 35, pp. 1057–1067.

Bibliography (continued)

- Figuières, C. (2000), “Dynamic Concepts of Interaction in Long Run Investment Games”, GREQAM working paper 00A01, University of Aix-Marseille 2.
- Figuières, C. (2002), “Complementarity, Substitutability and the Strategic Accumulation of Capital”, *International Game Theory Review*, Vol. 4(4), pp. 371–390.
- Itaya, J.-I. and Dasgupta, D. (1995), “Dynamics, Consistent Conjectures, and Heterogeneous Agents in the Private Provision of Public Goods”, *Public Finance/Finances Publiques*, Vol. 50(3), pp. 371–389.
- Itaya, J.-I. and Okamura, M. (2003), “Conjectural Variations and Voluntary Public Good Provision in a Repeated Game Setting”, *J. Public Economic Theory*, Vol. 5(1), pp. 51–66.
- Itaya, J.-I. and Shimomura, K. (2001), “A Dynamic Conjectural Variations Model in the Private Provision of Public Goods: a Differential Game Approach”, *Journal of Public Economics*, Vol. 81, pp. 153–172.
- Olsder, G.J. (1981), “A Critical Analysis of a New Equilibrium Concept”, Memorandum Nr. 329, Dept. Applied Maths., Twente University of Technology, The Netherlands.