On the Existence of Credible Incentive Equilibria

Alain Jean-Marie   Mabel Tidball

ajm@lirmm.fr   tidball@inra.ensam.fr

INRIA/LIRMM CNRS University of Montpellier 2

INRA/LAMETA Montpellier

Introduction

Necessary conditions for $\exists$ of credible incentive equilibria

We consider:
static games and dynamic games with open-loop strategies
We show:
two examples: stability of a Cartel, environmental problem

Results

- credible incentive equilibria with differentiable incentive do not exist without strong conditions on the payoff
- for piecewise-differentiable incentive functions, an infinity of credible incentive equilibria can be chosen.

Introduction

The incentive problem:

- construct a game in which players are induced to play a cooperative desired outcome E^* by defining an incentive rule
- E^* equilibrium of the game
- the incentive rule is credible

Credibility holds if every player, if faced with a deviation from her opponent, would prefer to follow the incentive rather than sticking to her equilibrium value.

Introduction

- Principle of incentive equilibria: developed for dynamic games by Ehtamo and Hämäläinen, inspired from the work of Osborne about the definition of a “quota rule” able to explain the stability of a Cartel.
- Used since for several applications in the Management of Natural Resources or in Marketing, by Ehtamo and Hämäläinen, by Jørgensen and Zaccour, and recently by Martín-Herrán and Zaccour.

Definitions

- Two-player game
- The strategy of player i will be denoted by $E_i \in \Sigma_i$
- The payoff function of player i

$$J_i : \Sigma_1 \times \Sigma_2 \longrightarrow \mathbb{R} .$$

- Both players agree, before playing the game, that a certain Pareto optimum E^* is a desired output of the game

Definitions

Definition 1 (Incentive equilibrium) *Consider a Pareto optimum* (E_1^*, E_2^*) *of the game. An incentive equilibrium strategy at this optimum is a pair of mappings* (Ψ_1, Ψ_2)*, with* $\Psi_1 : \Sigma_2 \rightarrow \Sigma_1$*,* $\Psi_2 : \Sigma_1 \rightarrow \Sigma_2$*, and such that:*

$$J_1(E_1, \Psi_2(E_1)) \leq J_1(E_1^*, \Psi_2(E_1^*)) \qquad \forall E_1 \in \Sigma_1$$

$$J_2(\Psi_1(E_2), E_2) \leq J_2(\Psi_1(E_2^*), E_2^*) \qquad \forall E_2 \in \Sigma_2$$

$$\Psi_1(E_2^*) = E_1^* \qquad \Psi_2(E_1^*) = E_2^* .$$

Definitions

Definition 2 (Credible incentive equilibrium) *The pair* (Ψ_1, Ψ_2) *is a credible incentive equilibrium at* (E_1^*, E_2^*) *if it is an incentive equilibrium, and if there exists a subset* $\Sigma_1' \times \Sigma_2'$ *of* $\Sigma_1 \times \Sigma_2$ *such that:*

$$J_1(\Psi_1(E_2), E_2) \quad \geq \quad J_1(E_1^*, E_2) \ ,$$

$$J_2(E_1, \Psi_2(E_1)) \quad \geq \quad J_2(E_1, E_2^*) \ ,$$

for all $E_1 \in \Sigma_1'$ *and* $E_2 \in \Sigma_2'$.

The static case

Incentive functions:

$$\Psi_i(E_j) = \begin{cases} \Psi_i^+(E_j) & \text{if } E_j \geq E_j^* \\ \Psi_i^-(E_j) & \text{if } E_j \leq E_j^* \end{cases},$$

where Ψ_i^+ and Ψ_i^- are differentiable, including at $E_j = E_j^*$.

$$a_i^+ = (\Psi_i^+)'(E_j^*)\,, \quad \text{and} \quad a_i^- = (\Psi_i^-)'(E_j^*)$$

$$A_i = -\frac{\partial J_i/\partial E_i}{\partial J_i/\partial E_j}(E_1^*, E_2^*)\,.$$

Necessary conditions for equilibria

E^* NOT NECESSARILY A PARETO OPTIMUM

$\partial J_1/\partial E_1$	$\partial J_2/\partial E_1$	Conditions
-	-	$A_2 = 0$ and $a_1^+ = a_1^- = 0$
+	+	$A_2 = 0$ and $a_1^+ = a_1^- = 0$
-	+	$a_1^+ \leq \min(A_2, 0) \leq \max(A_2, 0) \leq a_1^-$
+	-	$a_1^- \leq \min(A_2, 0) \leq \max(A_2, 0) \leq a_1^+$
0	-	$a_1^- \leq A_2 \leq a_1^+$
0	+	$a_1^+ \leq A_2 \leq a_1^-$
+	0	$\partial J_2/\partial E_2 = 0$ and $a_1^- \leq 0 \leq a_1^+$
-	0	$\partial J_2/\partial E_2 = 0$ and $a_1^+ \leq 0 \leq a_1^-$
0	0	$\partial J_2/\partial E_2 = 0$

Necessary conditions for equilibria (ctd)

Corollary 3 *Let E^* be a Pareto optimum,*

a/ $\partial J_1/\partial E_1 > 0$, $\partial J_1/\partial E_2 < 0$, $\partial J_2/\partial E_1 < 0$, $\partial J_2/\partial E_2 > 0$.

$$\rightarrow \quad a_1^- \leq 0 \leq A_2 \leq a_1^+, \quad \textit{and} \quad a_2^- \leq 0 \leq A_1 \leq a_2^+.$$

b/ $\partial J_1/\partial E_1 < 0$, $\partial J_1/\partial E_2 > 0$, $\partial J_2/\partial E_1 > 0$, $\partial J_2/\partial E_2 < 0$.

$$\rightarrow \quad a_1^+ \leq 0 \leq A_2 \leq a_1^-, \quad \textit{and} \quad a_2^+ \leq 0 \leq A_1 \leq a_2^-.$$

c/ $\partial J_1/\partial E_1 > 0$, $\partial J_1/\partial E_2 > 0$, $\partial J_2/\partial E_1 < 0$, $\partial J_2/\partial E_2 < 0$.

$$\rightarrow \quad a_1^- \leq A_2 \leq 0 \leq a_1^+, \quad \textit{and} \quad a_2^+ \leq A_1 \leq 0 \leq a_2^-.$$

Differentiable incentive equilibrium

E^* NOT NECESSARILY A PARETO OPTIMUM

a/ if $a_1 = 0$ and $a_2 = 0$, then necessarily

$$\frac{\partial J_i}{\partial E_i}(E_1^*, E_2^*) \;=\; 0 \;, \qquad i = 1, 2 \;;$$

b/ if $a_1 \neq 0$ and $a_2 \neq 0$, then necessarily

$$\frac{\partial J_i}{\partial E_j}(E_1^*, E_2^*) \;=\; 0 \;, \qquad i, j = 1, 2 \;;$$

c/ if $a_1 = 0$ and $a_2 \neq 0$, then necessarily

$$\frac{\partial J_2}{\partial E_2}(E_1^*, E_2^*) = 0 \;, \qquad \frac{\partial J_1}{\partial E_1}(E_1^*, E_2^*) \;+\; a_2 \, \frac{\partial J_1}{\partial E_2}(E_1^*, E_2^*) = 0$$

Differentiable credible incentive equilibria

Theorem 4 *Let (Ψ_1, Ψ_2) be a credible incentive equilibrium at a Pareto optimum, where the incentive functions Ψ_i are differentiable. Then, necessarily:*

$$\frac{\partial J_i}{\partial E_j}(E_1^*, E_2^*) \; = \; 0 \, , \qquad i, j = 1, 2 \, .$$

Nash equilibria and One-sided incentives

- A Nash equilibrium (E_1^N, E_2^N), with constant incentive functions: $\Psi_i(E_j) = E_i^N$, satisfies conditions for being a credible incentive equilibrium.
- One-sided incentives: We can also find on which side of the incentive equilibrium E^* it can be credible not to react, depending on the sign of the partial derivatives of the payoff functions.

Osborne's example

- Strategies E_i: level of production of the firms, $J_i(\cdot)$, their profit functions. With $\partial J_i / \partial E_i > 0$ and $\partial J_i / \partial E_j < 0$.
- The topic of Osborne's paper is the stability of a Cartel. In this context, the "incentive" function is actually a threat function, with which members of the Cartel would retaliate to potential cheaters.

$$\Psi_i(E_j) \;=\; \max\left\{E_i^*, E_i^* + \frac{E_i^*}{E_j^*}(E_j - E_j^*)\right\} \;,$$

This is a credible incentive equilibrium

The case of Nash Open-Loop equilibria

The state of the system evolves according to the differential equation

$$\dot{x}(t) \;=\; f(E_1(t), E_2(t), x(t))\;, \quad x(0) = x_0\;, \tag{1}$$

where $E_i(t)$ is the action of player i at time t according to her strategy E_i. Payoff of player i:

$$J_i(E_1, E_2; x_0) \;=\; \int_0^T e^{-\rho t} F_i(E_1(t), E_2(t), x(t))\, dt\;, \tag{2}$$

with a time horizon $T < +\infty$ and a discount factor $\rho \geq 0$.

Affine incentive equilibrium:

$\Psi_1(E_2)(t) \;=\; E_1^*(t) + v_1(t)(E_2(t) - E_2^*(t))$

Necessary conditions for the Open-Loop case

A credible affine incentive equilibrium at a Pareto optimum is a solution of the following system of equations, for some $\alpha_1 > 0$ and $\alpha_2 > 0$:

Conditions for being Pareto

$$\begin{cases} 0 = \alpha_1 \dfrac{\partial F_1}{\partial E_i} + \alpha_2 \dfrac{\partial F_2}{\partial E_i} + \lambda^* \dfrac{\partial f}{\partial E_i} & i = 1, 2 \\ \dot{\lambda}^* = -\alpha_1 \dfrac{\partial F_1}{\partial x} - \alpha_2 \dfrac{\partial F_2}{\partial x} - \lambda^* \dfrac{\partial f}{\partial x} + \rho \lambda^* \ ; & \lambda^*(T) = 0 \\ \dot{x}^* = f \ ; \qquad x(0) = x_0 & \end{cases}$$

Necessary conditions for the Open-Loop case

Conditions for being an incentive equilibria

$$\begin{cases} 0 &= \dfrac{\partial F_1}{\partial E_1} + v_2 \dfrac{\partial F_1}{\partial E_2} + \lambda^1 \left(\dfrac{\partial f}{\partial E_1} + v_2 \dfrac{\partial f}{\partial E_2} \right) \\ \dot{\lambda}^1 &= -\dfrac{\partial F_1}{\partial x} - \lambda^1 \dfrac{\partial f}{\partial x} + \rho \lambda^1 \, ; \quad \lambda^1(T) = 0 \\ 0 &= v_1 \dfrac{\partial F_2}{\partial E_1} + \dfrac{\partial F_2}{\partial E_2} + \lambda^2 \left(v_1 \dfrac{\partial f}{\partial E_1} + \dfrac{\partial f}{\partial E_2} \right) \\ \dot{\lambda}^2 &= -\dfrac{\partial F_2}{\partial x} - \lambda^2 \dfrac{\partial f}{\partial x} + \rho \lambda^2 \, ; \quad \lambda^2(T) = 0 \end{cases}$$

Necessary conditions for the Open-Loop case

Conditions for being credible

$$\begin{cases} 0 &= -v_1 \dfrac{\partial F_1}{\partial E_1} + \lambda^1 \dfrac{\partial f}{\partial E_2} + \lambda^{1c} \left(v_1 \dfrac{\partial f}{\partial E_1} + \dfrac{\partial f}{\partial E_2} \right) \\ \dot{\lambda}^{1c} &= \dfrac{\partial F_1}{\partial x} - \lambda^{1c} \dfrac{\partial f}{\partial x} + \rho \lambda^{1c} \; ; \qquad \lambda^{1c}(T) = 0 \\ 0 &= -v_2 \dfrac{\partial F_2}{\partial E_2} + \lambda^2 \dfrac{\partial f}{\partial E_1} + \lambda^{2c} \left(\dfrac{\partial f}{\partial E_1} + v_2 \dfrac{\partial f}{\partial E_2} \right) \\ \dot{\lambda}^{2c} &= \dfrac{\partial F_2}{\partial x} - \lambda^{2c} \dfrac{\partial f}{\partial x} + \rho \lambda^{2c} \; ; \qquad \lambda^{2c}(T) = 0 \end{cases}$$

Properties

- Necessary conditions stated $\rightarrow E^*$ must be a simultaneous maximum for both payoff functions J_i.
- Piecewise-differentiable incentive functions.
  $V_i(t, E_j(t)) = V_i^+(t, E_j(t))$ if $E_j(t) \geq E_j^*(t)$ and
  $V_i(t, E_j(t)) = V_i^-(t, E_j(t))$ if $E_j(t) \leq E_j^*(t)$.

  Left and right-derivatives: $v_i^\pm(t) = \partial V_i^\pm / \partial E_j(t, E_j^*(t))$.

  Transposition of the results of the static case: replace "$\partial J_i / \partial E_j$" by "$\partial F_i / \partial E_j + \lambda^i \partial f / \partial E_j$".

Environmental example

$$J_i(E_1(\cdot), E_2(\cdot); x_0) \;=\; \int_0^\infty e^{-\rho t}\left(\log(E_i(t)) - \phi_i x(t)\right)\, dt\ ,$$

$$\dot{x}(t) \;=\; E_1(t) + E_2(t) - \delta x(t), \quad x(0) = x_0\ .$$

Pareto solution, the maximization of $\sum_i \alpha_i J_i$, is:

$$E_i^* \;=\; \frac{\alpha_i(\delta+\rho)}{\alpha_1\phi_1 + \alpha_2\phi_2}\ .$$

The Pareto-optimal control does not depend on time

Environmental example. Static credibility

Consider only time-invariant strategies. The total payoff of player i is given by:

$$J_i(e_1, e_2; x_0) \;=\; \frac{1}{\rho}\log(e_i) \;-\; \frac{\phi_i}{\rho(\rho+\delta)}(e_1+e_2) \;-\; \frac{\phi_i x_0}{\rho+\delta}\,.$$

$$A_i \;=\; \frac{\alpha_j\phi_j}{\alpha_i\phi_i}\,, \qquad A_j \;=\; \frac{1}{A_i} \;=\; \frac{\alpha_i\phi_i}{\alpha_j\phi_j}\,.$$

$$a_i^- \le 0 \le \frac{\alpha_i\phi_i}{\alpha_j\phi_j} \le a_i^+.$$

Environmental example. Static credibility

We select the piecewise affine function:

$$\Psi_i(e_j) \;=\; \max\left\{ e_i^*, e_i^* + \frac{\alpha_i \phi_i}{\alpha_j \phi_j}(e_j - e_j^*) \right\} \; .$$

For Player 1, the credibility condition becomes: for $e_2 \geq e_2^*$:

$$0 \;\leq\; \log \frac{e_1^* + \alpha_1\phi_1/\alpha_2\phi_2(e_2 - e_2^*)}{e_1^*} \;-\; \phi_1 \, \frac{\alpha_1\phi_1}{\alpha_2\phi_2} \, (e_2 - e_2^*) \; .$$

$\exists$ interval $[e_2^*, \bar{e}_2]$ where the condition is satisfied.

$$\bar{e}_2 \;\geq\; e_2^* \left\{ 1 + 2 \, \frac{\alpha_2}{\alpha_1} \left(\frac{\phi_2}{\phi_1} \right)^2 \right\} \; .$$

Environmental example. Dynamic credibility

$\alpha_1 = \alpha_2 = 1$.

This implies that $e_1^* = e_2^* = e^* = (\delta + \rho)/(\phi_1 + \phi_2)$.

We select the incentive function:

$$\Psi_i(E_j)(t) \;=\; e^* \;+\; \max\left\{\frac{\phi_i}{\phi_j}(E_j(t) - e^*), 0\right\} \;.$$

Environmental example. Dynamic credibility

Condition of credibility for player 1

$$\int_0^{+\infty} \left[\log(\frac{\Psi_1(E_2(t))}{e^*}) - \phi_1 x^\Psi(t) + \phi_1 x^*(t) \right] e^{-\rho t} \, dt \ \geq \ 0 \ ,$$

where the two trajectories $x^\Psi(\cdot)$ and $x^*(\cdot)$ are the respective solutions of

$$\begin{aligned} \dot{x} &= e^* + \frac{\phi_1}{\phi_2} \max(0, E_2(t) - e^*) + E_2(t) - \delta x(t) \\ \dot{x} &= e^* + E_2(t) - \delta x(t) \end{aligned}$$

Environmental example. Dynamic credibility

Assume that there exists $M \geq 1$ such that for all t,

$$E_2(t) \;\leq\; M\, e^* \,.$$

credibility implies that $E_2(\cdot)$ verifies

$$\int_0^{+\infty} \left[\frac{\phi_1}{\phi_2} + \frac{2\phi_1^2}{\phi_2^2}\right] \frac{E_2(t)}{e^*} e^{-\rho t} dt \;\geq$$

$$\frac{1}{\rho}\left[\frac{\phi_1^2}{\phi_2^2}(M^2-1) + \frac{\phi_1}{\phi_2} + \frac{\phi_1^2}{\phi_2^2}\frac{M-1}{\rho+\delta}e^*\right] \,.$$

Environmental example. Dynamic credibility

For instance, it can be checked that the equilibrium is credible with respect to strategies of the form

$$E_2(t) \;=\; e^N + (e^* - e^N)e^{-\alpha t} \;,$$

or

$$E_2(t) \;=\; e^* + (e^N - e^*)e^{-\alpha t} \;,$$

where $e^N = (\rho + \delta)/2$ is the Nash equilibrium of the game (a time-invariant strategy as well).

Conclusion

- Credibility is difficult to obtain in static and continuous-time games: at a Pareto solution as well as elsewhere, if the incentive function is required to be differentiable. A credible incentive equilibria may happen only at critical points of both payoff functions simultaneously.
- With piecewise-differentiable incentive functions, (local) credibility is rather easy to obtain, and many slopes are generally allowed for these incentive functions. The actual challenge is to find incentive functions that provide a “domain of credibility” as large as possible.

Extensions

As logical continuations of this work, we mention:

- Study whether credibility of open-loop strategies may hold in a neighborhood of the equilibrium, not only in a particular subset of deviations.
- Extend the analysis to discrete-time problems.
- Investigate incentives defined on Nash-Feedback strategies.